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Second order two-species systems with nonlocal interactions and large damping

Valeria Iorio

joint work with M. Di Francesco and S. Fagioli

A bit of the state of the art

Let ρ be a population density and v its velocity field and consider the model

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0, \\ \dfrac{\partial}{\partial t}(\rho v) + \dfrac{\partial}{\partial x}(\rho v^2) = f[\rho], \end{cases}$$

with initial data in a space of probability measures.

- In [Natile, Savaré, '09], the one dimensional, $f = 0$ case is studied.
- In [Brenier, Gangbo, Savaré, Westdickenberg, '13], the one-dimensional, f-nonlocal case is investigated.
- In [Carrillo, Choi, Tse, '18], authors study the multi-dimensional, damped case with f made up by internal energy, confinement and nonlocal interaction terms.

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Let ρ_1 and ρ_2 be two species of agents interacting with each other. Let v_1 and v_2 be their velocity fields, respectively. Let $\sigma > 0$ be the damping parameter.
The one-dimensional system we deal with is

$$\begin{cases} \dfrac{\partial \rho_1}{\partial t} + \dfrac{\partial}{\partial x}(\rho_1 v_1) = 0, \\ \dfrac{\partial \rho_2}{\partial t} + \dfrac{\partial}{\partial x}(\rho_2 v_2) = 0, \\ \dfrac{\partial}{\partial t}(\rho_1 v_1) + \dfrac{\partial}{\partial x}(\rho_1 v_1^2) = -\sigma \rho_1 v_1 - \rho_1 [K'_{11} * \rho_1 + K'_{12} * \rho_2], \\ \dfrac{\partial}{\partial t}(\rho_2 v_2) + \dfrac{\partial}{\partial x}(\rho_2 v_2^2) = -\sigma \rho_2 v_2 - \rho_2 [K'_{22} * \rho_2 + K'_{21} * \rho_1]. \end{cases} \qquad \text{(Second order system)}$$

equipped with initial data

$$\begin{cases} (\rho_1, v_1)(t=0) = (\overline{\rho}_1, \overline{v}_1), \\ (\rho_2, v_2)(t=0) = (\overline{\rho}_2, \overline{v}_2). \end{cases}$$

Discrete particle counterpart

(Second order system) has a natural discrete *particle* counterpart. Let us consider $x_1, \ldots, x_N$ as N particles of the first species with masses $m_1, \ldots, m_N$, and $y_1, \ldots, y_M$ as M particles of the second species with masses $n_1, \ldots, n_M$. The dynamics of x_i and y_j are determined by

$$\begin{cases} \ddot{x}_i = -\sigma \dot{x}_i - \sum\limits_{k \neq i} m_k K_{11}'(x_i - x_k) - \sum\limits_{k} n_k K_{12}'(x_i - y_k), \\ \ddot{y}_j = -\sigma \dot{y}_j - \sum\limits_{k \neq j} n_k K_{22}'(y_j - y_k) - \sum\limits_{k} m_k K_{21}'(y_j - x_k), \end{cases} \qquad \text{(Second order particle system)}$$

with $i = 1, \ldots, N$ and $j = 1, \ldots, M$ and the following initial data

$$\begin{cases} x_i(0) = \overline{x}_i, \\ \dot{x}_i(0) = \overline{v}_i, \end{cases} \qquad \begin{cases} y_j(0) = \overline{y}_j, \\ \dot{y}_j(0) = \overline{w}_j. \end{cases}$$

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The goals are:

- prove a well-posedness result in the case of smooth interaction potentials;
- investigate the large damping limit, proving that, under a suitable rescaling, (Second order system) converges towards the corresponding first order system;
- show a large-time collapse result in the case of Newtonian self-interaction potentials.

