

A variational approach to a fuzzy Boltzmann equation

Matthias Erbar (Bielefeld)

Gradient flows face to face 2023

11 September 2023

joint work in progress with Zihui He (Bielefeld)

Outline

1. Models

2. Existence results

- Known results and motivation
- Existence and uniqueness results

3. Variational characterization

- GENERIC framework
- Variational characterisation

Models
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Existence results
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Variational characterization
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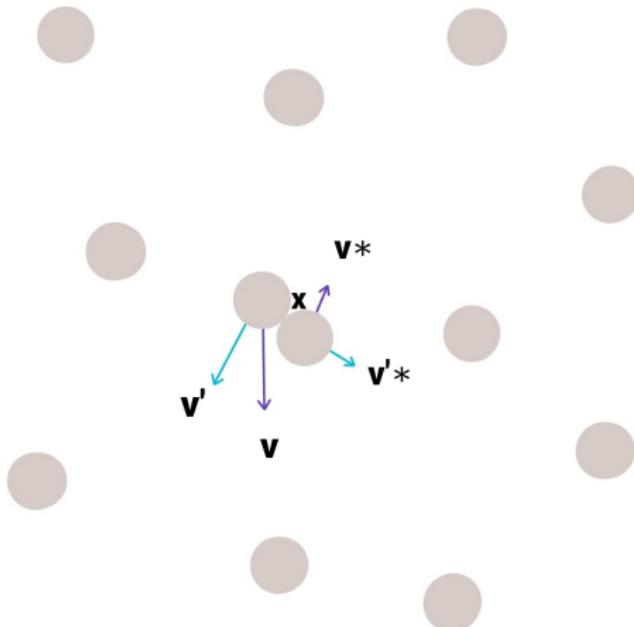
- GENERIC framework
- Variational characterisation

Models
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Existence results
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Variational characterization
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Collisional particle dynamics

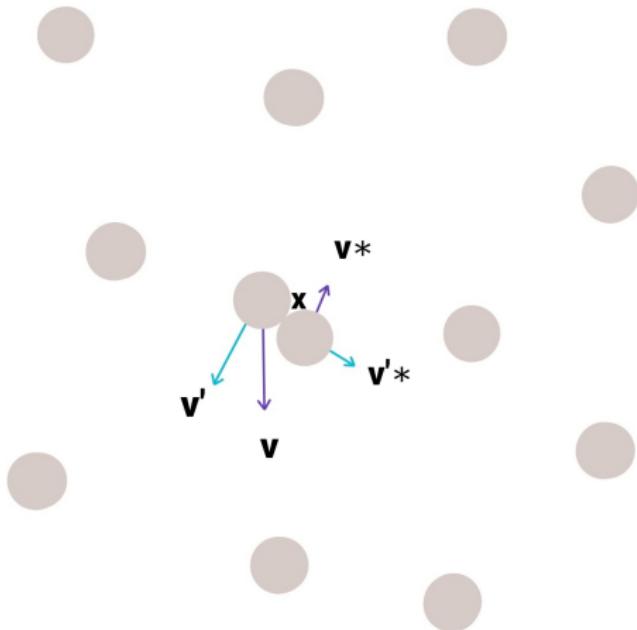


Models
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Variational characterization
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Collisional particle dynamics



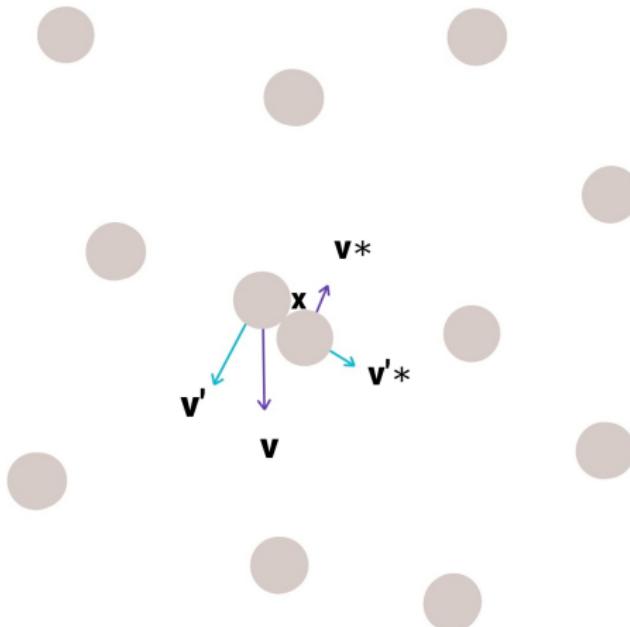
- Two-particle elastic collision
 - conservation of momentum
 $v + v_* = v' + v'_*$
 - conservation of energy
 $|v|^2 + |v_*|^2 = |v'|^2 + |v'|^2$

