Hypocoercivity for run and tumble equations

Havva Yoldaş
Delft University of Technology

based on joint works with Jo Evans (U. Warwick)

Gradient flows face-to-face 3, Université Claude Bernard Lyon 1, France

September 14, 2023

References

- J. Evans & H.Y., *On the asymptotic behaviour of the run and tumble equation for bacterial chemotaxis*, to appear on SIAM J. Mathematical Analysis.
- J. Evans & H.Y., *Trend to equilibrium for run and tumble equations with non-uniform tumbling kernels*, preprint on arXiv (2023).

E. coli in motion by Howard Berg

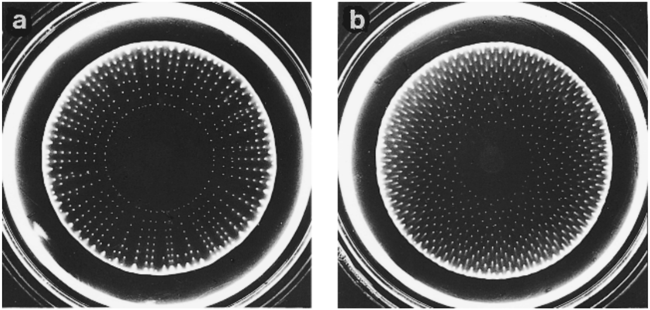
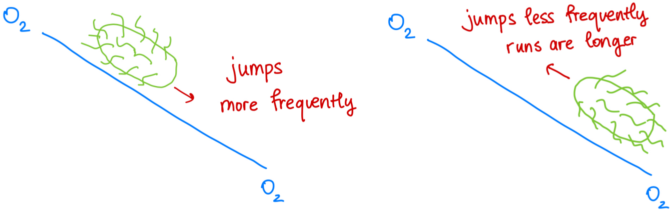


FIGURE 3.7. (a) Cells of a mutant of *E. coli* chemotactic to aspartate but not to serine that have spread outward in a soft-agar plate to form radial arrays of spots. (b) Cells of the same kind that have formed a hexagonal array of spots. The carbon source was α-ketoglutarate (2.5 mM), which is not a chemoattractant. Plate (a) contained, in addition, 2.5 mM hydrogen peroxide, and plate (b) 2.0 mM hydrogen peroxide. The plates were inoculated at the center and incubated for 40 hours at 25°C. They were illuminated slantwise from below and photographed against a dark background. The bright ring near the periphery is an illumination artifact.

Motion of chemotactic bacteria

- Run: Travel in a straight line
- Tumble/ Jump: Instantaneous change velocity
  - Post-tumbling velocity is uniform on a ball
  - Microorganism: *E. Coli*, [Adler '66, Berg, Brown '72]
- Bacteria jump faster when it goes away from high chemical concentration
- Bias in velocity towards high concentrations of chemoattractant
- In long-time: Aggregation of bacteria



How can we interpret this behaviour mathematically?

1. Stochastic models (tracking the position & the direction of each individual based on the experiments) [Adler ′66, Berg-Brown ′72, Boyarsky-Noble ′77, Stroock ′74] and many more...
2. PDE (macroscopic) models (density & mean flux of the whole population) [Patlak ′53, Keller-Segel ′71, ′73,...]

How can we interpret this behaviour mathematically?

1. Stochastic models (tracking the position & the direction of each individual based on the experiments) [Adler ′66, Berg-Brown ′72, Boyarsky-Noble ′77, Stroock ′74] and many more...
2. PDE (macroscopic) models (density & mean flux of the whole population) [Patlak ′53, Keller-Segel ′71, ′73,...]

Combining 1 & 2:

Run and Tumble Model for Chemotaxis [Stroock '74, Alt '80]

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d} \int_{\mathcal{V}} (T(t,x,v,v')f(t,x,v') - T(t,x,v',v)f(t,x,v))$$

where $x \in \mathbb{R}^d$ and $v \in \mathcal{V} = B(0, V_0)$, $\quad |\mathcal{V}| = 1$ and $f(0,x,v) = f_0(x,v)$.

Mesoscopic description: run and tumble equation

Run and Tumble Model for Chemotaxis [Stroock '74, Alt '80]

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d} \int_{\mathcal{V}} (T(t,x,v,v') f(t,x,v') - T(t,x,v',v) f(t,x,v))$$

where $x \in \mathbb{R}^d$ and $v \in \mathcal{V} = B(0, V_0)$, $\quad |\mathcal{V}| = 1$ and $f(0,x,v) = f_0(x,v)$.

