Fast diffusion equations, tails and convergence rates.

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Grandient Flows: face to face - 3
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A general framework

Nonnegative, integrable solutions to an evolution (of parabolic type) PDE

$$\begin{cases} \partial_t u = \nabla.\left(\mathcal{A}(t,x,u,\nabla u)\right) + \mathcal{B}(t,x,u,\nabla u) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,, \end{cases}$$

- Asymptotic behaviour given by a (family of) function(s) $V = V(t,x)$
- Let us suppose, for instance,

$$\|u(t,x) - V(t,x)\|_{L^1(\mathbb{R}^d)} \to 0 \quad \text{as} \quad t\to\infty\,,$$

Main question: understanding the behaviour of solutions for **large** $|x|$ and how it affects **convergence** to equilibrium ($V(t,x)$).

A first example: the heat equation - 1

Cauchy problem for nonnegative, integrable initial data

$$\begin{cases} \partial_t u = \Delta u & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) \geq 0 & \text{in } \mathbb{R}^d, \end{cases}$$

- The solution $u(t,x)$ is given by

$$u(t,x) = G * u_0 \quad \text{where} \quad G(t,x) = \frac{1}{(4\pi\,t)^{\frac{d}{2}}}\, e^{-\frac{|x|^2}{4\,t}}$$

- G also represents the **asymptotic behaviour** for large times, i.e., as $t \to \infty$

$$\|u(t) - MG(t)\|_{L^1(\mathbb{R}^d)} \to 0 \quad \text{and} \quad t^{\frac{d}{2}}\,\|u(t) - MG(t)\|_{L^\infty(\mathbb{R}^d)} \to 0\,,$$

where $M = \int_{\mathbb{R}^d} u_0(x)dx$.

- If $\int_{\mathbb{R}^d} |x|^2\, u_0(x)\,dx < \infty$ then we can have **rate of convergence**

$$\|u(t) - MG(t)\|_{L^1(\mathbb{R}^d)} \lesssim t^{-\frac{1}{2}} \quad \text{and} \quad t^{\frac{d}{2}}\,\|u(t) - MG(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{1}{2}}\,,$$

A first example: the heat equation - 2

Q_0) All solutions to the heat equation (with $L^1(\mathbb{R}^d)$ initial datum) develop **gaussian tails**?

- Let's take $u_0 \sim A|x|^{-\alpha}$ for $|x|$ large and $\alpha > d$. Then[1] on the set

  $S := \{(t,x) \in (0,\infty)\times\mathbb{R}^d : 2\,(\alpha-d+\delta)\,t\log(t) \le |x|^2 \le 2\,(\alpha-d+2\delta)\,t\log(t)\}$ for $\delta > 0$

  the solution $u(t,x)$ with initial datum u_0 behave as

$$u(t,x) \sim A\,|x|^{-\alpha} \quad \text{as } |x| \to \infty\,.$$

- Then, for $(t,x) \in S$, we have

$$G(t,x) \sim \frac{b}{t^{\frac{\alpha}{2}+\frac{\delta}{4}}} \quad \text{VS} \quad u(t,x) \sim \frac{A}{t^{\frac{\alpha}{2}}\log^{\frac{\alpha}{2}}(t)}$$

- **No uniform rate of convergence!**

$$t^{\frac{d}{2}}\,\|G(t,x) - u(t,x)\|_{L^\infty(\mathbb{R}^d)} \ge \frac{\mathsf{a}}{t^{\frac{\alpha-d}{2}}\log^{\frac{\alpha}{2}}(t)}$$

[1] See **Herraiz '99**

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Fast diffusion equation

Nonnegative, integrable solutions to the Cauchy problem for the FDE

$$\begin{cases} \partial_t u = \Delta(u^m) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \\ \dfrac{d-2}{d} < m < 1\,, & d \geq 3 \end{cases}$$

Main question: understanding the behaviour of solutions for **large** $|x|$ and how it affects **convergence** to equilibrium.

