Weak solutions to gradient flows in metric measure spaces

J.M. Mazón,
joint works with W. Gorny

Gradient Flow Face to Face 3 Lyon, September 2023

Introduction

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M. Kell, *q-Heat flow and the gradient flow of the Renyi entropy in the p-Wasserstein space*. Journal Funct. Anal. **271** (2016), 2045-2089.

Preliminaries (Sobolev Spaces)

We charactrize these subdifferential using the first-order differential structure on a metric measure space introduced by Gigli

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From now on we will assume that $(\mathbb{X}, d, \nu)$ is a complete and separable metric space and ν is a nonnegative Radon measure.

We say that a Borel function g is an upper gradient of a Borel function $u : \mathbb{X} \to \mathbb{R}$ if for all curves $\gamma : [0, l_\gamma] \to \mathbb{X}$ we have

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \leq \int_\gamma g := \int_0^{l_\gamma} g(\gamma(t))|\dot{\gamma}(t)|dt\, ds,$$

where

$$|\dot{\gamma}(t)| := \lim_{\tau \to 0} \frac{\gamma(t+\tau) - \gamma(t)}{\tau}$$

is the metric speed of γ.

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The Sobolev-Dirichlet class $D^{1,p}(\mathbb{X})$ consists of all Borel functions $u : \mathbb{X} \to \mathbb{R}$ for which there exists an upper gradient which lies in $L^p(\mathbb{X}, \nu)$. The Sobolev space $W^{1,p}(\mathbb{X}, d, \nu)$ is defined as

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For every $u \in D^{1,p}(\mathbb{X})$, there exists a minimal p-upper gradient $|Du| \in L^p(\mathbb{X}, \nu)$, i.e. we have

$$|Du| \leq g \quad \nu - \text{a.e.}$$

for all p-upper gradients $g \in L^p(\mathbb{X}, \nu)$. It is unique up to a set of measure zero.

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The space $W^{1,p}(\mathbb{X}, d, \nu)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\mathbb{X},d,\nu)} = \left(\int_{\mathbb{X}} |u|^p \, d\nu + \int_{\mathbb{X}} |Du|^p \, d\nu \right)^{1/p},$$

Preliminaries (The differential structure)

An $L^p(\nu)$-normed module is the structure $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}, \cdot, |\cdot|)$ where: $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}$ is a Banach space, $\cdot$ is a multiplication of elements of $\mathcal{M}$ with $L^\infty(\nu)$ functions satisfying

$$f(gv) = (fg)v, \quad \text{and} \quad \mathbf{1}v = v \quad \text{for every } f, g \in L^\infty(\nu), \; v \in \mathcal{M},$$

where $\mathbf{1}$ is the function identically equal to 1, and $|\cdot| : \mathcal{M} \to L^p(\nu)$ is the pointwise norm, i.e. a map assigning to every $v \in \mathcal{M}$ a non-negative function in $L^p(\nu)$ such that

$$\|v\|_{\mathcal{M}} = \||v|\|_{L^p(\nu)}, \quad |fv| = |f||v|, \quad \nu - a.e.$$

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Let M be an $L^p(\nu)$-normed module. The dual module M^* is defined by

$$M^* = \mathrm{HOM}(M, L^1(\mathbb{X}, \nu)),$$

where, $T \in \mathrm{HOM}(M, L^1(\mathbb{X}, \nu))$ if $T : M \to L^1(\mathbb{X}, \nu)$ is a bounded linear map satisfaying

$$T(f \cdot v) = f \cdot T(v) \qquad \forall v \in M, \ f \in L^\infty(\mathbb{X}, \nu). \tag{1}$$

Preliminaries (The differential structure)

To define the cotangent module to $\mathbb{X}$ we consider

$$\mathrm{PCM}_p = \left\{ \{(f_i, A_i)\}_{i\in\mathbb{N}} \ : \ (A_i)_{i\in\mathbb{N}} \subset \mathcal{B}(\mathbb{X}), \ f_i \in D^{1,p}(A_i), \ \sum_{i\in\mathbb{N}} \int_{A_i} |Df_i|^p \, d\nu < \infty \right\},$$

where A_i is a partition of $\mathbb{X}$.