- $f(t,x,v) \geq 0$: probability density of bacteria
- T describes the change in velocity from v to v':
  $T(t,x,v,v') := T(m,v,v') = \lambda(m)\kappa(v,v')$.
- $\lambda(m) : \mathbb{R} \to [0,\infty)$: tumbling rate
  $\mathbb{P}\,(\text{Tumble happens in } [t, t+\Delta t]) = \lambda(v_t \cdot \nabla_x M(x_t))\Delta t + \mathcal{O}(\Delta t)$.
- $m = v \cdot \nabla_x M$, M : external signal
- $M = m_0 + \log(S)$, $m_0 > 0$, S : chemoattractant concentration
- $\kappa(v,v')$: probability distribution of change in $v \to v'$, $\int_{\mathcal{V}} \kappa \, \mathrm{d}v' = 1$.

TU Delft

Run and tumble equation - a kinetic equation

Run and Tumble Model for Chemotaxis [Stroock '74, Alt '80]

$$\begin{cases} \partial_t f = \mathcal{L}[f] = -v \cdot \nabla_x f + \int_{\mathbb{R}^d} \int_{\mathcal{V}} \lambda(m')\kappa(v,v') f' - \lambda(m) f \\ f(0,x,v) = f_0(x,v) \in \mathcal{P}(\mathbb{R}^d, \mathcal{V}) \end{cases} \tag{RT}$$

where $x \in \mathbb{R}^d$ and $v \in \mathcal{V} = B(0, V_0)$ so that $|\mathcal{V}| = 1$.

- Tumbling frequency
  $T(x,v,v') = \lambda(m')\kappa(v,v') = 1 - \chi\psi(x,v'),\ \chi \in (0,1)$
- Remember: $m = v \cdot \nabla_x M$, and $M = \log(S)$,
- Fixed $S(x) \rightsquigarrow$ (RT) is a linear equation.
- Realistic case: (RT) + Poisson like coupling

$$-\Delta S + \alpha S = \rho(t,x) := \int_{\mathcal{V}} f(t,x,v)\,\mathrm{d}v, \quad \alpha \geq 0 \tag{P}$$

TU Delft

An introduction to kinetic theory

A generic kinetic equation

$$\partial_t f(t,x,v) + \underbrace{\mathcal{T}[f](t,x,v)}_{\text{Transport term}} = \underbrace{\mathcal{C}[f](t,x,v)}_{\text{Collision term}} \tag{KE}$$

- $f(t,x,v)$: probability of finding a particle at time $t > 0$ in a phase $z := (x,v) \in Z := \Omega \times \mathcal{V}$.
- Transport term
  - $\mathcal{T}[f] = v \cdot \nabla_x f$ or $\mathcal{T}[f] = v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f$.
- Collision term
  - Acts only on v variable
- Initial datum: $f(0,x,v) = f_0(x,v) \in \mathcal{P}(\Omega \times \mathcal{V})$.

Hypocoercivity

- We want to find $f_\infty > 0$ s.t. $\partial_t f_\infty = (\mathcal{C} - \mathcal{T}) f_\infty = 0$ and $f_t \to f_\infty$ as $t \to +\infty$.
- $\exists C > 0$, a positive function $\beta(t)$ such that $\beta(t) \to 0$ as $t \to +\infty$ and

$$\|f_t - f_\infty\|_* \leq C\beta(t)\|f_0 - f_\infty\|_*$$

$\Longrightarrow$ (KE) is "hypocoercive" in the distance $\|\cdot\|_*$

- $\beta(t) = e^{-\lambda t}$ for some $\lambda > 0 \rightsquigarrow$ geometric convergence
- $\beta(t)$ is a polynomial function $\rightsquigarrow$ sub-geometric convergence
- $\mathcal{C} \rightsquigarrow$ dissipation on v + $\mathcal{T} \rightsquigarrow$ transport in x

  "Mixing" of dissipation into x variable $\rightsquigarrow$ Hypocoercivity

  [Hérau & Nier '04]; [Hérau '06]; [Villani '09];...

[1] D. Bakry, P. Cattiaux, A. Guillin, *Rate of convergence for ergodic continuous Markov processes: Lyapunov vs. Poincaré*, J. Funct. Anal. (2008).