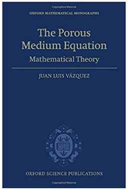
- M. Bonforte, N. S., Fine properties of solutions to the Cauchy problem for a Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights,
  to appear in **Annales de l' Institut Henri Poincaré C - Analyse Non Linéaire**
- M. Bonforte, D. Stan, N. S. The Cauchy problem for the fast p-Laplacian evolution equation. Characterization of the global Harnack principle and fine asymptotic behaviour,
  Journales de Mathématiques Pures et Appliquées
- M. Bonforte, J. Dolbeault, B. Nazaret, N.S
  Stability in Gagliardo-Nirenberg-Sobolev inequalities,
  Memoirs of the American Mathematical Society

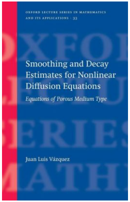
Porous Medium and Fast Diffusion Equations

$$u_t = \Delta u^m = \nabla.(m\, u^{m-1} \nabla u) \quad \text{where} \quad m > 0$$

- $m > 1$, Porous Medium Equation, slow diffusion (finite speed of propagation)
- $m = 1$, Heat Equation[2]
- $0 < m < 1$, Fast Diffusion Equation, Fat tails

Main references: two monographs of **J. L. Vázquez**





[2] do you know that Fourier was a leading figure also in another branch of science?

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PME $(m > 1)$: Free boundaries and finite speed of propagation

Nonnegative, integrable solutions to the Cauchy problem

$$\text{(CP)}\quad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d, \quad m > 1\,, \\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,, \end{cases}$$

- ▷ Fundamental solution

$$u(x,t) = t^{-\frac{1}{m-1}}\left(C\,t^{2\,\vartheta} - k\,\xi^2\right)_+^{\frac{1}{m-1}}\,, \quad \vartheta = \frac{1}{d(m-1)+2} > 0\,, \quad k = \frac{(m-1)\vartheta}{2m} > 0$$

- ▷ Solutions are not classical!
- ▷ If $u \in C_c^\infty(\mathbb{R}^d)$ then $u(x,t) \in C^\alpha(\mathbb{R}^d)$ and it is compactly supported for any $t > 0$ (finite speed of propagation).
- ▷ $\int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0\,\mathrm{d}x$, for any $t > 0$ (mass conservation)
- ▷ $u(x,t) \to 0$ as $t \to 0$

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- ▷ $m > 1$, slow diffusion, Porous Medium Equation
- ▷ $m = 1$, Heat Equation[3]

▷ $0 < m < 1$, Fast Diffusion Equation, Fat tails

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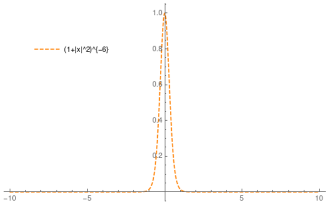
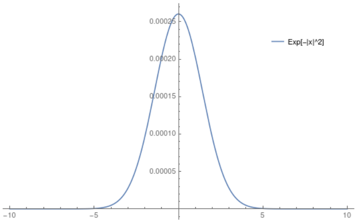
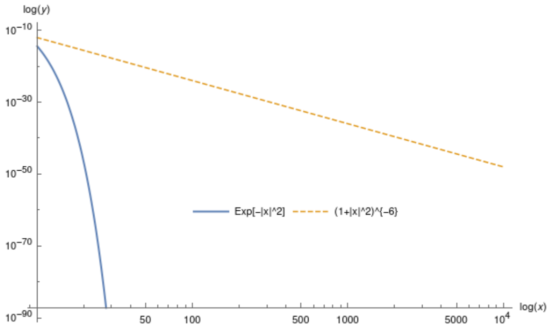
Fat tail ($(1+|x|^2)^{-6}$) vs Gaussian tail($e^{-|x|^2}$)







Fast Diffusion Equation

Nonnegative, integrable solutions to the Cauchy problem

$$\text{(CP)} \quad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,, \end{cases}$$