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where A_i is a partition of $\mathbb{X}$.

We define the equivalence relation $\sim$ as

$$\{(A_i, f_i)\}_{i\in\mathbb{N}} \sim \{(B_j, g_j)\}_{j\in\mathbb{N}} \quad \text{if} \quad |D(f_i - g_j)| = 0 \ \ \nu-\text{a.e. on } A_i \cap B_j.$$

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Consider the map $|\cdot|_* : \mathrm{PCM}_p/\sim \to L^p(\mathbb{X}, \nu)$ given by

$$|\{(f_i, A_i)\}_{i\in\mathbb{N}}|_* := |Df_i| \quad \nu\text{-a.e. on } A_i, \ \forall\, i \in \mathbb{N}$$

ν-everywhere on A_i for all $i \in \mathbb{N}$, namely the pointwise norm on $\mathrm{PCM}_p/\sim$.

In $\mathrm{PCM}_p/\sim$ we define the norm $\|\cdot\|$ as

$$\|\{(f_i, A_i)\}_{i\in\mathbb{N}}\|^p = \sum_{i\in\mathbb{N}} \int_{A_i} |Df_i|^p$$

and set $L^p(T^*\mathbb{X})$ to be the closure of $\mathrm{PCM}_p/\sim$ with respect to this norm, i.e. we identify functions which differ by a constant and we identify possible rearranging of the sets A_i.

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$L^p(T^*\mathbb{X})$ is called the cotangent module and its elements will be called p-cotangent vector field. $L^p(T^*\mathbb{X})$ is a $L^p(\nu)$-normed module.

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We will assume that $\frac{1}{p} + \frac{1}{q} = 1$ and we denote by $L^q(T\mathbb{X})$ the dual module of $L^p(T^*\mathbb{X})$, namely $L^q(T\mathbb{X}) := \mathrm{HOM}(L^p(T^*\mathbb{X}), L^1(\mathbb{X},\nu))$, which is a $L^q(\nu)$-normed module. $L^q(T\mathbb{X})$ is called the tangent module.

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The elements of $L^q(T\mathbb{X})$ will be called q-vector fields on $\mathbb{X}$.

Preliminaries (The differential structure)

The duality between $\omega \in L^p(T^*\mathbb{X})$ and $X \in L^q(T\mathbb{X})$ will be denoted by $\omega(X) \in L^1(\mathbb{X}, \nu)$.

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Definition

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from the definition of the pointwise norm, it is clear that

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If $X \in L^q(T\mathbb{X})$, we have $|X| \in L^q(\mathbb{X}, \nu)$. From now on, to simplify, we will write

$$\|X\|_q := \||X|\|_{L^q(\mathbb{X},\nu)}.$$

Preliminaries (Divergence of vector field)

For $q \in (1, \infty]$ and $\frac{1}{r} + \frac{1}{s} = 1$, we set

$$\mathcal{D}^{q,r}(\mathbb{X}) = \Big\{ X \in L^q(T\mathbb{X}) : \ \exists f \in L^r(\mathbb{X}, \nu) \quad \forall g \in W^{1,p}(\mathbb{X}, d, \nu) \cap L^s(\mathbb{X}, \nu)$$

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The function f, which is unique by the density of $W^{1,p}(\mathbb{X}, d, \nu)$ in $L^p(\mathbb{X}, \nu)$, will be called the (q, r)-divergence of X. We will write $\mathrm{div}(X) = f$.

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Furthermore, whenever Lipschitz functions are dense in $W^{1,p}(\mathbb{X}, d, \nu)$, then the divergence does not depend on r in the following sense: if f is the (q, r)-divergence of X and $f \in L^{r'}(\mathbb{X}, \nu)$, then it is also the (q, r')-divergence of X.

