Γ-Convergence of an Ambrosio-Torterelli approximation scheme for image segmentation

Lisa Maria Kreusser

University of Bath

Joint work with
I. Fonseca (Carnegie Mellon), C.-B. Schönlieb (Cambridge) and M. Thorpe (Warwick)

Gradient Flows face-to-face 3
September 12, 2023

Image Segmentation

Motivation

- Image data is one of the **largest and fastest growing sources of information**
- **Partitioning an image** into disjoint regions with certain characteristics

- One of the most **fundamental and ubiquitous** tasks in image analysis
- **Examples:** Object detection, scene parsing, organ reconstruction, tumor detection, etc.
- **Mathematical model** for image segmentation

Variational models for image segmentation

Question: How to incorporate boundary in segmentation problem?

Variational models for image segmentation

Question: How to incorporate boundary in segmentation problem?
$\Rightarrow$ Implicit/explicit representation of boundaries, e.g. active contours, level-sets, graph cut, etc.

Variational models for image segmentation

Question: How to incorporate boundary in segmentation problem?
$\Rightarrow$ Implicit/explicit representation of boundaries, e.g. active contours, level-sets, graph cut, etc.

Variational approaches:

- Mumford-Shah model
- Chan-Vese active contour model without edges
- Chan-Vese multiphase level set framework

Implementation via the level set method of Osher and Sethian

Mumford-Shah model

- **Notation:**
  - Domain $\Omega \subset \mathbb{R}^d$ with $d \geqslant 1$
  - Given image $u_0 : \Omega \to \mathbb{R}^m$ with $m \geqslant 1$ to be segmented into two regions, e.g. bounded scalar (gray-scale) or vector-valued (color) image
  - Closed subset C in Ω, made up of a finite set of smooth curves
  - Connected components Ω_i of $\Omega \backslash C$, i.e. $\Omega = \cup_i \Omega_i \cup C$
- **Goal:** Find a decomposition Ω_i of Ω and an optimal piecewise smooth approximation u of a given image u_0 such that
  - u varies smoothly within each Ω_i
  - u varies rapidly or discontinuously across the boundaries of Ω_i
- **Mathematical formulation:** Minimisation of the energy functional

$$\mathbb{E}^{MS}(C, u) = \int_\Omega (u - u_0)^2 \, \mathrm{d}x + \mu \int_{\Omega \backslash C} |\nabla u|^2 \, \mathrm{d}x + \nu |C|$$

for fixed parameters $\mu, \nu > 0$

Mumford-Shah model

- **Mathematical formulation:** Minimisation of the energy functional

$$\mathbb{E}^{MS}(C,u) = \int_\Omega (u-u_0)^2\,\mathrm{d}x + \mu \int_{\Omega\setminus C} |\nabla u|^2\,\mathrm{d}x + \nu|C|$$

- **Interpretation:** For minimizer (u, C):
  - u is an 'optimal' piecewise smooth approximation of the possibly noisy image u_0
  - C can be regarded as approximating the edges of u_0
- **Theoretical results on the existence/regularity of minimizers:** Mumford and Shah, Morel and Solimini and De Giorgi et al., ...
- **Analysis based on weak formulation of Mumford-Shah model:** Ambrosio, Chambolle, Dal Maso, De Giorgi, March, Tortorelli, ...

Chan-Vese active contours model

Motivation:

- particular case of the Mumford-Shah model by restricting the segmented image u to piecewise constant functions
  $\Rightarrow$ Neglect $\mu \int_{\Omega \setminus C} |\nabla u|^2 \, \mathrm{d}x$ for now, i.e.

$$\mathbb{E}^{MS}(C, u) = \int_{\Omega} (u - u_0)^2 \, \mathrm{d}x + \nu |C|$$

- motivates the generalized, widely used multiphase level set model

Chan-Vese active contours model

Motivation:

- particular case of the Mumford-Shah model by restricting the segmented image u to piecewise constant functions
  $\Rightarrow$ Neglect $\mu \int_{\Omega \setminus C} |\nabla u|^2 \, \mathrm{d}x$ for now, i.e.