Parameters and main features:

- m in the range $$\frac{d-2}{d} < m < 1\,, \quad \text{with} \quad d \geq 3\,.$$
- Initial data in $$u_0 \in \mathrm{L}^1_+(\mathbb{R}^d) = \{u_0 : \mathbb{R}^d \to \mathbb{R} : u_0 \geq 0\,, \int_{\mathbb{R}^d} u_0 \,\mathrm{d}x < \infty\}\,.$$
- Existence and uniqueness in $\mathrm{L}^1_{\mathrm{loc}}$ are settled, solutions are C^∞, see **Herrero-Pierre** '85.
- **Mass** is conserved, namely for all $t > 0$,

$$\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x$$

Fast Diffusion Equation

Nonnegative, integrable solutions to the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases}$$

- (CP) admits the self-similar solution (called **Barenblatt**)

$$\mathcal{B}_M(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0 \frac{t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{1-m}}} = t^{-d\vartheta}\,\mathsf{B}_M(x\,t^{-\vartheta}),$$

where $\vartheta^{-1} = 2 - d(1-m) > 0$, and

$$\mathsf{B}_M(x) = \left[\frac{b_0}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{m-1}}$$

- Asymptotic behaviour (relaxation to self-similarity) as $t \to \infty$

$$\|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \to 0 \qquad \text{and} \qquad t^{d\vartheta}\,\|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \to 0$$

Rates of convergence towards the Barenblatt profile

A very popular question in the late 90's and 00's was[4]

Can we get rates of convergence towards the Barenblatt profile?

Carrillo-Vázquez 2003 under hp. of *finite relative entropy* ($\int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x < \infty$)

$$\|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \lesssim t^{-\frac{1}{2}}$$

$\triangleright$ However, in general, **there is no rate of convergence in** $\mathrm{L}^1(\mathbb{R}^d)$! Consider the solution $u_\alpha(t,x)$ with initial datum

$$u_0(x) := \frac{A}{(1 + B|x|^2)^\alpha}$$

For any $\delta > 0$ there exist $\alpha = \alpha(\delta)$ such that $u_\alpha(t,x)$ as $t \to \infty$

$$t^\delta \,\|u_\alpha(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \longrightarrow \infty$$

The rate of convergence towards the Barenlatt profile **depend** on the **tail behaviour** of the solution!

[4] Long story: look at the book of Jüngel or our paper **M.Bonforte, J.Dolbeault, B. Nazaret, and N.S. 2021**

Refined asymptotic behaviour: questions

Main question: understanding the behaviour of solutions for large $|x|$.

We consider the *uniform relative error*, for any $t > 0$

$$\left\| \frac{u(t)}{\mathcal{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$$

Q_1) For which initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$ the solution $u(t,x)$ to (CP) converges to $\mathcal{B}_M(t,x)$ in *uniform relative error*, i.e.

$$\lim_{t\to\infty} \left\| \frac{u(t)}{\mathcal{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0$$

Q_2) When Q_1) has a positive answer, can we compute rate of convergence? Does it exist $g(t)$ such that $g(t) \to \infty$ as $t \to \infty$ such that

$$\left\| \frac{u(t)}{\mathcal{B}_M(t)} - 1 \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \frac{1}{g(t)}$$

If yes, then $\|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} \lesssim \frac{1}{g(t)}$ and $t^{d\vartheta} \, \|u(t) - \mathcal{B}_M(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim \frac{1}{g(t)}$

A heat (equation) intermezzo

For solutions to $u_t = \Delta u$ uniform convergence in relative error does **not** hold,

$$\sup_{\mathbb{R}^d} \left| \frac{u(t,x)}{e^{\frac{-|x|^2}{4t}}} \right| = \sup_{\mathbb{R}^d} \left| e^{\frac{-|x_0|^2 + 2\, x \cdot x_0}{4t}} \right| = +\infty\,.$$

Take for instance $(4\pi)^{\frac{d}{2}} u(1,x) = e^{\frac{-|x+x_0|^2}{4t}}$.