The p-Laplacian evolution equation

Let $1 < p < \infty$ and we assume that $(\mathbb{X}, d)$ is complete and separable and that ν is a nonnegative measure which is finite on bounded sets.

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The p-Cheeger energy (restricted to $L^2(\mathbb{X}, \nu)$) $\mathrm{Ch}_p : L^2(\mathbb{X}, \nu) \to [0, +\infty]$ is defined by the formula

$$\mathrm{Ch}_p(u) = \begin{cases} \displaystyle\frac{1}{p} \int_{\mathbb{X}} |Du|^p \, d\nu & u \in W^{1,p}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu) \\ +\infty & u \in L^2(\mathbb{X}, \nu) \setminus W^{1,p}(\mathbb{X}, d, \nu). \end{cases} \tag{2}$$

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The abstract Cauchy problem

$$\begin{cases} u'(t) + \partial \mathrm{Ch}_p(u(t)) \ni 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{3}$$

has a unique strong solution for any initial datum $u_0 \in L^2(\mathbb{X}, \nu)$.

Definition

$(u, v) \in \mathcal{A}_p$ if and only if $u, v \in L^2(\mathbb{X}, \nu)$, $u \in W^{1,p}(\mathbb{X}, d, \nu)$ and there exists a vector field $X \in \mathcal{D}^{q,2}(\mathbb{X})$ such that the following conditions hold:

$$-\mathrm{div}(X) = v \quad \text{in } \mathbb{X}; \tag{4}$$

$$du(X) = |du|_*^p = |X|^q \quad \nu\text{-a.e. in } \mathbb{X}. \tag{5}$$

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Theorem

$\partial \mathrm{Ch}_p = \mathcal{A}_p$. *Furthermore, the operator $\mathcal{A}_p$ is completely accretive and the domain of $\mathcal{A}_p$ is dense in $L^2(\mathbb{X}, \nu)$.*

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The more diffucult part is to prove that $\mathcal{A}_p$ satisfies the range condition, i.e.

$$\text{Given } g \in L^2(\mathbb{X}, \nu), \ \exists\, u \in D(\mathcal{A}_p) \ s.t. \ \ g \in u + \mathcal{A}_p(u). \tag{6}$$

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Finally we show that $\mathcal{A}_p$ is complely accretive in $L^2(\mathbb{X}, \nu)$.

Corollary

The following conditions are equivalent:

(a) $(u, v) \in \partial \mathrm{Ch}_p$;

(b) $u, v \in L^2(\mathbb{X}, \nu)$, $u \in W^{1,p}(\mathbb{X}, d, \nu)$ *and there exists a vector field* $X \in \mathcal{D}^{q,2}(\mathbb{X})$ *with* $|X|^q \leq |du|_*^p$ ν*-a.e. such that* $-div(X) = v$ *in* $\mathbb{X}$ *and for every* $w \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu)$

$$\int_{\mathbb{X}} v(w-u)\, d\nu \leq \int_{\mathbb{X}} dw(X)\, d\nu - \int_{\mathbb{X}} |du|_*^p\, d\nu; \qquad (7)$$

(c) $u, v \in L^2(\mathbb{X}, \nu)$, $u \in W^{1,p}(\mathbb{X}, d, \nu)$ *and there exists a vector field* $X \in \mathcal{D}^{q,2}(\mathbb{X})$ *with* $|X|^q \leq |du|_*^p$ ν*-a.e. such that* $-div(X) = v$ *in* $\mathbb{X}$ *and for every* $w \in L^2(\mathbb{X}, \nu) \cap W^{1,p}(\mathbb{X}, d, \nu)$

$$\int_{\mathbb{X}} v(w-u)\, d\nu = \int_{\mathbb{X}} dw(X)\, d\nu - \int_{\mathbb{X}} |du|_*^p\, d\nu. \qquad (8)$$