$$\mathbb{E}^{MS}(C, u) = \int_{\Omega} (u - u_0)^2 \, \mathrm{d}x + \nu |C|$$

- motivates the generalized, widely used multiphase level set model

Mathematical model: Minimisation of the energy

$$\mathbb{E}^{PC}(C, c^{(1)}, c^{(2)}) = \int_{E} (c^{(1)} - u_0)^2 \, \mathrm{d}x + \int_{\Omega \setminus E} (c^{(2)} - u_0)^2 \, \mathrm{d}x + \nu |C|$$

with respect to $c^{(1)}, c^{(2)}$ and C where $\nu > 0$ is a given parameter and set $E \subset \Omega$ depends on C

Chan-Vese active contours model

- **Notation:** Let $E \subset \Omega$ be an open subset of Ω such that the set E is the area inside the boundary curve $C = \partial E$ of length $|C|$ and let $c^{(1)}, c^{(2)}$ be unknown constants
- **Minimisation of the energy functional**

$$\mathbb{E}^{PC}(C, c^{(1)}, c^{(2)}) = \int_E (c^{(1)} - u_0)^2 \,\mathrm{d}x + \int_{\Omega \setminus E} (c^{(2)} - u_0)^2 \,\mathrm{d}x + \nu |C|$$

with respect to constants $c^{(1)}, c^{(2)}$ and C where $\nu > 0$ is a given parameter

Level set formulation of the Chan-Vese model

Original energy:

$$\mathbb{E}^{PC}(C, c^{(1)}, c^{(2)}) = \int_E (c^{(1)} - u_0)^2 \,\mathrm{d}x + \int_{\Omega \setminus E} (c^{(2)} - u_0)^2 \,\mathrm{d}x + \nu |C|$$

Representation of C as the **zero-crossing of a level set function** $\phi\colon \Omega \to \mathbb{R}$, i.e. $C = \{x \in \Omega \colon \phi(x) = 0\}$, and

$$\phi(x) > 0 \quad \text{in } E, \qquad \phi(x) < 0 \quad \text{in } \Omega \setminus E, \qquad \phi(x) = 0 \quad \text{on } \partial E.$$

Level set formulation of the Chan-Vese model

Original energy:

$$\mathbb{E}^{PC}(C, c^{(1)}, c^{(2)}) = \int_E (c^{(1)} - u_0)^2 \,\mathrm{d}x + \int_{\Omega \setminus E} (c^{(2)} - u_0)^2 \,\mathrm{d}x + \nu |C|$$

Representation of C as the **zero-crossing of a level set function** $\phi \colon \Omega \to \mathbb{R}$, i.e. $C = \{x \in \Omega \colon \phi(x) = 0\}$, and

$$\phi(x) > 0 \quad \text{in } E, \qquad \phi(x) < 0 \quad \text{in } \Omega \setminus E, \qquad \phi(x) = 0 \quad \text{on } \partial E.$$

Level-set energy $E^{PC}(\phi, c^{(1)}, c^{(2)})$

$$= \int_\Omega (c^{(1)} - u_0)^2 H(\phi) \,\mathrm{d}x + \int_\Omega (c^{(2)} - u_0)^2 (1 - H(\phi)) \,\mathrm{d}x + \nu \int_\Omega |\nabla H(\phi)| \,\mathrm{d}x$$

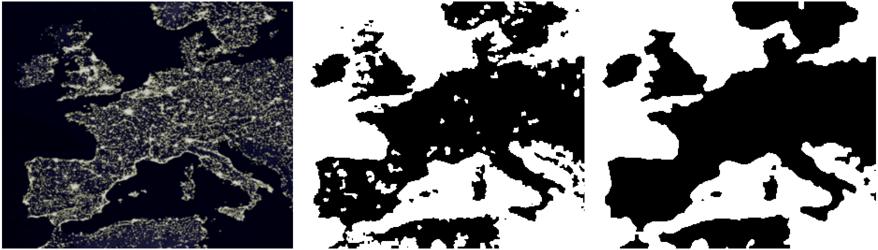
for $u(x) = c^{(1)} H(\phi(x)) + c^{(2)}(1 - H(\phi(x)))$ and Heaviside function H