Answer to Q1: it is a matter of tails!

The **relative error**

$$\left|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right|$$

is not always uniformly bounded in $\mathbb{R}^d$

- ▷ Fast Diffusion : any solution develop a **fat tail** $u(t,x) \gtrsim |x|^{-\frac{2}{1-m}}$ for $|x|$ large
- ▷ However, let the initial datum be

$$u_0(x) = \frac{1}{(1+|x|^2)^{\frac{m}{1-m}}} > \mathcal{B}_M(t,x)\,,$$

then the solution $u(t,x)$ to (CP) with initial data u_0 satisfies

$$\mathcal{B}_M(t,x) < \frac{1}{\left[(c\,t+1)^{\frac{1}{1-m}} + |x|^2\right]^{\frac{m}{1-m}}} \le u(t,x) \le \frac{(1+t)^{\frac{m}{1-m}}}{(1+t+|x|^2)^{\frac{m}{1-m}}}\,,$$

Recall that $\mathcal{B}_M(t,x) \sim |x|^{-\frac{2}{1-m}}$

Answer to Q1: the path to the Global Harnack Principle

We can reformulate the problem as an inequality for $x \in \mathbb{R}^d$ and t large of the form

$$\mathcal{B}_{M_1}(t - \tau_1, x) \le u(t,x) \le \mathcal{B}_{M_2}(t + \tau_2, x) \qquad \textbf{(GHP)}$$

The **GHP** holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$, **Vázquez 2003/ Bonforte - Vázquez 2006**

Let us define

$$|f|_{\mathcal{X}_m} := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d \setminus B_R(0)} |f(x)| \, \mathrm{d}x < \infty \,,$$

and the space

$$\mathcal{X}_m := \{f \in \mathrm{L}^1_+(\mathbb{R}^d) : |f|_{\mathcal{X}} < +\infty\}.$$

Theorem [M. Bonforte, N.S. - 2020]

Under the running assumptions, **GHP** holds, i.e.,

$$\mathcal{B}_{M_1}(t - \tau_1, x) \le u(t,x) \le \mathcal{B}_{M_2}(t + \tau_2, x) \,,$$

if and only if the initial data $u_0 \in \mathcal{X}_m \setminus \{0\}$.

Our contribution: we found the maximal set of initial data for which **GHP** holds!
However: see **Vázquez 2003** where a similar condition is introduced.

Answer to Q1: difference between pointwise and integral assumptions

The two assumptions

$$u_0 \lesssim |x|^{-\frac{2}{1-m}} \quad \text{and} \quad \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d \setminus B_R(0)} |u_0(x)| \, \mathrm{d}x < \infty \,,$$

are different ! The pointwise implies the integral one but not viceversa !

$$g_{\alpha,\beta}(y) := \sum_{k=2}^{\infty} \frac{\chi_{B_k^\beta}(y)}{||y| - k|^\alpha} \,,$$

where $\chi_{B_k^\beta}$ is the characteristic function of the set

$$B_k^\beta := \{x \in \mathbb{R}^d : k \leq |x| \leq k + k^{-\beta}\}$$

Answer to Q1: convergence in uniform relative error

$$\text{(CP)}\quad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \qquad \frac{d-2}{d} < m < 1\,, \quad \text{with} \quad d \geq 3\,.$$

Theorem-1 [M. Bonforte, N.S. - 2020]

Under the running assumption, a solution $u(t,x)$ to (CP) converges to $\mathcal{B}_M(t,x)$ in *uniform relative error*, i.e.