The p-Laplacian evolution equation

Definition

We define in $L^2(\mathbb{X}, \nu)$ the multivalued operator $\Delta_{p,\nu}$ by

$$(u, v) \in \Delta_{p,\nu} \text{ if and only if } -v \in \partial \mathrm{Ch}_p(u).$$

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We have that the abstract Cauchy problem (3) corresponds to the Cauchy problem for the p-Laplacian, i.e.,

$$\begin{cases} \partial_t u(t) \in \Delta_{p,\nu}(u(t)), & t \in [0, T] \\ u(0) = u_0. \end{cases} \tag{9}$$

Definition

Given $u_0 \in L^2(\mathbb{X}, \nu)$, we say that u is a *weak solution* of the Cauchy problem (9) in $[0, T]$, if $u \in W^{1,1}(0, T; L^2(\mathbb{X}, \nu))$, $u(0, \cdot) = u_0$, and for almost all $t \in (0, T)$

$$u_t(t, \cdot) \in \Delta_{p,\nu} u(t, \cdot). \tag{10}$$

In other words, if $u(t) \in W^{1,p}(\mathbb{X}, d, \nu)$ and there exist vector fields $X(t) \in \mathcal{D}^{q,2}(\mathbb{X})$ such that for almost all $t \in [0, T]$ the following conditions hold:

$$\mathrm{div}(X(t)) = u_t(t, \cdot) \quad \text{in } \mathbb{X};$$

$$|X(t)|^q = du(t)(X(t)) = |du(t)|_*^p \quad \nu\text{-a.e. in } \mathbb{X}.$$

The p-Laplacian evolution equation

Theorem

For any $u_0 \in L^2(\mathbb{X}, \nu)$ *and all* $T > 0$ *there exists a unique weak solution* $u(t)$ *of the Cauchy problem* (9) *in* $[0, T]$, *with* $u(0) = u_0$. *Moreover, the following comparison principle holds: if* u_1, u_2 *are weak solutions for the initial data* $u_{1,0}, u_{2,0} \in L^2(\mathbb{X}, \nu) \cap L^r(\mathbb{X}, \nu)$, *respectively, then*

$$\|(u_1(t) - u_2(t))^+\|_r \leq \|(u_{1,0} - u_{2,0})^+\|_r \quad \text{for all } 1 \leq r \leq \infty. \tag{11}$$

Some important particular cases

p-Laplacian in weighted Euclidean spaces

Endow $\mathbb{R}^N$ with the Euclidean distance d_{Eucl}. For a nonnegative Radon measure ν in $(\mathbb{R}^N, d_{Eucl})$, we refer to the metric measure space $(\mathbb{R}^N, d_{Eucl}, \nu)$ as a weighted Euclidean space.

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p-Laplacian in Finsler manifolds

Let (M, F) by a geodesically complete, reversible Finsler manifold, with metric with metric

$$d_F(x,y) := \inf \left\{ \ell_F(\gamma) \ : \ \gamma : [0,1] \to M \text{ piecewise } C^1 \text{ with } \gamma(0) = x, \ \gamma(1) = y \right\}$$

where

$$\ell_F(\gamma) := \int_0^1 F(\gamma(t), \dot{\gamma}(t))\, dt.$$

If ν is non-negative Radon measure on (M, d_F), the metric measure space (M, d_F, ν) satisfes our assumptions

The total variation flow

The Cauchy problem

$$\begin{cases} u_t(t,x) = \operatorname{div}\left(\frac{Du(t,x)}{|Du(t,x)|_\nu}\right) & \text{in } (0,T)\times\mathbb{X}, \\ u(0,x) = u_0(x) & \text{in } \mathbb{X}. \end{cases} \tag{12}$$

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We need to assume that the metric space $(\mathbb{X}, d)$ is complete, separable, equipped with a doubling measure ν, and that the metric measure space $(\mathbb{X}, d, \nu)$ supports a weak $(1,1)$-Poincaré inequality.