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Time scaling and formal large damping limit

Consider the new time variable $\tau = t/\sigma$, introduce the scaled particle trajectories:

$$x_i(t) = \chi_i(\tau) = \chi_i(t/\sigma), \qquad y_j(t) = \xi_j(\tau) = \xi_j(t/\sigma)$$

and notice that

$$\dot{\chi}_i(0) = \sigma \overline{v}_i, \qquad \dot{\xi}_j(0) = \sigma \overline{w}_j.$$

Thus, (Second order particle system) becomes

$$\begin{cases} \sigma^{-2}\ddot{\chi}_i(\tau) = -\dot{\chi}_i(\tau) - \sum_{k\neq i} m_k K'_{11}(\chi_i(\tau) - \chi_k(\tau)) - \sum_k n_k K'_{12}(\chi_i(\tau) - \xi_k(\tau)) \\ \sigma^{-2}\ddot{\xi}_j(\tau) = -\dot{\xi}_j(\tau) - \sum_{k\neq j} n_k K'_{22}(\xi_j(\tau) - \xi_k(\tau)) - \sum_k m_k K'_{21}(\xi_j(\tau) - \chi_k(\tau)) \end{cases}$$

Time scaling and formal large damping limit

A formal limit $\sigma \to +\infty$ leads to the first order ODE system

$$\begin{cases} \dot{\chi}_i(\tau) = -\sum_{k\neq i} m_k K'_{11}(\chi_i(\tau) - \chi_k(\tau)) - \sum_k n_k K'_{12}(\chi_i(\tau) - \xi_k(\tau)), \\ \dot{\xi}_j(\tau) = -\sum_{k\neq i} n_k K'_{22}(\xi_j(\tau) - \xi_k(\tau)) - \sum_k m_k K'_{21}(\xi_j(\tau) - \chi_k(\tau)). \end{cases}$$

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Considering the same definition of τ, at continuous level, as $\sigma \to +\infty$ we formally get the first order PDE system

$$\begin{cases} \dfrac{\partial \rho_1}{\partial \tau} = \dfrac{\partial}{\partial x}[\rho_1 K'_{11} * \rho_1 + \rho_1 K'_{12} * \rho_2], \\ \dfrac{\partial \rho_2}{\partial \tau} = \dfrac{\partial}{\partial x}[\rho_2 K'_{22} * \rho_2 + \rho_2 K'_{21} * \rho_1]. \end{cases} \qquad \text{(First order system)}$$

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Sticky particle assumption

- Since we consider a second order pressure-less Euler system, densities ρ_i are not forced to be absolutely continue with respect to Lebesgue measure, so collisions may occur.
- The dynamics we adopt is the sticky particle dynamics, according to which when two particles of the same species collide, then the stick and are not allowed to split.
- So that, in dimension one, we will introduce an operator that assures us that particles cannot cross each other.

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Pseudo-inverses and Lagrangian description

Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$ with finite second moment, i.e., $\int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < \infty$ for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. The 2-Wasserstein distance $W_2(\mu, \eta)$ between two measures $\mu, \eta \in \mathcal{P}_2(\mathbb{R}^d)$ is defined by

$$W_2^2(\mu, \eta) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x, y) \qquad \text{with } \gamma \in \Pi_o(\mu, \eta),$$

where $\Pi_0(\mu, \eta)$ denotes the class of the optimal plans between μ and η.

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where $\Pi_0(\mu,\eta)$ denotes the class of the optimal plans between μ and η.
In the one-dimensional case, there exists a unique optimal plan $\gamma \in \Pi_o(\mu,\eta)$, and it can be characterised by the pseudo-inverses of μ and η: given $\mu \in \mathcal{P}(\mathbb{R})$, its pseudo-inverse is

$$X_\mu(m) := \inf\{x \,:\, M_\mu(x) > m\} \qquad \text{for all } m \in \Omega,$$

where $\Omega := (0,1)$ and M_μ is the cumulative distribution of the measure μ, i.e.,

$$M_\mu(x) := \mu((-\infty, x]) \qquad \text{for all } x \in \mathbb{R}.$$

The map X_μ is right-continuous and non-decreasing. Moreover,

$$\mu \in \mathcal{P}_2(\mathbb{R}) \text{ if and only if } X_\mu \in L^2(\Omega).$$