Hypocoercivity

- We want to find $f_\infty > 0$ s.t. $\partial_t f_\infty = (\mathcal{C} - \mathcal{T}) f_\infty = 0$ and $f_t \to f_\infty$ as $t \to +\infty$.
- $\exists C > 0$, a positive function $\beta(t)$ such that $\beta(t) \to 0$ as $t \to +\infty$ and

$$\|f_t - f_\infty\|_* \leq C\beta(t)\|f_0 - f_\infty\|_*$$

$\implies$ (KE) is "hypocoercive" in the distance $\|\cdot\|_*$

- $\beta(t) = e^{-\lambda t}$ for some $\lambda > 0 \rightsquigarrow$ geometric convergence
- $\beta(t)$ is a polynomial function $\rightsquigarrow$ sub-geometric convergence
- $\mathcal{C} \rightsquigarrow$ dissipation on v + $\mathcal{T} \rightsquigarrow$ transport in x

  "Mixing" of dissipation into x variable $\rightsquigarrow$ Hypocoercivity

  [Hérau & Nier '04]; [Hérau '06]; [Villani '09];...

- Two approaches for quantitative ergodicity estimates [1]
  1. Poincaré-type inequalities $\rightsquigarrow$ Integral bounds on the generator
  2. Harris-type theorems $\rightsquigarrow$ Lyapunov functions

[1] D. Bakry, P. Cattiaux, A. Guillin, *Rate of convergence for ergodic continuous Markov processes: Lyapunov vs. Poincaré*, J. Funct. Anal. (2008).

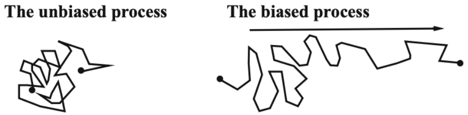
Back to the run and tumble - how does it differ?

I. Confinement mechanism

- Run and tumble vs. linear Boltzmann euqations

$$\textbf{RT:} \quad \partial_t f + v \cdot \nabla_x f = \int_{\mathcal{V}} \lambda'(m') f' \, \mathrm{d}v' - \lambda(m) f$$

$$\textbf{BGK:} \quad \partial_t f + v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f = \mathcal{M}(v) \int_{\mathbb{R}^d} f' \, \mathrm{d}v' - f$$



Back to the run and tumble - how does it differ?

II. Nature of steady states

- Boltzmann-type equations $\rightsquigarrow$ Maxwellian velocity distribution

$$\partial_t f + \underbrace{v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f}_{\mathcal{T}[f]} = \underbrace{\mathcal{M}(v) \int_{\mathbb{R}^d} f' \, \mathrm{d}v' - f}_{\mathcal{C}[f]}$$

- [DMS '15][2] Condition: $f_\infty \in \mathrm{Ker}(\mathcal{T}) \cap \mathrm{Ker}(\mathcal{C})$.
- Classical Hypocoercivity:
  $\frac{\mathrm{d}}{\mathrm{d}t} H[f] \leq -\lambda H[f] \implies \|f_t - f_\infty\|_* \leq C e^{-\lambda t} \|f_0 - f_\infty\|_*$
- (RT) has complex, non-explicit steady states!!
- Classical hypocoercivity methods are difficult to apply!

[2] J. Dolbeault, C. Mouhot, C. Schmeiser, *Hypocoercivity for linear equations conserving mass*, Trans. Am. Math. Soc. (2015)

Harris-type theorems

- Harris-type theorems: Ergodicity of Markov Processes
- Markov $\rightsquigarrow$ transition probabilities
- Transition probabilities $\rightsquigarrow$ Semigroup of linear operators
- Spectral properties of the semigroup $\rightsquigarrow$ Ergodicity of Markov Processes
- [Doeblin '40] $\rightsquigarrow$ Transition probabilities $> 0 \implies$ Mixing property book by [Stroock '14]
- [Harris '56] $\rightsquigarrow$ Conditions for $\exists \& !$ equilibrium state
- [Meyn-Tweedie '90s] $\rightsquigarrow$ Exponential convergence to a unique invariant measure
- [Douc, Fort, Guillin '09,'10]; [Fort, Roberts '05] $\rightsquigarrow$ Sub-geometric case
- [Hairer & Mattingly '11] $\rightsquigarrow$ Quantitative hypocoercivity, alternative proof using mass transport distances
- [Cañizo & Mischler '21] $\rightsquigarrow$ Proofs based on PDE (semigroup) arguments
- Spectral gaps of integro-differential operators $\sim$ PDMPs