Level set formulation of the Chan-Vese model

$\mathbb{E}^{PC}(\phi, c^{(1)}, c^{(2)})$

$$= \int_\Omega (c^{(1)} - u_0)^2 H(\phi)\, \mathrm{d}x + \int_\Omega (c^{(2)} - u_0)^2 (1 - H(\phi))\, \mathrm{d}x + \nu \int_\Omega |\nabla H(\phi)|\, \mathrm{d}x$$

for $u(x) = c^{(1)} H(\phi(x)) + c^{(2)}(1 - H(\phi(x)))$ and Heaviside function H:



(a) Input image (b) $\nu = 0.2$ (c) $\nu = 0.6$

Figure: Image segmentation results for different parameter values $\nu > 0$

Extension of the model to piecewise smooth segmentations

- Replacing the constants $c^{(1)}, c^{(2)}$ by **smooth functions** on E and $\Omega \backslash E$ proposed independently by Vese and Chan, and Tsai et al.
- Extension to **vector-valued functions** such as color images
- **Energy functional:**

$$
\begin{aligned}
&\mathbb{E}^{PS}(\phi, c^{(1)}, c^{(2)}) \\
&= \int_\Omega |c^{(1)} - u_0|^2 H(\phi)\,\mathrm{d}x + \int_\Omega |c^{(2)} - u_0|^2 (1 - H(\phi))\,\mathrm{d}x \\
&\quad + \mu \int_\Omega |\nabla c^{(1)}|^2 H(\phi) + |\nabla c^{(2)}|^2 (1 - H(\phi))\,\mathrm{d}x + \nu \int_\Omega |\nabla H(\phi)|\,\mathrm{d}x
\end{aligned}
$$

- **Numerical results** have been obtained independently and contemporaneously by Vese and Chan, and Tsai et al.
- **Very good reconstruction** of piecewise smooth regions possible with the model, jumps are well located and without smearing, and the piecewise constant case can be recovered.

Reformulation of the energy functional via the Ambrosio-Tortorelli approximation

- **Numerical minimization difficult**:
  - Non-smoothness of energy functional, particularly $\nu \int_\Omega |\nabla H(\phi)| \, \mathrm{d}x$
    $\Rightarrow$ Replace by suitable approximation
  - Dependency on the unknown form of the level set function ϕ
- **Ambrosio-Tortorelli approximation**
  - one of the most computationally efficient approximations of the Mumford-Shah functional
  - uses the Ginzburg-Landau functional

$$\mathbb{E}_\epsilon^{GL}(v) = \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{\epsilon} W(v) \, \mathrm{d}x$$

where $\epsilon > 0$ is a positive constant and $W \colon \mathbb{R} \to [0, +\infty)$ is a double well potential with wells at 0 and 1, e.g. $W(x) = x^2(x-1)^2$

Reformulation of the energy functional via the Ambrosio-Tortorelli approximation

- **Reformulated energy functional**

$$\bar{\mathbb{E}}_{\mu_\epsilon,\epsilon}(v, c^{(1)}, c^{(2)}) = \int_\Omega |c^{(1)} - u_0|^p |v| + |c^{(2)} - u_0|^p |1 - v| \,\mathrm{d}x$$
$$+ \mu_\epsilon \int_\Omega |\nabla c^{(1)}|^p |v| + |\nabla c^{(2)}|^p |1 - v| \,\mathrm{d}x + \frac{\nu}{c_W} \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{\epsilon} W(v) \,\mathrm{d}x$$

  where $c_W := 2 \int_0^1 \sqrt{W(t)} \,\mathrm{d}t > 0$
- **Aim:** Study convergence of minimisers as $\epsilon \to 0$ to show consistency of numerical method
- **Problem:** For piecewise smooth approximations $c^{(1)}, c^{(2)}$ any Γ-convergence result requires $c^{(1)}, c^{(2)}$ to be defined only for $x \in \Omega$ such that $v(x) \neq 0$ and $1 - v(x) \neq 0$, respectively
  $\Rightarrow$ introduce appropriate definition of differentiability, appropriate definition of domain functions, ...