$$\lim_{t\to\infty}\left\|\frac{u(t)-\mathcal{B}_M(t)}{\mathcal{B}_M(t)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0$$

if and only if

$$u_0 \in \mathcal{X}_m \setminus \{0\} \quad \text{and} \quad M = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}$$

where

$$\mathcal{B}_M(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0\,\frac{t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{1-m}}} \text{ and } \qquad |f|_{\mathcal{X}_m} := \sup_{R>0} R^{\frac{2}{1-m}-d}\int_{\mathbb{R}^d\setminus B_R(0)} |f(x)|\,\mathrm{d}x$$

Answer to Q2: Rate of convergence

What is known about the *relative error* in the range $\frac{d-2}{d} < m < 1$, $\tau > 0$ constant

$$\left\| \frac{u(t,x) - \mathcal{B}_M(t+\tau,x)}{\mathcal{B}_M(t+\tau,x)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \frac{1}{g(t)} .$$

- ▷ Optima: $g(t) = Ct$
- ▷ [Carrillo-Vazquez 2003], radial data $g(t) = Ct$ and $u_0(x) \lesssim |x|^{-2/(1-m)}$ pointwise
- ▷ [Kim-McCann 2006], $g(t) = Ct$ and $\int_{\mathbb{R}^d} |x|^\alpha u_0\, dx = \int_{\mathbb{R}^d} |x|^\alpha \mathcal{B}_M(,x)\, dx$ for $\alpha \leq \frac{2}{1-m} - d$
- ▷ [Blanchet, Bonforte, Dolbeault, Grillo, Vazquez], better rates if

$$\mathcal{B}_{M_1}(\tau,x) \leq u_0(x) \leq \mathcal{B}_{M_2}(\tau,x)$$

Theorem-2 [M. Bonforte, J. Dolbeault, B. Nazaret, N.S. - 2021]

Assume $\frac{d}{d+2} < m < 1$ and $u_0 \in \mathcal{X}_m$ then

$$\left\| \frac{u(t) - \mathcal{B}_M(t)}{\mathcal{B}_M(t)} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \frac{C}{t^{\mathsf{a}}}$$

where $\mathsf{a} < 1$.

▷ If we assume radiality on the initial datum then $\mathsf{a} = 1$.

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What is known so far?

The main ingredient is the **GHP**; we know that (**in the correct range of parameters**) and in the correct space of initial data $\mathcal{X}_m$ it holds for

- $u_t = \Delta(u^m)$;
- $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, in the range $d \geq 3$, $\frac{2d}{d+1} < p < 2$
- $u_t = \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m)$, the doubly non linear eq.
- $\partial_t u = \nabla.\left(\mathcal{A}(t,x,u,\nabla u)\right) + \mathcal{B}(t,x,u,\nabla u)$, under hps. in which the eq. is similar to the above

This phenomenon depends mostly on **fat tails**, the **GHP** also holds for **fractional** heat equation

$$u_t + (-\Delta)^s u = 0\,, \quad 0 < s < 1\,,$$

▷ [Bonforte, Sire, Vazquez 2017] and [Vazquez 2018]

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The main ingredient is the **GHP**; we know that (**in the correct range of parameters**) and in the correct space of initial data $\mathcal{X}_m$ it holds for

- $u_t = \Delta(u^m)$;
- $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, in the range $d \geq 3$, $\frac{2d}{d+1} < p < 2$
- $u_t = \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m)$, the doubly non linear eq.
- $\partial_t u = \nabla.\left(\mathcal{A}(t,x,u,\nabla u)\right) + \mathcal{B}(t,x,u,\nabla u)$, under hps. in which the eq. is similar to the above

This phenomenon depends mostly on **fat tails**, the **GHP** also holds for **fractional** heat equation

$$u_t + (-\Delta)^s u = 0\,, \quad 0 < s < 1\,,$$

▷ [Bonforte, Sire, Vazquez 2017] and [Vazquez 2018]

Thank you for your attention!

Generalized Global Harnack principle

What happens for if the initial data $u_0 \notin \mathcal{X}$?