TU Delft

Harris-type theorems

(M) Uniform mixing property in a region
- "Minorisation condition"
- "local Doeblin condition"
- "Irreducibility"

(FL) Geometric drift condition
- "Foster-Lyapunov condition"
- "Confinement"

$\Longrightarrow$
- ∃&! stationary state
- Exponential or algebraic convergence

Harris-type theorems II

Harris's theorem

Let $(S_t)_{t\geq 0}$ be a Markov semigroup defined on $\mathcal{M}(Z)$ satisfying

$$\exists \sigma > 0,\ D \geq 0,\ \phi : Z \to [1, +\infty) \text{ s.t. } \mathcal{L}^*\phi(z) \leq -\sigma\phi(z) + D \tag{FL}$$

$$\exists \alpha \in (0,1), \eta \in \mathcal{P},\ \tau > 0,\ \text{ s.t. } S_\tau \mu \geq \alpha\eta, \quad \forall \mu \in \mathcal{P}(\mathcal{A}) \tag{M}$$

where

$$\mathcal{A} := \{z \mid \phi(z) \leq R\}, \quad R > 2D/(1-\alpha).$$

Then $\exists!$ stationary solution μ_∞ and $\forall \mu \in \mathcal{P}(Z)$, $\exists C > 0,\ \lambda > 0$ s.t.

$$\|S_t(\mu - \mu_\infty)\|_\phi \leq Ce^{-\lambda t}\|\mu - \mu_\infty\|_\phi.$$

$$\|f\|_\phi := \int_\Omega \phi(z)|f(z)|(\mathrm{d}z)$$

Harris-type theorems III

Subgeometric Harris's theorem

Let $(S_t)_{t\geq 0}$ be a Markov semigroup (+ Feller) defined on $\mathcal{M}(Z)$ satisfying

$$\exists \sigma > 0,\ D \geq 0, \phi : Z \to [1, +\infty) \text{ with pre-compact sub-level sets} \\ \mathcal{L}^*\phi(z) \leq -\sigma V(\phi) + D \tag{FL$_s$}$$

where V strictly concave, positive, increasing, $\lim_{u\to\infty} V'(u) = 0$.

$$\forall R > 0\, \exists \alpha \in (0,1), \eta \in \mathcal{P},\ \tau > 0,\ \text{s.t. } S_\tau \mu \geq \alpha\eta, \quad \forall \mu \in \mathcal{P}(\mathcal{A}) \tag{M$_s$}$$

where $\mathcal{A} := \{z \mid \phi(z) \leq R\}$. Then $\exists !$ stationary solution μ_∞ s.t. $\int V(\phi(z))\mu_\infty(\,\mathrm{d}z) \leq D$ and $\forall \mu \in \mathcal{P}(Z)$, $\exists C > 0$,

$$\|S_t(\mu - \mu_\infty)\|_{TV} \leq \frac{C\mu(\phi)}{(H_V^{-1})(t)} + \frac{C}{(V \circ H_V^{-1})(t)}, \quad H_V = \int_0^t \frac{\mathrm{d}s}{V(s)}.$$

TU Delft

Back to the RT equation: linear case

(H1) Uniform tumbling kernel: $\kappa \equiv 1$.

(H2) Tumbling rate increases as the bacteria move away from the regions with higher density of chemoattractant.

$$\lambda(m) = 1 - \chi\psi(m), \quad m = v' \cdot \nabla_x M, \, \chi \in (0,1),$$

where ψ is an odd, non-decreasing function, $\|\psi\|_\infty \leq 1$ and $m\psi(m)$ is differentiable.

(H3) Chemoattractant density decreases as $|x| \to \infty$.

- $M(x) \to -\infty$ as $|x| \to \infty$,
- $\exists R \geq 0$ and $m_* > 0$ s.t. when $|x| > R$, $|\nabla_x M(x)| \geq m_*$.
- $\mathrm{Hess}(M)(x) \to 0$ as $|x| \to \infty$ and $\mathrm{Hess}(M)(x)$ is bounded.