Energy functionals for Γ-convergence

Approximative energy functional:

$$\mathbb{E}_{\mu_\epsilon,\epsilon}(v, c^{(1)}, c^{(2)}) = \|c^{(1)} - u_0\|_{L^p(\nu_{|v|};\mathbb{R}^m)} + \|c^{(2)} - u_0\|_{L^p(\nu_{|1-v|};\mathbb{R}^m)} + \mu_\epsilon \|c^{(1)}\|^p_{L^{1,p}(\nu_{|v|})} + \mu_\epsilon \|c^{(2)}\|^p_{L^{1,p}(\nu_{|1-v|})} + \frac{\nu}{c_W} \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{\epsilon} W(v)\,\mathrm{d}x$$

Two cases: $\mu_\epsilon \to \mu$ with $\mu > 0$, and $\mu_\epsilon \to +\infty$ as $\epsilon \to 0$

Limiting energy functional (as $\epsilon \to 0$):

$$\mathbb{E}_\mu(v, c^{(1)}, c^{(2)}) = \|c^{(1)} - u_0\|_{L^p(\nu_{|v|};\mathbb{R}^m)} + \|c^{(2)} - u_0\|_{L^p(\nu_{|1-v|};\mathbb{R}^m)} + \mu \|c^{(1)}\|^p_{L^{1,p}(\nu_{|v|})} + \mu \|c^{(2)}\|^p_{L^{1,p}(\nu_{|1-v|})} + \nu \,\mathrm{TV}(v)$$

for any $v = \chi_E \in \mathrm{BV}(\Omega; \{0,1\})$ with $E = \{x \in \Omega \colon v(x) = 1\}$, $c^{(1)} \in W^{1,p}((\Omega, \nu_{|v|}); \mathbb{R}^m)$ and $c^{(2)} \in W^{1,p}((\Omega, \nu_{|1-v|}); \mathbb{R}^m)$, and $\mathbb{E}_\mu(v, c^{(1)}, c^{(2)}) = +\infty$ otherwise.

Γ-convergence

Definition

Let (X, d) be a metric space and let $\mathbb{E}_n$ be a sequence of functions $\mathbb{E}_n \colon X \to [-\infty, +\infty]$. We say that $\{\mathbb{E}_n\}$ **Γ-converges** to a function $\mathbb{E} \colon X \to [-\infty, +\infty]$ if the following two properties are satisfied:

- **(Liminf inequality)** For every $x \in X$ and every sequence $\{x_n\} \subset X$ such that $x_n \to x$ with respect to d,

$$\mathbb{E}(x) \leqslant \liminf_{n\to\infty} \mathbb{E}_n(x_n).$$

- **(Limsup inequality)** For every $x \in X$ there exists a sequence $\{x_n\} \subset X$ such that $x_n \to x$ with respect to d and

$$\limsup_{n\to\infty} \mathbb{E}_n(x_n) \leqslant \mathbb{E}(x).$$

The limit function $\mathbb{E}$ is called the **Γ-limit** of the sequence $\{\mathbb{E}_n\}$.

Compactness property

Definition

A sequence of nonnegative functionals $\{\mathbb{E}_n\}$ satisfies the **compactness property** if for any increasing subsequence $\{n_k\}$ of natural numbers and any bounded sequence $\{x_k\} \subset X$ such that

$$\sup_{k \in \mathbb{N}} \mathbb{E}_{n_k}(x_k) < \infty,$$

the sequence $\{x_k\}$ is relatively compact in X.

Compactness, Γ-convergence and the convergence of minimizers

Proposition

Let $\mathbb{E}_n \colon X \to [0,\infty]$ be a sequence of nonnegative functionals which are not identically equal to $+\infty$, satisfy the compactness property and Γ-converge to the functional $\mathbb{E} \colon X \to [0,\infty]$ which is not identically equal to $+\infty$. Then,

$$\lim_{n\to\infty} \inf_{x\in X} \mathbb{E}_n(x) = \min_{x\in X} \mathbb{E}(x).$$

Γ-convergence for piecewise constant segmentations

- For constants $c^{(1)}, c^{(2)}$, we define $\bar{\mathbb{E}}_\epsilon \colon L^1(\Omega;\mathbb{R}) \times \mathbb{R}^m \times \mathbb{R}^m$ by

$$\bar{\mathbb{E}}_\epsilon(v, c^{(1)}, c^{(2)}) := \int_\Omega |c^{(1)} - u_0|^p |v| + |c^{(2)} - u_0|^p |1 - v| \,\mathrm{d}x + \frac{\nu}{c_W} \int_\Omega \epsilon |\nabla v|^2 + \frac{1}{\epsilon} W(v) \,\mathrm{d}x,$$

$$\bar{\mathbb{E}}(v, c^{(1)}, c^{(2)}) := \begin{cases} \int_E |c^{(1)} - u_0|^p \,\mathrm{d}x + \int_{\Omega\setminus E} |c^{(2)} - u_0|^p \,\mathrm{d}x + \nu \,\mathrm{TV}(v), & v = \chi_E \in \mathrm{BV}(\Omega;\{0,1\}), \\ +\infty, & \text{otherwise.} \end{cases}$$