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For $u \in L^1(\mathbb{X}, \nu)$, we define the total variation of u on an open set $\Omega \subset \mathbb{X}$ by the formula

$$|Du|_\nu(\Omega) := \inf\left\{ \liminf_{n\to\infty} \int_\Omega |\nabla u_n|\, d\nu \ : \ u_n \in Lip_{loc}(\Omega),\ u_n \to u \text{ in } L^1(\Omega,\nu) \right\}, \tag{13}$$

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$$|\nabla u|(x) := \limsup_{y\to x} \frac{|u(y)-u(x)|}{d(x,y)},$$

is the slope of u, and

$$BV(\mathbb{X}, d, \nu) := \{ u \in L^1(\mathbb{X},\nu) \ : \ |Du|_\nu(\mathbb{X}) < \infty \}.$$

The total variation flow

The energy functional $\mathcal{TV} : L^2(\mathbb{X}, \nu) \to [0, +\infty]$ defined by

$$\mathcal{TV}(u) := \begin{cases} |Du|_\nu(\mathbb{X}) & \text{if } u \in BV(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu), \\ +\infty & \text{if } u \in L^2(\mathbb{X}, \nu) \setminus BV(\mathbb{X}, d, \nu). \end{cases} \tag{14}$$

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We need a Green formula of the Anzellotti type.

Definition

Suppose that the pair (X, u) satisfies

$$\mathrm{div}(X) \in L^p(\mathbb{X}, \nu), \quad u \in BV(\mathbb{X}, d, \nu) \cap L^q(\mathbb{X}, \nu), \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{15}$$

Then, given a Lipschitz function $f \in \mathrm{Lip}(\mathbb{X})$ with compact support, we set

$$\langle (X, Du), f \rangle := -\int_{\mathbb{X}} u\, df(X)\, d\nu - \int_{\mathbb{X}} uf \,\mathrm{div}(X)\, d\nu.$$

The total variation flow

Theorem

Suppose that the pair (X, u) satisfies the condition (15). Then

$$\int_{\mathbb{X}} u \, div(X) \, d\nu + \int_{\mathbb{X}} (X, Du) = 0.$$

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Definition

$(u, v) \in \mathcal{A}_1$ if and only if $u, v \in L^2(\mathbb{X}, \nu)$, $u \in BV(\mathbb{X}, d, \nu)$ and there exists a vector field $X \in \mathcal{D}^{\infty,2}(\mathbb{X})$ with $\|X\|_\infty \leq 1$ such that the following conditions hold:

$$-\mathrm{div}(X) = v \quad \text{in } \mathbb{X};$$

$$(X, Du) = |Du|_\nu \quad \text{as measures.}$$

The total variation flow

Theorem

$\partial \mathcal{TV} = \mathcal{A}_1$. Furthermore, the operator $\mathcal{A}_1$ is completely accretive and the domain of $\mathcal{A}_1$ is dense in $L^2(\mathbb{X}, \nu)$.

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$\partial\mathcal{TV} = \mathcal{A}_1$. Furthermore, the operator $\mathcal{A}_1$ is completely accretive and the domain of $\mathcal{A}_1$ is dense in $L^2(\mathbb{X},\nu)$.

Definition

We define in $L^2(\mathbb{X},\nu)$ the multivalued operator $\Delta_{1,\nu}$ by

$$(u,v) \in \Delta_{1,\nu} \text{ if and only if, } -v \in \partial\mathcal{TV}(u).$$