Models
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Collisional particle dynamics



- Two-particle elastic collision
 - conservation of momentum
 $v + v_* = v' + v'_*$
 - conservation of energy
 $|v|^2 + |v_*|^2 = |v'|^2 + |v'|^2$
- Consider scaling limit of dilute gas
 - $x = \varepsilon y, t = \varepsilon \tau$
 - $N \rightarrow \infty, N\varepsilon^2 \sim 1$

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The Boltzmann equation



Ludwig Boltzmann
1844–1906

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The Boltzmann equation



Ludwig Boltzmann
1844–1906

$$\partial_t f_t + \underbrace{v \cdot \nabla_x f_t}_{\text{transport term}} = \underbrace{Q(f_t)}_{\text{collision term}}$$

Unknown: $f_t(x, v) : [0, T] \times (\mathbb{R}^3)^2 \rightarrow \mathbb{R}_+$ probability density of particles

$x \in \mathbb{R}^3$ position, $v \in \mathbb{R}^3$ velocity

Collision term:

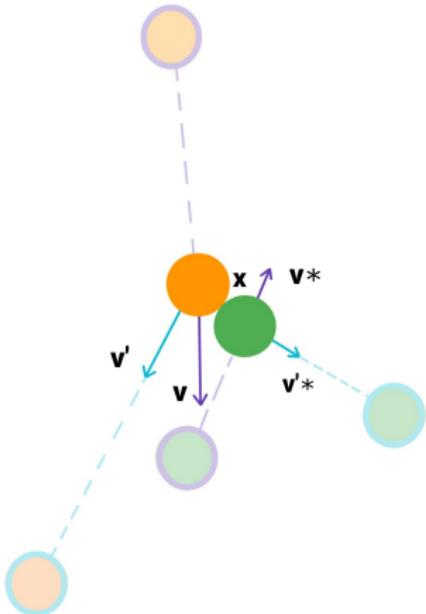
$$Q(f_t) = \int_{\mathbb{R}^3 \times S^2} (f_t(x, v') f_t(x, v'_*) - f_t(x, v) f_t(x, v_*)) B(v - v_*, w) dv_* dw$$

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Collision term

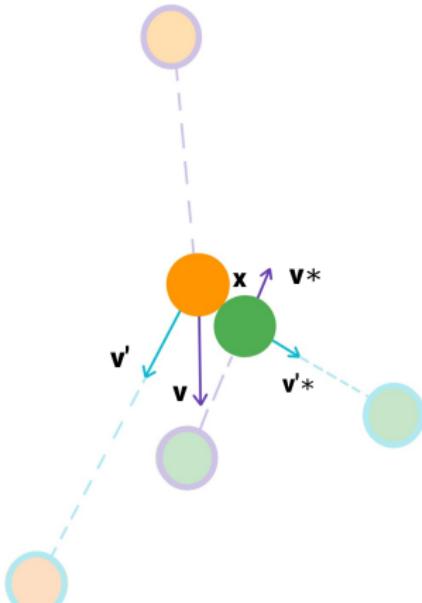


Models
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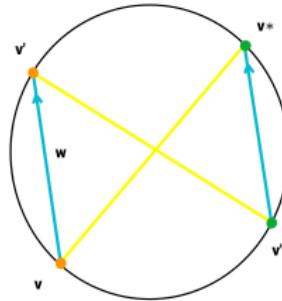
Existence results
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Collision term