(H4) $\exists \tilde{\lambda} > 0$ (depends on $\psi, \|\nabla_x M\|_\infty$) and $\exists k > 0$ (depends on ψ)

$$\int_{\mathcal{V}} m'\psi(m') \, \mathrm{d}v' \geq \tilde{\lambda} |\nabla_x M(x)|^k.$$

TU Delft

Theorem I: linear Case [J. Evans, H. Y., SIMA (2023)]

Suppose that $t \mapsto f_t$ is the solution to (RT) with $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$ and that (H1)-(H4) are satisfied.

- There exist $C, \rho > 0$ (independent from f_0) such that

$$\|f_t - f_\infty\|_* \leq Ce^{-\sigma t}\|f_0 - f_\infty\|_*, \qquad (\bigstar)$$

  where f_∞ is the unique steady state solution of (RT) and

$$\|\mu\|_* = \int_{\mathbb{R}^d}\int_{\mathcal{V}} \Psi(m, \psi(m))e^{-\gamma M(x)}|\mu| \,\mathrm{d}v\,\mathrm{d}x.$$

- If there exist $C_1, C_2, \alpha > 0$ s.t. $C_1 - \alpha\langle x\rangle \leq M(x) \leq C_2 - \alpha\langle x\rangle$ then $(\bigstar)$ holds with $\|\mu\|_{**} = \int_{\mathbb{R}^d}\int_{\mathcal{V}} e^{\delta\langle x\rangle}|\mu| \,\mathrm{d}v\,\mathrm{d}x$, where δ is a constant small enough depends on M and $\langle x\rangle := \sqrt{1+|x|^2}$.

TU Delft

Sketch of the proof - linear case

Minorisation/local Doeblin condition:
Find $t_* > 0$ and $\alpha \in (0,1)$ such that for any $f_0 \in \mathcal{P}$, $f_{t_*} \geq \alpha\mu$, μ probability measure.

Sketch of the proof - linear case

Minorisation/local Doeblin condition:
Find $t_* > 0$ and $\alpha \in (0,1)$ such that for any $f_0 \in \mathcal{P}$, $f_{t_*} \geq \alpha\mu$, μ probability measure.

- (RT) $\implies f_t = \mathcal{S}_t f_0 = \mathcal{T}_t f_0 + \int_0^t \mathcal{T}_{t-s}(\mathcal{J} f_s)\,\mathrm{d}s$
- Transport $(\mathcal{T})_{t\geq 0}$: $\quad \partial_t f + v \cdot \nabla_x M + \lambda(x,v) f = 0.$

$$\mathcal{T}_t \delta_{(x_0,v_0)}(x,v) \geq e^{-(1+\chi)t} \delta_{(x_0+v_0 t, v_0)}(x,v)$$

Sketch of the proof - linear case

Minorisation/local Doeblin condition:
Find $t_* > 0$ and $\alpha \in (0,1)$ such that for any $f_0 \in \mathcal{P}$, $f_{t_*} \geq \alpha\mu$, μ probability measure.

- (RT) $\implies$ $f_t = \mathcal{S}_t f_0 = \mathcal{T}_t f_0 + \int_0^t \mathcal{T}_{t-s}\left(\mathcal{J} f_s\right) \mathrm{d}s$
- Transport $(\mathcal{T})_{t\geq 0}$: $\quad \partial_t f + v \cdot \nabla_x M + \lambda(x,v) f = 0.$

$$\mathcal{T}_t \delta_{(x_0,v_0)}(x,v) \geq e^{-(1+\chi)t} \delta_{(x_0+v_0 t, v_0)}(x,v)$$

- Tumble/Jump $\mathcal{J}[f] := \int_{\mathcal{V}} \lambda'(m') f' \, \mathrm{d}v'$.

$$\mathcal{J}\mathcal{T}_t \delta_{(x_0,v_0)}(x,v) \geq (1-\chi) e^{-(1+\chi)t} \delta_{(x_0+v_0 t)}(x) \mathbb{1}_{\{|v|\leq V_0\}}(v).$$

TU Delft

Sketch of the proof of Theorem I - linear case

Lemma - Minorisation condition for (RT)

For every $R > 0$, taking $t_* = 3 + R/V_0$

$$
\begin{aligned}
f(t_*, x, v) &\geq \int_0^t \int_0^s \mathcal{T}_{t-s} \mathcal{J} \mathcal{T}_{s-r} \mathcal{J} \mathcal{T}_r f_0(x, v) \, \mathrm{d}r \, \mathrm{d}s \\
&\geq \cdots \\
&\geq (1-\chi)^2 e^{-(1+\chi)t_*} \frac{1}{t_*^d |B(V_0)|} \mathbb{1}_{\{|x| \leq V_0\}} \mathbb{1}_{\{|v| \leq V_0\}}
\end{aligned}
$$

for any $f_0(x, v) \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$ with $\int_{|x| \leq R} \int_{\mathcal{V}} f_0 \, \mathrm{d}x \, \mathrm{d}v = 1$.