Definition

Given $u_0 \in L^2(\mathbb{X}, \nu)$, we say that u is a *weak solution* of the Cauchy problem (12) in $[0, T]$, if $u \in W^{1,1}(0, T; L^2(\mathbb{X}, \nu))$, $u(0, \cdot) = u_0$, and for almost all $t \in (0, T)$

$$u_t(t, \cdot) \in \Delta_{1,\nu}(t, \cdot). \tag{16}$$

In other words, $u(t) \in BV(\mathbb{X}, d, \nu)$ and there exist vector fields $X(t) \in \mathcal{D}^{\infty,2}(\mathbb{X})$ with $\|X(t)\|_\infty \leq 1$ such that for almost all $t \in [0, T]$ the following conditions hold:

$$\mathrm{div}(X(t)) = u_t(t, \cdot) \quad \text{in } \mathbb{X};$$

$$(X(t), Du(t)) = |Du(t)|_\nu \quad \text{as measures.}$$

Theorem

For any $u_0 \in L^2(\mathbb{X},\nu)$ *and* $T > 0$ *there exists a unique weak solution* $u(t)$ *of the Cauchy problem* (12) *with* $u(0) = u_0$. *Moreover, the following comparison principle holds: if* u_1, u_2 *are weak solutions for the initial data* $u_{1,0}, u_{2,0} \in L^2(\mathbb{X},\nu) \cap L^r(\mathbb{X},\nu)$, *respectively, then*

$$\|(u_1(t) - u_2(t))^+\|_r \leq \|(u_{1,0} - u_{2,0})^+\|_r \quad \text{for all } 1 \leq r \leq \infty. \tag{17}$$

We also have

$$\left\| \frac{d}{dt} u(t) \right\|_{L^2(\mathbb{X},\nu)} \leq \frac{\|u_0\|_{L^2(\mathbb{X},\nu)}}{t}, \quad \text{for every} \quad t > 0, \tag{18}$$

and

$$\frac{d}{dt} u(t) \leq \frac{u(t)}{t}, \quad \nu\text{-a.e. on } \mathbb{X} \text{ for every } t > 0 \text{ if } u_0 \geq 0. \tag{19}$$

Asymptotic Behaviour

L. Bungert and M. Bunger, Asymptotic Profiles of Nonlinear Homogeneous Evolution. Journal of Evolution Equation **20** (2020), 1061–1092.

L. Bungert and M. Bunger, Asymptotic Profiles of Nonlinear Homogeneous Evolution. Journal of Evolution Equation **20** (2020), 1061–1092.

Assume $\nu(\mathbb{X}) < \infty$, we have that Ch_p is coercive if satisfies to the following Poincaré inequality

$$\|u - \overline{u}\|^p_{L^2(\mathbb{X},\nu)} \leq M\,\mathrm{Ch}_p(u) \quad \forall\, u \in W^{1,p}(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu), \tag{20}$$

where

$$\overline{u} := \frac{1}{\nu(\mathbb{X})} \int_{\mathbb{X}} u d\nu$$

for $1 < p < \infty$; and

$$\|u - \overline{u}\|_{L^2(\mathbb{X},\nu)} \leq M\,\mathcal{TV}(u) \quad \forall\, u \in BV(\mathbb{X}, d, \nu) \cap L^2(\mathbb{X}, \nu), \tag{21}$$

for $p = 1$.

Asymptotic Behaviour

Theorem

Assume that $\nu(\mathbb{X}) < \infty$ *and the Poincaré inequality* (20) *holds, for* $1 < p < \infty$ *and* (21)*, for* $p = 1$*. For* $u_0 \in L^2(\mathbb{X}, \nu)$*, let* $u(t)$ *be the weak solution of the Cauchy problem* (9)*, for* $1 < p < \infty$*, and the weak solution of the Cauchy problem* (14)*, for* $p = 1$*. Then, we have*

(i) *(Finite extinction time) For* $1 \leq p < 2$*,*

$$T_{\rm ex}(u_0) \leq \frac{\| u_0 - \overline{u_0} \|_{L^2(\mathbb{X},\nu)}^{p-2}}{(2-p)\lambda_1(\mathrm{Ch}_p)},$$

where

$$T_{\rm ex}(u_0) := \inf\{ T > 0 \ : \ u(t) = \overline{u_0}, \quad \forall \, t \geq T \}.$$

(ii) *(Infinite extinction time) For* $p \geq 2$*,*

$$T_{\rm ex}(u_0) = +\infty.$$

Total variation flow on bounded domains

We consider the Neumann problem, i.e.