- Γ-convergence of $\bar{\mathbb{E}}_\epsilon$ to $\bar{\mathbb{E}}$ for piecewise constant segmentations

Compactness property for piecewise constant approximations

Theorem (Compactness)

Let $\Omega \subset \mathbb{R}^d$ be an open set with finite measure, let $\epsilon_n \to 0$ and let $\{v_n\} \subset W^{1,2}(\Omega;\mathbb{R})$, $\{c_n^{(1)}\}, \{c_n^{(2)}\} \subset \mathbb{R}^m$ such that

$$M := \sup_{n\in\mathbb{N}} \bar{\mathbb{E}}_{\epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}) < +\infty.$$

Then, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v \in \mathrm{BV}(\Omega;\{0,1\})$ with $v = \chi_E$ for some Lebesgue measurable set $E \subset \Omega$ such that $v_{n_k} \to v$ in $L^1(\Omega;\mathbb{R})$. If $\lambda^d(E) > 0$, then there exists a converging subsequence $\{c_{n_k}^{(1)}\}$ of $\{c_n^{(1)}\}$ with limit $c^{(1)} \in \mathbb{R}^m$. If $\lambda^d(\Omega \setminus E) > 0$ then there exists a converging subsequence $\{c_{n_k}^{(2)}\}$ of $\{c_n^{(2)}\}$ with limit $c^{(2)} \in \mathbb{R}^m$.

Idea of compactness proof

- Set

$$f(t) := \frac{2\nu}{c_W}\int_0^t \sqrt{W(s)}\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

- For every $n \in \mathbb{N}$ we have

$$M \geqslant \mathbb{E}_{\mu_{\epsilon_n},\epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}) \geqslant \frac{\nu}{c_W}\int_\Omega \epsilon|\nabla v|^2 + \frac{1}{\epsilon}W(v)\,\mathrm{d}x$$
$$\geqslant \frac{2\nu}{c_W}\int_\Omega \sqrt{W(v_n)}|\nabla v_n|\,\mathrm{d}x = \int_\Omega |\nabla(f \circ v_n)|\,\mathrm{d}x$$

- Rellich–Kondrachov theorem implies that $\{f \circ v_n\}$ has a converging subsequence, i.e. there exists a subsequence $\{v_n\}$ (not relabeled) and a function $w \in \mathrm{BV}(\Omega; \mathbb{R})$ such that $w_n := f \circ v_n \to w$ in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$
- Hence, f^{-1} is continuous with

$$v_n(x) = f^{-1}(w_n(x)) \to f^{-1}(w(x)) =: v(x) \quad \lambda^d\text{-a.e. } x \in \Omega$$

- Since $W(v_n) \to 0$ λ^d-a.s., we have $v(x) \in \{0, 1\}$ for λ^d-a.e. $x \in \Omega$

Liminf inequality for piecewise constant approximations

Theorem (Liminf inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set. Let $v \in L^1(\Omega; \mathbb{R})$, $c^{(1)}, c^{(2)} \in \mathbb{R}^m$ and consider a sequence $\epsilon_n \to 0$. Assume that $\{v_n\} \subset L^1(\Omega; \mathbb{R})$ such that $v_n \to v$ in $L^1(\Omega; \mathbb{R})$. Further, let $\{c_n^{(1)}\}, \{c_n^{(2)}\} \subset \mathbb{R}^m$ such that $c_n^{(1)} \to c^{(1)}$, $c_n^{(2)} \to c^{(2)}$. Then,

$$\bar{\mathbb{E}}(v, c^{(1)}, c^{(2)}) \leqslant \liminf_{n\to\infty} \bar{\mathbb{E}}_{\epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}).$$