TU Delft

Sketch of the proof of Theorem I - linear case

Foster-Laypunov condition: Find $\gamma, D > 0$ and ϕ such that $\mathcal{L}^*\phi \leq -\gamma\phi + D$ where $\mathcal{L}^*$ is the adjoint operator

$$\mathcal{L}^*[\phi] = v \cdot \nabla_x \phi + \lambda(v \cdot \nabla_x M) \left(\int_{\mathcal{V}} \phi(x, v')\, \mathrm{d}v' - \phi(x, v) \right)$$

Lemma - Foster-Laypunov condition for (RT)

For $\beta = \chi/(1+\chi)$ and γ sufficiently small, $m := v \cdot \nabla_x M(x)$,

$$\phi(x, v) = (1 - \gamma m(1 + \beta\psi(m))e^{-\gamma M(x)}$$

satisfies (FL).

- Idea: Compute the action of $\mathcal{L}^*$ on $e^{-\gamma M(x)}$, $me^{-\gamma M(x)}$ and $m\psi(m)e^{-\gamma M(x)}$ and put them together.

TU Delft

Towards more realistic models..

Non-uniform tumbling kernel

- The realistic one [Berg-Brown 72', Macnab 80', Othmer-Hillen 02', Frymier-Ford-Cummings 93']:

$$\kappa_1(v, v') = \kappa_1(\theta) = \frac{g(\theta)}{2\pi \sin\theta} \quad \text{where} \quad \theta = \arccos\left(\frac{v \cdot v'}{|v||v'|}\right),$$

where $g(\theta)$ is the sixth order polynomial satisfying $g(0) = g(\pi) = 0$.

Unbounded velocity space: $v \in \mathbb{R}^d$

- The tumbling kernel is given by the Maxwellian distribution on the post-tumbling velocities independently from the pre-tumbling velocities, i.e.,

$$\kappa_2(v, v') = \kappa_2(v') = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v'|^2}{2}}.$$

TU Delft

Theorem II: linear equation [J. Evans, H. Y., preprint(2023)]

Case I. Angle dependent kernel $\kappa_1(\theta)$

Under the previous assumptions on $\lambda, m\psi(m), M(x)$ with $\kappa_1(\theta)$ and that $x \in \mathbb{R}^2$ and $v \in \mathbb{S}^1$, then there exist positive constants C, σ (independent of f_0) such that

$$\|f_t - f_\infty\|_\phi \leq Ce^{-\sigma t}\|f_0 - f_\infty\|_\phi,$$

where f_∞ is the unique steady state solution to the RT equation. The norm $\|\cdot\|_*$ is the weighted total variation norm with the weight

$$\phi(x,v) = \left(1 - \frac{\gamma}{1 - C_K} v \cdot \nabla_x M - Av \cdot \nabla_x M \psi(v \cdot \nabla_x M)\right) e^{-\gamma M}$$

where $\gamma, A, C_K > 0$ are constants which can be computed explicitly.

Theorem II: linear equation [J. Evans, H. Y., preprint (2023)]

Case II. Unbounded velocity space

Under the previous assumptions on $\lambda, m\psi(m), M(x)$ with initial data $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\kappa_2(v')$, then there exist positive constants $C > 0$ such that

$$\|f_t - f_\infty\|_{TV} \leq Ct^{-1/2} M_{f_0},$$

where

$$M_{f_0} = \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,v)\phi_2(x,v)\mathrm{d}v\mathrm{d}x$$

with

$$\phi_2(x,v) = 1 + M^2 + 2c \cdot \nabla_x M M \left(1 + \frac{\chi}{1+\chi}\psi(v \cdot \nabla_x M)\right) + Av^2,$$

where $c, A > 0$ are constants which can be computed explicitly.

TU Delft

Nonlinear models...

We consider..