Models
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$$v' = v - \langle v - v_*, w \rangle w$$
$$v'_* = v_* + \langle v - v_*, w \rangle w$$

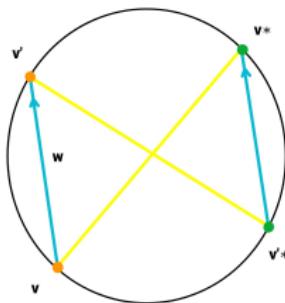
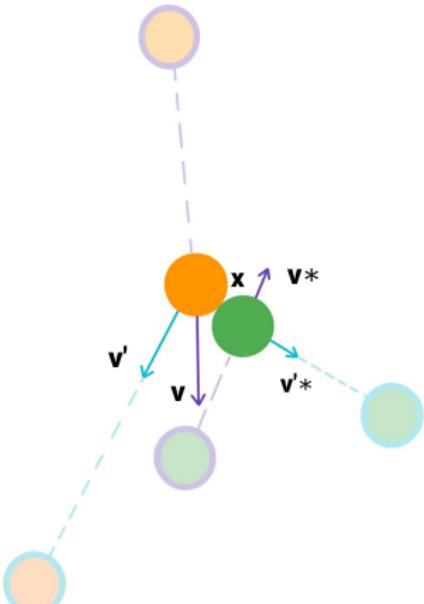
conservation of momentum
 $v + v_* = v' + v'_*$

conservation of energy
 $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$

Existence results
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Variational characterization
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Collision term



$$\begin{aligned} v' &= v - \langle v - v_*, w \rangle w \\ v'_* &= v_* + \langle v - v_*, w \rangle w \end{aligned}$$

conservation of momentum
 $v + v_* = v' + v'_*$

conservation of energy
 $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$

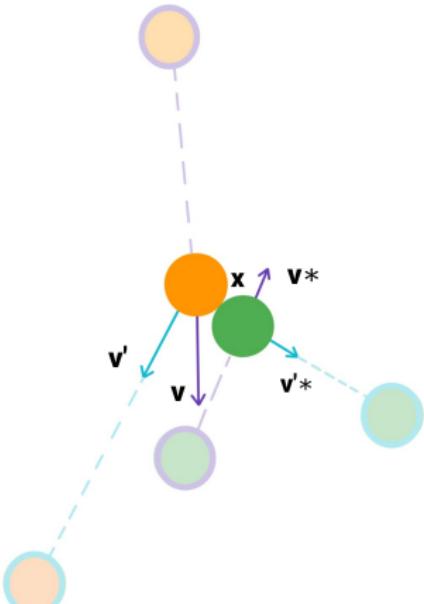
Collision term

$$Q(f_t)(x, v) = \int_{\mathbb{R}^3 \times S^2} \left(\underbrace{(f_t(x, v') f_t(x, v'_*)}_{\text{gain}} \right. \\ \left. - \underbrace{f_t(x, v) f_t(x, v_*)}_{\text{loss}} \right) B(v - v_*, w) dv_* dw$$

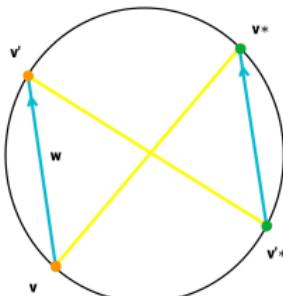
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Collision term



Models
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$$v' = v - \langle v - v_*, w \rangle w$$
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conservation of momentum
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Existence results
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Collision term

$$Q(f_t)(x, v) = \int_{\mathbb{R}^3 \times S^2} \left(\underbrace{f_t(x, v') f_t(x, v'_*)}_{\text{gain}} - \underbrace{f_t(x, v) f_t(x, v_*)}_{\text{loss}} \right) B(v - v_*, w) dv_* dw$$

Collision kernel (physically)

$$B(v - v_*, w) = |v - v_*|^\mu b(\theta)$$

Variational characterization
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Fuzzy Boltzmann equation