Limsup inequality for piecewise constant approximations

Theorem (Limsup inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary. For every $v \in L^1(\Omega; \mathbb{R})$ and $c^{(1)}, c^{(2)} \in \mathbb{R}^m$, there exist sequences $\{v_n\} \subset L^1(\Omega; \mathbb{R})$ and $\{c_n^{(1)}\}, \{c_n^{(2)}\} \subset \mathbb{R}^m$ such that $v_n \to v$ in $L^1(\Omega; \mathbb{R})$, $c_n^{(1)} \to c^{(1)}$, $c_n^{(2)} \to c^{(2)}$, and

$$\limsup_{n\to\infty} \bar{\mathbb{E}}_{\epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}) \leqslant \bar{\mathbb{E}}(v, c^{(1)}, c^{(2)}),$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Characterization of $W^{1,p}$ functions

Standard definitions of Sobolev spaces:

$$W^{1,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) \; : \; f \in L^p(\Omega), \nabla f \in L^p(\Omega)\},$$
$$L^{1,p}(\Omega) = \{f \in \mathcal{D}'(\Omega) \; : \; \nabla f \in L^p(\Omega)\}$$

Theorem (Characterization of $W^{1,p}$)

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and $1 < p < +\infty$. Then $f \in W^{1,p}(\Omega)$, where $1 < p < +\infty$, ***if and only if*** *$f \in L^p(\Omega)$ and there is $0 \leqslant g \in L^p(\Omega)$ so that*

$$|f(x) - f(y)| \leqslant |x - y|(g(x) + g(y)) \;\; a.e.$$

Moreover, $\|f\|_{L^{1,p}} \approx \inf_g \|g\|_{L^p}$, i.e. there exists a constant $C \geqslant 1$ such that $\frac{1}{C}\|f\|_{L^{1,p}} \leqslant \inf_g \|g\|_{L^p} \leqslant C\|f\|_{L^{1,p}}$, where the infimum is taken over the class of all functions g satisfying the above inequality.

Definition of metric measure spaces

Definition

Let (Ω, d, ν) be a metric space (Ω, d) with finite diameter

$$\operatorname{diam}\Omega = \sup_{x,y\in\Omega} d(x,y) < +\infty$$

and a finite positive Borel measure ν. Let $1 < p \leqslant +\infty$. The Sobolev spaces $L^{1,p}(\Omega, d, \nu)$ and $W^{1,p}(\Omega, d, \nu)$ are defined as

$$L^{1,p}(\Omega, d, \nu) = \{f : \Omega \to \mathbb{R} \colon f \text{ is measurable and there exists } 0 \leqslant g \in L^p(\nu) \text{ such that } |f(x) - f(y)| \leqslant d(x,y)(g(x) + g(y)) \text{ a.e.}\}$$

and

$$W^{1,p}(\Omega, d, \nu) = \{f \in L^{1,p}(\Omega, d, \nu) \ : \ f \in L^p(\nu)\},$$

respectively.

Definition of the space CL^p

- For open set $\Omega \subset \mathbb{R}^d$ define **function space**

$$CL^p(\Omega) := \{(v, c^{(1)}, c^{(2)})\colon v \in L^1((\Omega, \lambda^d|_\Omega); \mathbb{R}), c^{(1)} \in L^p((\Omega, \nu_{|v|}); \mathbb{R}^m), c^{(2)} \in L^p((\Omega, \nu_{|1-v|}); \mathbb{R}^m)\}$$

where $\nu_{|v|}$ and $\nu_{|1-v|}$ are probability measures which have Lebesgue densities $\frac{|v|}{\|v\|_{L^1(\Omega;\mathbb{R})}}$ and $\frac{|1-v|}{\|1-v\|_{L^1(\Omega;\mathbb{R})}}$ restricted to Ω, respectively.