- Linear case with $\psi(m) = \text{sgn}(m)$ & ψ Lipschitz $(d \geq 1)$.
- Non-linear toy model

$$S(x) = S_\infty(x)(1 + \eta N(x) * \rho), \quad \rho(t,x) = \int_{\mathcal{V}} f(t,x,v)\,\mathrm{d}v,$$

where $\eta > 0$ a small constant, N a compactly supported positive smooth function, S_∞ a smooth function.

Why to consider this toy model?

- Intermediate case between more realistic non-linear couplings and the linear one.
- S can be considered as a perturbation of the linear equation when $N * \rho$ is decreasing and η small.

TU Delft

Theorem III: non-linear equation [Evans, Y., SIMA (2023)]

Suppose that $t \mapsto f_t$ is the solution to nonlinear (RT) where

$$S(x) = S_\infty(x)(1 + \eta N(x) * \rho),$$

where N is a smooth function with a compact support, $\eta > 0$ and S_∞ is a smooth function satisfying for $C_1, C_2, \alpha > 0$

$$C_1 - \alpha\langle x\rangle \le M_\infty(x) := \log(S_\infty(x)) \le C_2 - \alpha\langle x\rangle,$$

where $\langle x\rangle := \sqrt{1 + |x|^2}$. Suppose also that (H1)-(H4) are satisfied and ψ is a Lipschitz function.

- There exists $\tilde{C}$ (dep. on C_1, C_2, α) s.t. if $\eta < \tilde{C}$ there exists a unique steady state solution f_∞.
- Any f_0 satisfying $\|f_0\|_{**} \le K$ (K dep. on $\sigma, \chi, V_0, \eta, \cdots$) then we have

$$\|f_t - f_\infty\|_{**} \le Ce^{-\sigma t/2}\|f_0 - f_\infty\|_{**}. \qquad (\bigstar\bigstar)$$

Sketch of the proof of Theorem III - non-linear case

- Build a stationary solution
  - Consider the nonlinear problem as a perturbation of the linear problem
  - Fixed point argument: $G(M) = \log\big(S_\infty(1 + \eta N * \rho^M)\big)$, $\rho^M = \int f_\infty^M \, \mathrm{d}v'$.

Sketch of the proof of Theorem III - non-linear case

- Build a stationary solution
  - Consider the nonlinear problem as a perturbation of the linear problem
  - Fixed point argument: $G(M) = \log\big(S_\infty(1+\eta N * \rho^M)\big)$, $\rho^M = \int f_\infty^M \,\mathrm{d}v'$.
- Contraction argument
  - $f = \mathcal{L}_{M_t} f = \mathcal{L}_{\tilde{M}} f - (\mathcal{L}_{\tilde{M}} - \mathcal{L}_{M_t}) f, \quad \tilde{M}$ fixed point of G.

$$f_t = \mathcal{S}_t^{\tilde{M}} f_0 + \int_0^t \mathcal{S}_{t-s}^{\tilde{M}} (\mathcal{L}_{\tilde{M}} - \mathcal{L}_{M_s}) f_s \,\mathrm{d}s.$$

$$\|f_t - f_\infty\|_{**} = \|\mathcal{S}_t^{\tilde{M}} f_0 - f_\infty\|_{**} + \left\| \int_0^t \mathcal{S}_{t-s}^{\tilde{M}} (\mathcal{L}_{\tilde{M}} - \mathcal{L}_{M_s}) f_s \,\mathrm{d}s \right\|_{**}$$

TU Delft

Summary

- Extension of [Mischler, Weng 2017] on the linear equation to $d \geq 1$, also for smooth ψ.
- Introducing the weakly non-linear model
  - A unique stationary solution
  - Exponential convergence
- First results concerning the hypocoercivity for non-uniform kernels
- Constructive proofs
- Quantifiable convergence rates
- Convergence results are in weighted TV norms with exponential weights, i.e. $e^{-\gamma M} = S^{-\gamma}$, $\gamma > 0$ small constant.
- Providing perspectives to treat the more realistic non-linear couplings.

Thank you!

Announcement:

- A new call for **a postdoc position** (1 year - possibility of extension) and **a PhD position** (4 years) at TU Delft under my supervision.
- Topics: in the broad area of analyis of PDEs arising from structured population dynamics and kinetic theory: study of well-posedness, long-time behaviour, numerical analysis and derivation problems