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Pseudoinverses and Lagrangian description

Finally, introducing

$$\mathcal{K} := \{X \in L^2(\Omega) \ : \ X \text{ is non-decreasing}\},$$

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Pseudoinverses and Lagrangian description

Finally, introducing

$$\mathcal{K} := \{X \in L^2(\Omega) \ : \ X \text{ is non-decreasing}\},$$

we have that the map

$$\Psi : \mathcal{P}_2(\mathbb{R}) \ni \mu \mapsto X_\mu \in \mathcal{K}$$

is a distance-preserving bijection between the space $\mathcal{P}_2(\mathbb{R})$ and the convex cone $\mathcal{K}$.

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Let $I_{\mathcal{K}} : L^2(\Omega) \to [0, +\infty)$ be the indicator function of the L^2-convex cone $\mathcal{K}$, that is

$$I_{\mathcal{K}}(X) = \begin{cases} 0 & \text{if } X \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}$$

The sub-differential of $I_{\mathcal{K}}$ in X is

$$\partial I_{\mathcal{K}}(X) = \begin{cases} \{Z \in L^2(\Omega) \ : \ 0 \geq \int_\Omega Z(\widetilde{X} - X)dm, \text{ for all } \widetilde{X} \in \mathcal{K}\} & \text{if } X \in \mathcal{K}, \\ \emptyset & \text{otherwise.} \end{cases}$$

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Lagrangian description

Let $\varepsilon := \sigma^{-2}$. Considering $(\rho_1, \rho_2, v_1, v_2)$ solution to (Second order system) and defining the maps X_i, V_i, $i = 1, 2$, as

$$X_i(t,\cdot) = \Psi(\rho_i(t,\cdot)), \qquad V_i(t,\cdot) = v_i(t, X_i(t,\cdot)) = \partial_t X_i(t,\cdot),$$

(Second order system) can be reformulated in Lagrangian terms as

$$\begin{cases} \varepsilon \dot{X}_1(t,m) + X_1(t,m) + \partial I_{\mathcal{K}}(X_1(t,m)) \ni \varepsilon \overline{V}_1(m) + \overline{X}_1(m) \\ \qquad\qquad + \displaystyle\int_0^t F_1[X_1(\cdot,r), X_2(\cdot,r)](m)\,dr, \\ \varepsilon \dot{X}_2(t,m) + X_2(t,m) + \partial I_{\mathcal{K}}(X_2(t,m)) \ni \varepsilon \overline{V}_2(m) + \overline{X}_2(m) \\ \qquad\qquad + \displaystyle\int_0^t F_2[X_1(\cdot,r), X_2(\cdot,r)](m)\,dr, \end{cases} \tag{Lagrangian system}$$

where

$$F_i[X_i, X_j](m) = -\int_\Omega K_{ii}'(X_i(r,m) - X_i(r,m'))\,dm' - \int_\Omega K_{ij}'(X_i(r,m) - X_j(r,m'))\,dm'.$$

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Assumption and definition of solution

- A function $K : \mathbb{R} \to \mathbb{R}$ is called an *admissible potential* if

$$K \in W^{2,\infty}(\mathbb{R}), \qquad K(0) = 0, \qquad K(-x) = K(x). \tag{Adm}$$

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- A function $K : \mathbb{R} \to \mathbb{R}$ is called an *admissible potential* if

$$K \in W^{2,\infty}(\mathbb{R}), \qquad K(0) = 0, \qquad K(-x) = K(x). \tag{Adm}$$

Let the kernels K_{ij} be as in (**Adm**). Let $\overline{X}_1, \overline{X}_2 \in \mathcal{K}$ and $\overline{V}_1, \overline{V}_2 \in L^2(\Omega)$ be given. A Lagrangian solution to (Lagrangian system) with initial data $(\overline{X}_1, \overline{X}_2, \overline{V}_1, \overline{V}_2)$ is a pair $(X_1, X_2) \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty); \mathcal{K}) \times \mathrm{Lip}_{\mathrm{loc}}([0,\infty); \mathcal{K})$ satisfying $X_1(0) = \overline{X}_1$, $X_2(0) = \overline{X}_2$ and (Lagrangian system) for a. e. $t \in [0,\infty)$.