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{|Du(t,x)|_\nu}\right) & \text{in } (0,T)\times\Omega; \\ \frac{\partial u}{\partial \eta} := \frac{Du}{|Du|_\nu}\cdot \eta = 0 & \text{in } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{22}$$

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In order to study the Neumann problem (22), consider the associated energy functional $\mathcal{TV}_{\mathcal{N}} : L^2(\Omega,\nu) \to [0,+\infty]$ defined by

$$\mathcal{TV}_{\mathcal{N}}(u) := \begin{cases} |Du|_\nu(\Omega) & \text{if } u \in BV(\Omega,d,\nu)\cap L^2(\Omega,\nu); \\ +\infty & \text{if } u \in L^2(\Omega,\nu)\setminus BV(\Omega,d,\nu). \end{cases} \tag{23}$$

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Then, by the Brezis-Komura theorem there exists a unique strong solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\mathcal{TV}_{\mathcal{N}}(u(t)) \ni 0 & \text{for } t\in[0,T]; \\ u(0) = u_0, \end{cases} \tag{24}$$

where $u_0 \in L^2(\Omega,\nu)$.

Total variation flow on bounded domains

Theorem

For any $u_0 \in L^2(\Omega,\nu)$ and all $T > 0$, there exists a unique weak solution of the Neumann problem (22) in $[0,T]$. Moreover, the following comparison principle holds: for all $q \in [1,\infty]$, if u_1, u_2 are weak solutions for the initial data $u_{1,0}, u_{2,0} \in L^2(\Omega,\nu) \cap L^q(\Omega,\nu)$ respectively, then

$$\|(u_1(t) - u_2(t))^+\|_q \leq \|(u_{1,0} - u_{2,0})^+\|_q. \tag{25}$$

We also have

$$\left\| \frac{du(t)}{dt} \right\|_{L^2(\Omega,\nu)} \leq \frac{\|u_0\|_{L^2(\Omega,\nu)}}{t} \quad \text{for every } t > 0,$$

and if $u_0 \geq 0$, then additionally

$$\frac{du(t)}{dt} \leq \frac{u(t)}{t} \quad \nu - \text{a.e. on } \Omega \text{ for every } t > 0.$$

Total variation flow on bounded domains

We consider the Dirichlet problem

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{|Du(t,x)|_\nu}\right) & \text{in } (0,T)\times\Omega; \\ u(t,x) = f(x) & \text{in } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{26}$$

We consider the Dirichlet problem

$$\begin{cases} u_t(t,x) = \mathrm{div}\left(\frac{Du(t,x)}{|Du(t,x)|_\nu}\right) & \text{in } (0,T)\times\Omega; \\ u(t,x) = f(x) & \text{in } (0,T)\times\partial\Omega; \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{26}$$

Theorem

Let $f \in L^1(\partial\Omega, |D\chi_\Omega|_\nu)$. For any $u_0 \in L^2(\Omega,\nu)$ and $T > 0$ there exists a unique weak solution of the Dirichlet problem (26) in $[0,T]$. Moreover, the following comparison principle holds: for any $q \in [1,\infty]$, if u_1, u_2 are weak solutions for the initial data $u_{1,0}, u_{2,0} \in L^2(\Omega,\nu) \cap L^q(\Omega,\nu)$ respectively, then

$$\|(u_1(t) - u_2(t))^+\|_q \le \|(u_{1,0} - u_{2,0})^+\|_q. \tag{27}$$

Applications to related problems

The techniques we have developed have served us for study the following problems:

(1) Least gradient functions on metric measure spaces
(2) The Cheeger problem: Cheeger and calibrable sets in metric measure spaces
(3) The eigenvalue problem associated with the 1-Laplacian
(4) The Cheeger cut problem in metric mesure spaces

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THANKS FOR YOUR ATTENTION