Equilibration in space \rightsquigarrow spatially homogeneous Boltzmann equation

$$\partial_t f_t(v) = Q_H(f_t)(v)$$

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Fuzzy Boltzmann equation

Equilibration in space \leadsto spatially homogeneous Boltzmann equation

$$\partial_t f_t(v) = Q_H(f_t)(v)$$

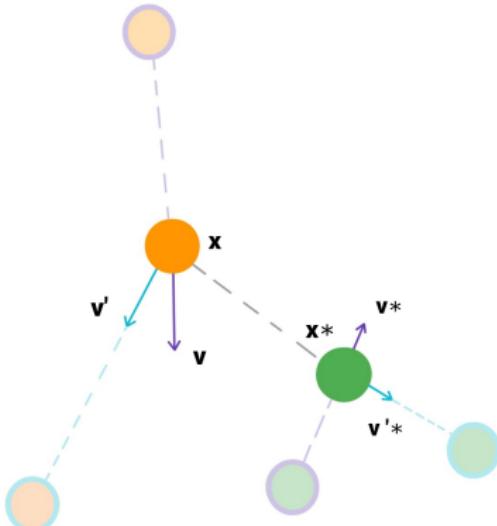
Delocalized collision \leadsto Fuzzy Boltzmann equation

$$\partial_t f_t(x, v) + v \cdot \nabla_x f_t(x, v) = Q_F(f_t)(x, v)$$

Delocalized collision term:

$$Q_F(f_t) = \int_{(\mathbb{R}^3)^2 \times \mathcal{S}^2} (f' f'_* - f f_*) B(v - v_*, w) k(x - x_*) dx_* dv_* dw$$

$$f = f_t(x, v), \quad f_* = f_t(x_*, v_*), \quad f' = f_t(x, v'), \quad f'_* = f_t(x_*, v'_*)$$



Delocalized collision

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Variational characterization
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Questions

$$\mathbb{P}_{x,v} := \{f \in L^1_{x,v} \mid f \geq 0, \|f\|_{L^1} = 1\}$$

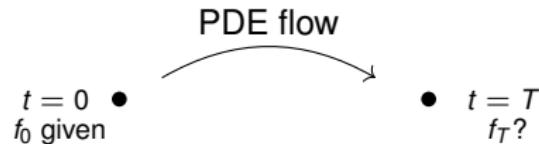
Models
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Questions

Existence of weak solutions



- Given $f_0 \in \mathbb{P}_{x,v}$
- Exists $(f_t) \subset \mathbb{P}_{x,v}$ solving fuzzy Boltzmann equation?

$$\mathbb{P}_{x,v} := \{f \in L^1_{x,v} \mid f \geq 0, \|f\|_{L^1} = 1\}$$

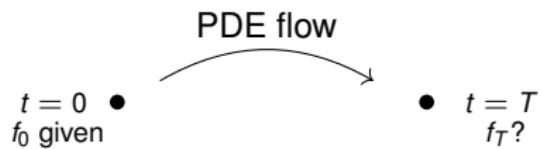
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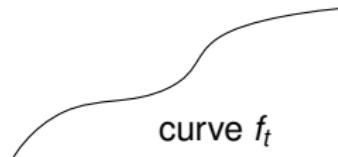
Questions

Existence of weak solutions



- Given $f_0 \in \mathbb{P}_{x,v}$
- Exists $(f_t) \subset \mathbb{P}_{x,v}$ solving fuzzy Boltzmann equation?

Characterization of solution curves



- Find variational characterization of the solution among curves $(f_t)_t$ in $\mathbb{P}_{x,v}$.
- Hidden structure?

$$\mathbb{P}_{x,v} := \{f \in L^1_{x,v} \mid f \geq 0, \|f\|_{L^1} = 1\}$$

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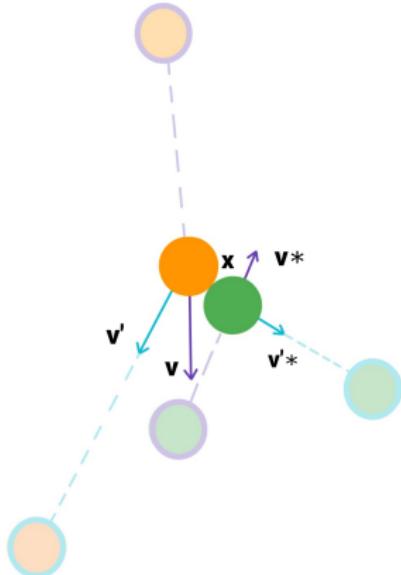
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Known results