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Existence and uniqueness

Proposition

Let $T > 0$ and suppose that the kernels K_{ij} satisfy (**Adm**). *Then, for every* $(\overline{X}_1, \overline{X}_2, \overline{V}_1, \overline{V}_2) \in \mathcal{K}^2 \times L^2(0,1)^2$ *there exists a unique Lagrangian solution* (X_1, X_2) *to* (Lagrangian system) *in* $[0, T]$.

As in [BGSW], the proof is based on standard Maximal Monotone Operators Theory by Brezis. In particular, (Lagrangian system) can be understood as a Lipschitz perturbation to a Maximal Monotone Operator.

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Theorem

*Let $T > 0$ and suppose that the kernels K_{ij} satisfy (**Adm**). Let $\bar{\rho}_1, \bar{\rho}_2 \in \mathcal{P}_2(\mathbb{R})$ and $\bar{v}_1 \in L^2(d\bar{\rho}_1)$ and $\bar{v}_2 \in L^2(d\bar{\rho}_2)$. Then, there exists a unique quadruple*

$$(\rho_1, \rho_2, v_1, v_2) \in Lip([0, T]; \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \times L^2(d\rho_1(t)) \times L^2(d\rho_2(t)))$$

that is a distributional solutions to (Second order system) such that, for $i = 1, 2$,

$$\lim_{t \downarrow 0} \rho_i(t, \cdot) = \bar{\rho}_i \quad in \quad \mathcal{P}_2(\mathbb{R}), \qquad \lim_{t \downarrow 0} \rho_i(t, \cdot) v_i(t, \cdot) = \bar{\rho}_i \bar{v}_i \quad in \quad \mathcal{M}(\mathbb{R}).$$

Theorem

Let $T > 0$ and suppose that the kernels K_{ij} satisfy (**Adm**). *Let* $(\rho_1^\varepsilon, \rho_2^\varepsilon, v_1^\varepsilon, v_2^\varepsilon)$ *be solution to* (Second order system) *with* $\varepsilon = \sigma^{-2}$ *under the initial condition* $(\bar\rho_1^\varepsilon, \bar\rho_2^\varepsilon, \bar v_1^\varepsilon, \bar v_2^\varepsilon)$ *and let* (ρ_1, ρ_2) *be solution to* (First order system) *with initial data* $(\bar\rho_1, \bar\rho_2)$. *Furthermore, assume that*

(i) $\bar\rho_1^\varepsilon \to \bar\rho_1$ *and* $\bar\rho_2^\varepsilon \to \bar\rho_2$ *as* $\varepsilon \to 0$ *in* $\mathcal{P}_2(\mathbb{R})$;

(ii) $\bar v_1^\varepsilon = o(1/\varepsilon)$ *in* $L^2(d\bar\rho_1^\varepsilon)$ *and* $\bar v_2^\varepsilon = o(1/\varepsilon)$ *in* $L^2(d\bar\rho_2^\varepsilon)$ *as* $\varepsilon \to 0$.

Then,

$$\lim_{\varepsilon\to 0} \int_0^T \mathcal{W}_2^2((\rho_1^\varepsilon, \rho_2^\varepsilon), (\rho_1, \rho_2))\, dt = 0.$$

Observe that initial data are not well-prepared in the velocity variable.
Indeed $\bar v_i^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \bar v_i$, then assumption (ii) is satisfied in case $\bar v_i \in L^2(d\bar\rho_i)$, $i = 1, 2$, are given and independent of ε.