If the initial data

$$\frac{1}{(A+|x|)^{\alpha}} \le u_0(x) \le \frac{1}{(B+|x|)^{\alpha}} \quad \text{where } d < \alpha < \frac{2}{1-m},$$

then the solution

$$u(t,x) \asymp \frac{1}{|x|^{\alpha}} \qquad \text{for large |x|.}$$

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p-Laplace evolution equation

(Almost) everything holds for the problem

$$(\mathrm{p-CP}) \quad \begin{cases} \partial_t u = \Delta_p(u) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,. \end{cases}$$

Recall that $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, we focus on the range $d \geq 3$, $\frac{2d}{d+1} < p < 2$.
Let us define

$$|f|_{\mathcal{X}_p} := \sup_{R>0} R^{\frac{p}{2-p}-d} \int_{B_R^c(0)} |f(x)|\,\mathrm{d}x < \infty\,,$$

and the space

$$\mathcal{X}_p := \{u \in \mathrm{L}^1_+(\mathbb{R}^d) : |u|_{\mathcal{X}_p} < +\infty\}.$$

Theorem [M. Bonforte, N.S., D. Stan]

The **GHP** holds if and only if $u_0 \in \mathcal{X}_p \setminus \{0\}$.

Convergence of the **relative error** holds if and only if $u_0 \in \mathcal{X}_p \setminus \{0\}$.

On the heat equation and fractional heat equation

For solutions to $u_t = \Delta u$ uniform convergence in relative error does **not** hold,

$$\sup_{\mathbb{R}^d} \left| \frac{u(t,x)}{e^{\frac{-|x|^2}{4t}}} \right| = +\infty\,.$$

Take for instance $(4\pi)^{\frac{d}{2}} u(1,x) = e^{\frac{-|x+x_0|^2}{4t}}$. For the **fractional** heat equation

$$u_t + (-\Delta)^s u = 0\,, \quad 0 < s < 1\,, \tag{F}$$

we have

Theorem (J.L. Vázquez, 2018)

Let $u(t,x)$ be a solution to (F) with initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and compactedly supported. Then

$$\sup_{\mathbb{R}^d} \left| \frac{u(t,x) - M\,P_t(x)}{M\,P_t(x)} \right| \le C\,M\,R\,t^{-2s}$$

where M is the mass of u_0, P_t is the fundamental solutions to (F), t large and u_0 is supporeted in the ball of radius R.

On the space $\mathcal{X}$

In 2003, **Vázquez** also introduced the following condition for which a form of GHP holds

$$\int_{B_{\frac{|x|}{2}}(x)} u_0(y)\,\mathrm{d}y = O\left(|x|^{d-\frac{2}{1-m}}\right)$$

which is "a posteriori" equivalent to

$$|f|_{\mathcal{X}} := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{B_R^c(0)} |f(x)|\,\mathrm{d}x < \infty\,,$$

The proof of the equivalence uses the GHP!

Convergence of the relative error-1

The **relative error**

$$\left|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right|$$

is not always uniformly bounded in $\mathbb{R}^d$ (recall the solution $w(t,x)$).

However, for initial data in $\mathcal{X}$ it is!

Carrillo and Vázquez, proved for **radial** solution whose initial data satisfy $u_0(|x|) \lesssim |x|^{-\frac{2}{1-m}}$

$$\left\|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \frac{C}{t}\,.$$

Later Kim and McCann get rid of the **radial** assumption, but no result are available for the whole space $\mathcal{X}$.

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Convergence of the relative error-2

Theorem [M. Bonforte, N.S.]

Under the running assumption, a solution $u(t,x)$ to (CP) converges to $\mathcal{B}_M(t,x)$ in *uniform relative error*, i.e.

$$\lim_{t\to\infty}\left\|\frac{u(t,x)-\mathcal{B}_M(t,x)}{\mathcal{B}_M(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}=0$$

if and only if

$$u_0\in\mathcal{X}\setminus\{0\}$$

In the case of radial initial data we find the estimate of Carrillo and Vázquez for the whole $\mathcal{X}$

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