Homogeneous Boltzmann equation (x degenerated)



Inhomogeneous Boltzmann equation

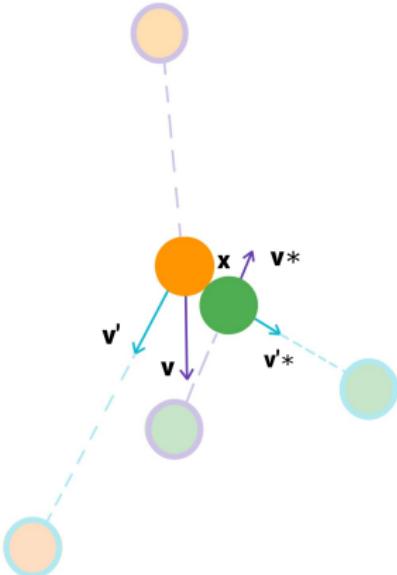
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Variational characterization
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Known results

Homogeneous Boltzmann equation (x degenerated)



$$\partial_t f_t(v) = Q_H(f_t)(v)$$

$$Q_H(f_t)(v) = \int_{\mathbb{R}^3 \times S^2} (f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)) B dv_* dw$$

A priori estimate: $\|Q_H(f_t)\|_{L_v^1} \lesssim 2 \|f_t\|_{L_v^1}^2$

Existence & uniqueness: Arkeryd 1972, $f_t \in C_T \mathbb{P}_v$

Inhomogeneous Boltzmann equation

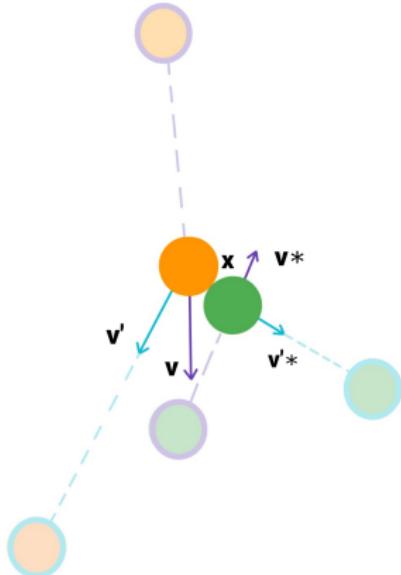
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Inhomogeneous Boltzmann equation

$$\partial_t f_t(x, v) + v \cdot \nabla_x f_t(x, v) = Q_I(f_t)(x, v)$$

$$Q_I(f_t)(x, v) = \int_{\mathbb{R}^3 \times S^2} (f_t(\textcolor{orange}{x}, v') f_t(\textcolor{orange}{x}, v'_*) - f_t(\textcolor{orange}{x}, v) f_t(\textcolor{orange}{x}, v_*)) B dv_* dw$$

Difficulty: lack of a priori estimate in L^1 -framework

Existence: DiPerna–Lions 1989, renormalized solutions Uniqueness: open

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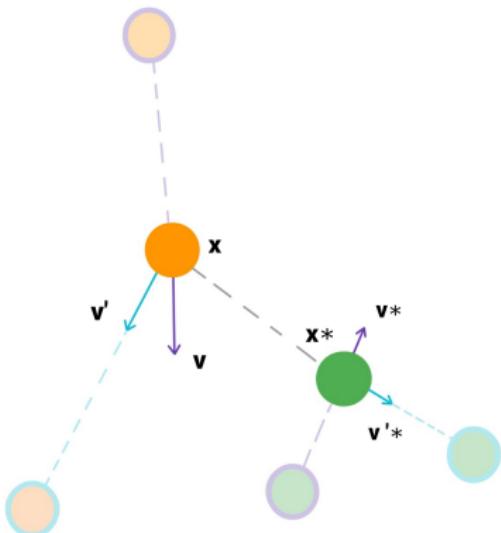
Variational characterization
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Motivation