- **Metric** for $(v, c^{(1)}, c^{(2)})$ and $(\tilde{v}, \tilde{c}^{(1)}, \tilde{c}^{(2)})$ in $CL^p(\Omega)$

$$\inf_{\pi \in \Pi(\nu_{|v|}, \nu_{|\tilde{v}|})} \left(\int\int_{\Omega\times\Omega} |x-y|^p + |c^{(1)}(x) - \tilde{c}^{(1)}(x)|^p d\pi(x,y) \right)^{\frac{1}{p}}$$

$$+ \inf_{\pi \in \Pi(\nu_{|1-v|}, \nu_{|1-\tilde{v}|})} \left(\int\int_{\Omega\times\Omega} |x-y|^p + |c^{(2)}(x) - \tilde{c}^{(2)}(x)|^p d\pi(x,y) \right)^{\frac{1}{p}}$$

Properties of the space CL^p

We can show:

- $(CL^p(\Omega), d_{CL^p})$ is a metric space.
- **Characterization of the convergence in** $CL^p(\Omega)$**:** Convergence in $CL^p(\Omega)$ can be regarded as a generalization of
  - weak convergence of measures
  - L^p convergence of functions

Transportation theory

Definition

Given a Borel map $T\colon \Omega \to \Omega$ and $\nu \in \mathcal{P}(\Omega)$ the **push-forward** of ν by T is denoted by $T_{\#}\nu \in \mathcal{P}(\Omega)$ and is given by

$$T_{\#}\nu(E) := \nu(T^{-1}(E)),\ E \in \mathcal{B}(\Omega).$$

For any bounded Borel function $\phi\colon \Omega \to \mathbb{R}$ the following change of variables holds:

$$\int_{\Omega} \phi(x)\,\mathrm{d}(T_{\#}\nu)(x) = \int_{\Omega} \phi(T(x))\,\mathrm{d}\nu(x).$$

Definition

A Borel map $T\colon \Omega \to \Omega$ is called a **transportation map** between the measures $\nu \in \mathcal{P}(\Omega)$ and $\tilde{\nu} \in \mathcal{P}(\Omega)$ if $\tilde{\nu} = T_{\#}\nu$.

Compactness property for piecewise smooth approximations

Theorem (Compactness)

Let $\Omega \subset \mathbb{R}^d$ with $d \geqslant 2$ be an open set with finite measure. Let $\epsilon_n \to 0$ and let $\{v_n\} \subset W^{1,2}((\Omega, \lambda^d|_\Omega); \mathbb{R}), \{c_n^{(1)}\}, \{c_n^{(2)}\}$ such that $c_n^{(1)} \in W^{1,p}((\Omega, \nu_{|v_n|}); \mathbb{R}^m), c_n^{(2)} \in W^{1,p}((\Omega, \nu_{|1-v_n|}); \mathbb{R}^m)$ and $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_{\epsilon_n},\epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}) < +\infty$ where $\lim_{n\to\infty} \mu_{\epsilon_n} \in (0, +\infty]$. Then there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $v \in \mathrm{BV}(\Omega; \{0, 1\})$ with $v = \chi_E$ for some Lebesgue measurable set $E \subset \Omega$ such that $v_{n_k} \to v$ in $L^1(\Omega; \mathbb{R})$. For appropriate assumptions on the perimeter of E and $\Omega \backslash E$, $0 < \lambda^d(E) < \lambda^d(\Omega)$ and transportation maps $\{T_{n_k}^{(1)}\}, \{T_{n_k}^{(2)}\}$ satisfying $\lim_{n_k\to\infty} \|T_{n_k}^{(1)} - I\|_{L^\infty} = \lim_{n_k\to\infty} \|T_{n_k}^{(2)} - I\|_{L^\infty} = 0$, there exists a subsequence $(v_{n_k}, c_{n_k}^{(1)}, c_{n_k}^{(2)})$ converging to $(v, c^{(1)}, c^{(2)}) \in CL^p(\Omega)$ in $CL^\alpha(\Omega)$ for any $1 \leqslant \alpha < p$.

Liminf inequality for piecewise smooth approximations

Theorem (Liminf inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set. Let $(v, c^{(1)}, c^{(2)}) \in CL^p(\Omega)$ *and consider positive sequences* $\{\epsilon_n\}, \{\mu_{\epsilon_n}\}$ *with* $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\lim_{n\to\infty} \mu_{\epsilon_n} \in (0, +\infty]$*. Assume that* $\{(v_n, c_n^{(1)}, c_n^{(2)})\} \subset CL^p(\Omega)$ *such that* $(v_n, c_n^{(1)}, c_n^{(2)}) \to (v, c^{(1)}, c^{(2)})$ *in* $CL^p(\Omega)$*. Then,*

$$\mathbb{E}_\mu(v, c^{(1)}, c^{(2)}) \leqslant \liminf_{n\to\infty} \mathbb{E}_{\mu_{\epsilon_n}, \epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}).$$