Fuzzy Boltzmann equation

$$\partial_t f_t(x, v) + v \cdot \nabla_x f_t(x, v) = Q_F(f_t)(x, v)$$

$$Q_F(f_t)(x, v) = \int_{(\mathbb{R}^3)^2 \times S^2} (f_t(\mathbf{x}, v') f_t(\mathbf{x}_*, v'_*) - f_t(\mathbf{x}, v) f_t(\mathbf{x}_*, v_*)) B k dx_* dv_* dw$$



Example of delocalized collision

Models
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Existence results
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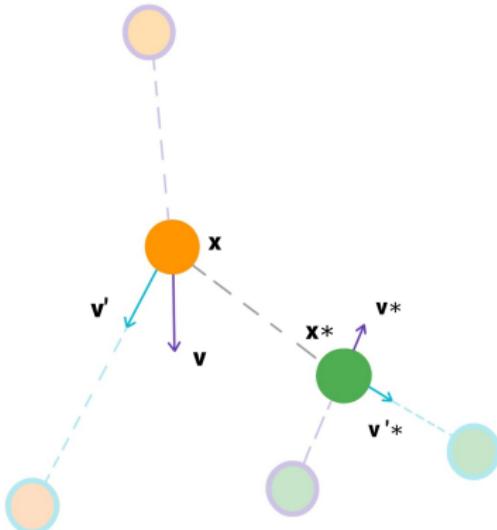
Variational characterization
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Motivation

Fuzzy Boltzmann equation

$$\partial_t f_t(x, v) + v \cdot \nabla_x f_t(x, v) = Q_F(f_t)(x, v)$$

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Example of delocalized collision

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Existence and uniqueness results

Assume $B(v - v_*, w) \sim \langle v - v_* \rangle^\mu$ for $\mu \in (-\infty, 1]$ and $k(x - x_*)$ bounded

Theorem 1

For $f_0 \in \mathbb{P}_{x,v}$ $\exists!$ weak solution $(f_t) \subset \mathbb{P}_{x,v}$ of fuzzy Boltzmann equation in the sense

$$\frac{d}{dt} \int_{(\mathbb{R}^3)^2} \varphi f_t - \int_{(\mathbb{R}^3)^2} f_t v \cdot \nabla_x \varphi = \int_{(\mathbb{R}^3)^4 \times S^2} \varphi \left(f' f'_* - f f_* \right) B k, \quad \forall \varphi \in C_c^\infty.$$

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$$\frac{d}{dt} \int_{(\mathbb{R}^3)^2} \varphi f_t - \int_{(\mathbb{R}^3)^2} f_t v \cdot \nabla_x \varphi = -\frac{1}{4} \int_{(\mathbb{R}^3)^4 \times S^2} \underbrace{\left(\varphi' + \varphi'_* - \varphi_* - \varphi \right)}_{=: \bar{\nabla} \varphi} \left(f' f'_* - f f_* \right) B k, \quad \forall \varphi \in C_c^\infty.$$

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- **momentum conservation:** $v f_0 \in L^1 \Rightarrow \|v f_t\|_{L^1} = \|v f_0\|_{L^1}$
- **energy conservation:** $|v|^2 f_0 \in L^1 \Rightarrow \||v|^2 f_t\|_{L^1} = \||v|^2 f_0\|_{L^1}$

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\mathcal{H} -Theorem

Entropy $\mathcal{H}(f_t) := \int_{(\mathbb{R}^3)^2} f_t \log f_t$

Entropy dissipation $\mathcal{D}(f_t) := - \int_{(\mathbb{R}^3)^4 \times S^2} \log f \left(f' f'_* - f f_* \right) B k = \frac{1}{4} \int_{(\mathbb{R}^3)^4 \times S^2} \bar{\nabla} \log f \left(f' f'_* - f f_* \right) B k$

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\mathcal{H} -Theorem

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$$= \frac{1}{4} \int_{(\mathbb{R}^3)^4 \times S^2} \underbrace{\Lambda(f' f'_*, f f_*)}_{\Lambda(a,b) = \frac{a-b}{\log a - \log b} \text{ logarithm mean}} (\bar{\nabla} \log f_t)^2 B k$$

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\mathcal{H} -Theorem

Entropy $\mathcal{H}(f_t) := \int_{(\mathbb{R}^3)^2} f_t \log f_t$

Entropy dissipation $\mathcal{D}(f_t) := \frac{1}{4} \int_{(\mathbb{R}^3)^4 \times S^2} \Lambda(f' f'_*, ff_*)(\bar{\nabla} \log f_t)^2$

Theorem 2

For a weak solution $(f_t)_t \subset \mathbb{P}_{x,v}$ to fuzzy Boltzmann equation with $\mathcal{H}(f_0) < \infty$:

$$\mathcal{H}(f_t) - \mathcal{H}(f_s) = - \int_s^t \mathcal{D}(f_r) dr \leq 0.$$

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GENERIC framework

(General Equations for Non-Equilibrium Reversible-Irreversible Coupling)

$$\partial_t f_t = \underbrace{L(f_t) D\mathcal{E}(f_t)}_{\text{conservative Hamiltonian}} - \underbrace{K(f_t) D\mathcal{H}(f_t)}_{\text{dissipative gradient flow}}$$

L anti-symmetric operator, \mathcal{E} energy, K symmetric positive-definite operator, \mathcal{H} entropy

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L anti-symmetric operator, \mathcal{E} energy, K symmetric positive-definite operator, \mathcal{H} entropy

key: non-interaction condition $L(f_t)D\mathcal{H}(f_t) = K(f_t)D\mathcal{E}(f_t) = 0$

$$\Rightarrow \frac{d}{dt} \mathcal{E}(f_t) = 0 \quad \text{and} \quad \frac{d}{dt} \mathcal{H}(f_t) = \langle D\mathcal{H}(f_t), K(f_t)D\mathcal{H}(f_t) \rangle \leq 0$$

GENERIC framework

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$$\Rightarrow \frac{d}{dt} \mathcal{E}(f_t) = 0 \quad \text{and} \quad \frac{d}{dt} \mathcal{H}(f_t) = \langle D\mathcal{H}(f_t), K(f_t)D\mathcal{H}(f_t) \rangle \leq 0$$

$$\partial_t f_t = -K(f_t) D\mathcal{H}(f_t) \quad \text{gradient flow}$$

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Reformulation of fuzzy Boltzmann equation

fuzzy Boltzmann equation $\partial_t f_t = -v \cdot \nabla_x f_t + Q_F(f_t)$
formally \rightsquigarrow GENERIC $\partial_t f_t = L(f_t)D\mathcal{E}(f_t) - K(f_t)D\mathcal{H}(f_t)$

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fuzzy Boltzmann equation $\partial_t f_t = -v \cdot \nabla_x f_t + Q_F(f_t)$

formally \rightsquigarrow GENERIC $\partial_t f_t = L(f_t)D\mathcal{E}(f_t) - K(f_t)D\mathcal{H}(f_t)$

$$\mathcal{E}(f_t) = \frac{1}{2} \int |v|^2 f_t, \quad \mathcal{H}(f_t) = \int f_t \log f_t, \quad L(f_t)g_t = -\operatorname{div} \left(f_t \begin{pmatrix} 0 & \operatorname{id} \\ -\operatorname{id} & 0 \end{pmatrix} \nabla g_t \right), \quad K(f_t)g_t = -\frac{1}{4} \overline{\operatorname{div}}(\Lambda(f_t) \bar{\nabla} g_t B k)$$

Models
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Reformulation of fuzzy Boltzmann equation

fuzzy Boltzmann equation $\partial_t f_t = -v \cdot \nabla_x f_t + Q_F(f_t)$

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Coincidence with analytic results: $\mathcal{E}(f_t)$ conserved, $\mathcal{H}(f_t)$ decreased

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Homogeneous Boltzmann equations $\partial_t f_t = Q_H(f_t)$ has gradient flow structure (E. '21)

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A variational characterization of GENERIC

Let us set: $\Psi(f, \xi) := \frac{1}{2} \langle \xi, K(f)\xi \rangle$ and $\Psi^*(f, \eta) := \frac{1}{2} \langle \eta, K(f)^{-1}\eta \rangle$