Limsup inequality for piecewise smooth approximations

Theorem (Limsup inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary. Let $(v, c^{(1)}, c^{(2)}) \in CL^p(\Omega)$ and consider positive sequences $\{\epsilon_n\}, \{\mu_{\epsilon_n}\}$ with $\lim_{n\to\infty} \epsilon_n = 0$ and $\lim_{n\to\infty} \mu_{\epsilon_n} \in (0, +\infty]$. Then, there exists a sequence $\{(v_n, c_n^{(1)}, c_n^{(2)})\} \subset CL^p(\Omega)$ such that $(v_n, c_n^{(1)}, c_n^{(2)}) \to (v, c^{(1)}, c^{(2)})$ in $CL^p(\Omega)$, and

$$\limsup_{n\to\infty} \mathbb{E}_{\mu_{\epsilon_n},\epsilon_n}(v_n, c_n^{(1)}, c_n^{(2)}) \leqslant \mathbb{E}_\mu(v, c^{(1)}, c^{(2)}).$$

Γ-convergence for piecewise smooth approximations

Theorem

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set, let $1 < p < +\infty$. Then, the functional $\mathbb{E}_{\mu_\epsilon,\epsilon} \colon CL^p(\Omega) \to [0, +\infty]$ Γ-converges with respect to the $CL^p(\Omega)$ topology to the functionals $\mathbb{E}_\mu(v, c^{(1)}, c^{(2)})$ whose form depends on $\mu_\epsilon \to \mu$ with $\mu > 0$, and $\mu_\epsilon \to +\infty$, respectively, as $\epsilon \to 0$.

Convergence of minimisers for piecewise smooth approximations

Corollary (Convergence of minimisers)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with $d \geqslant 2$. Suppose that $(v_n, c_n^{(1)}, c_n^{(2)}) \in CL^p(\Omega)$ is a minimizer of the energy $\mathbb{E}_{\mu_{\epsilon_n}, \epsilon_n}$ for positive sequences $\{\epsilon_n\}, \{\mu_{\epsilon_n}\}$ with $\lim_{n\to\infty} \epsilon_n = 0$ and $\lim_{n\to\infty} \mu_{\epsilon_n} = \mu \in (0, +\infty]$. If there exists $v = \chi_E$ for some Lebesgue measurable set $E \subset \Omega$, if there exist $\kappa > 0$, $r_0 > 0$ such that $P(E; B_r(x)) \geqslant \kappa r^d$ for every $x \in \partial^* E$, if there exist $\kappa > 0$, $r_0 > 0$ such that $P(\Omega \backslash E; B_r(x)) \geqslant \kappa r^d$ for every $x \in \partial^*(\Omega \backslash E)$, and if there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to v$ in $L^1(\Omega; \mathbb{R})$, then there exists $(v, c^{(1)}, c^{(2)}) \in CL^p(\Omega)$ such that, up to a subsequence (not relabeled), $(v_n, c_n^{(1)}, c_n^{(2)})$ converges to $(v, c^{(1)}, c^{(2)})$ in $CL^p(\Omega)$, and $(v, c^{(1)}, c^{(2)})$ minimizes the energy $\mathbb{E}_\mu$ for $\mu < +\infty$ and $\mu = +\infty$, respectively, over $CL^p(\Omega)$.

Conclusion

- **Challenging mathematical models**, requiring the development of new mathematical tools
- **Diverse applications** of image segmentation

- **Rigorous analysis** of the Mumford-Shah model
- **Combining techniques from a variety of fields of mathematics**
- Theoretical **foundations for substantial progress in applications**

Thank you very much for your attention!

Happy to answer any questions!

References

I. Fonseca, LMK, C.-B. Schönlieb and M. Thorpe,
Γ-*Convergence of an Ambrosio-Torterelli approximation scheme for image segmentation*,
arXiv:2202.04965, to appear in Indiana University Mathematics Journal

More information: https://people.bath.ac.uk/lmk54/
Email: Lmk54@bath.ac.uk