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Then for any smooth curve $(f_t)_{t \in [0, T]}$ formally

$$J_T(f_t) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \int_0^T \Psi(f_t, -D\mathcal{H}(f_t)) dt + \int_0^T \Psi^*(f_t, \partial_t f_t - L(f_t)D\mathcal{E}(f_t)) dt \geq 0.$$

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Furthermore, $J_T(f_t) = 0$ iff

$$\partial_t f_t - L(f_t)D\mathcal{E}(f_t) = -K(f_t)D\mathcal{H}(f_t)$$

i.e. $(f_t)_{t \in [0, T]}$ solves the GENERIC equation.

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A variational characterisation of the fuzzy Boltzmann equation

For the **fuzzy Boltzmann equation** choose formally

$$\Psi(f, \xi) := \frac{1}{2} \langle \xi, K(f)\xi \rangle_{L^2} = \frac{1}{8} \int |\bar{\nabla} \xi|^2 \Lambda(f) ,$$

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This leads to

$$\Psi(f_t, -D\mathcal{H}(f_t)) = \frac{1}{2} \mathcal{D}(f_t) \quad \text{entropy dissipation}$$

$$\Psi^*(f_t, \partial_t f_t - L(f_t)D\mathcal{E}(f_t)) = \inf_{U_t} \left\{ \frac{1}{8} \int \frac{|U_t|^2}{\Lambda(f'_t(f_t)_*, f_t(f_t)_*)} \mid \underbrace{\partial_t f_t + v \cdot \nabla_x f_t = -\frac{1}{4} \overline{\operatorname{div}} U_t}_{\text{transport collision rate equation}} \right\} =: \frac{1}{2} \mathcal{A}(f_t) \quad \text{curve action}$$

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Rmk: Also can obtain generalised GENERIC structures for non-quadratic Ψ, Ψ^*

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A variational characterisation of the fuzzy Boltzmann equation

Assume $B(v - v_*, w) \sim \langle v - v_* \rangle^\mu$ for $\mu \in (-\infty, 1]$ and $k(x - x_*) \sim \langle x - x_* \rangle^\nu$ for $\nu \leq 0$

Theorem 3

Let $(f_t)_{t \in [0, T]} \subset \mathbb{P}_{x,v}$ be weakly continuous in time. If

$$\mathcal{H}(f_0), \quad \int_0^T \int (|v|^{2+\mu_+} + |x|^2) f_t dt, \quad \int_0^T \sqrt{\mathcal{A}(f_t) \mathcal{D}(f_t)} dt \quad \text{are finite}$$

then

$$J_T(f_t) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2} \int_0^T \underbrace{\mathcal{D}(f_t)}_{\text{entropy dissipation}} dt + \frac{1}{2} \underbrace{\mathcal{A}_T(f_t)}_{\text{curve action}} \geq 0.$$

Furthermore, $J_T(f_t) = 0 \Leftrightarrow (f_t)_{t \in [0, T]}$ is a weak solution of fuzzy Boltzmann equation.

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Ideas of proof

Consider transport collision rate equations

$$\partial_t f_t + v \cdot \nabla_x f_t = -\frac{1}{4} \overline{\text{div}} U_t, \quad U_t : (\mathbb{R}^3)^4 \times \mathcal{S}^2 \rightarrow \mathbb{R},$$

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Key:

$$\frac{d}{dt} \mathcal{H}(f_t) = \frac{1}{4} \int_{(\mathbb{R}^3)^4 \times S^2} \bar{\nabla} \log(f_t) U_t$$

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Cauchy–Schwarz inequality $\Rightarrow \mathcal{H}(f_T) - \mathcal{H}(f_0) \geq -\frac{1}{2} \int_0^T \mathcal{D}(f_t) dt - \frac{1}{2} \mathcal{A}_T(f_t)$

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“=” holds $\Leftrightarrow U_t = -\bar{\nabla} \log f_t \Lambda(f_t) \rightsquigarrow$ fuzzy Boltzmann equation

Justification requires careful regularisation.

Models
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Thank you for your attention!

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