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Sketch of the proof

If (X_0^1, X_0^2) a solution to (First order system) via pseudo-inverses variables, then $Z_0 = (X_0^1, X_0^2)$ fulfills

$$Z_0(t,m) + \partial I_{\mathcal{K}^2}(Z_0(t,m)) \ni \overline{Z}_0(m) + \int_0^t L(Z_0(r,m))\,dr.$$

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If (X_1, X_2) is the solution to (Second order system), then $Z_\varepsilon = (X_1, X_2)$ satisfies

$$\varepsilon \dot{Z}_\varepsilon(t,m) + Z_\varepsilon(t,m) + \partial I_{\mathcal{K}^2}(Z_\varepsilon(t,m)) \ni \varepsilon \overline{U}_\varepsilon(m) + \overline{Z}_\varepsilon(m) + \int_0^t L(Z_\varepsilon(r,m))\,dr,$$

where $U_\varepsilon = (V_1, V_2)$ and $L((X_1, X_2)) = (F_1[X_1, X_2], F_2[X_1, X_2])$.

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where $U_\varepsilon = (V_1, V_2)$ and $L((X_1, X_2)) = (F_1[X_1, X_2], F_2[X_1, X_2])$. Considering the difference of the two inclusions above, after some manipulations we have

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega \left(Z_\varepsilon(t,m) - Z_0(t,m)\right)^2 dm + \frac{1-\varepsilon}{2}\int_\Omega \left(Z_\varepsilon(t,m) - Z_0(t,m)\right)^2 dm \\ &\leq \frac{1}{2}\int_\Omega \left[\varepsilon \overline{U}_\varepsilon(m) + \overline{Z}_\varepsilon(m) - \overline{Z}_0(m)\right]^2 dm + \frac{\varepsilon}{2}\int_\Omega \dot{Z}_0^2(t,m)\,dm \\ &\qquad + C\frac{1}{2}\int_0^t \int_\Omega \left(Z_\varepsilon(t,m) - Z_0(t,m)\right)^2 dm\,dr\,, \end{aligned}$$

where C is a fixed constant depending on the operator L. Finally, we integrate in time and apply Gronwall's Lemma.

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Newtonian potentials

Assume $K_{12}(x) = K_{22}(x) = N(x) = |x|$ and $K_{12} = K_{21} = H$. Consider also two uniformly convex external potentials A_1 and A_2 such that

$$A_i \geq \lambda_i |x|^2 \quad \text{and} \quad xA_i'(x) \geq \alpha_i |x|^2 \tag{Coer}$$

with positive α_i and λ_i, for i=1, 2. The system is

$$\begin{cases} \dfrac{\partial \rho_1}{\partial t} + \dfrac{\partial}{\partial x}(\rho_1 v_1) = 0, \\ \dfrac{\partial \rho_2}{\partial t} + \dfrac{\partial}{\partial x}(\rho_2 v_2) = 0, \\ \dfrac{\partial}{\partial t}(\rho_1 v_1) + \dfrac{\partial}{\partial x}(\rho_1 v_1^2) = -\sigma \rho_1 v_1 - \rho_1 [N' * \rho_1 + H' * \rho_2 + A_1], \\ \dfrac{\partial}{\partial t}(\rho_2 v_2) + \dfrac{\partial}{\partial x}(\rho_2 v_2^2) = -\sigma \rho_2 v_2 - \rho_2 [N' * \rho_2 + H' * \rho_1 + A_2]. \end{cases}$$

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Lagrangian description

The system in Lagrangian coordinates is

$$
\begin{cases}
\partial_t X_1 = V_1, \\
\partial_t X_2 = V_2, \\
\partial_t V_1 = -\displaystyle\int_\Omega \operatorname{sign}(X_1(m) - X_1(m'))\, dm' \\
\qquad\qquad -\displaystyle\int_\Omega H'(X_1(m) - X_2(m'))\, dm' - \sigma V_1 - A_1'(X_1), \\
\partial_t V_2 = -\displaystyle\int_\Omega \operatorname{sign}(X_2(m) - X_2(m'))\, dm' \\
\qquad\qquad -\displaystyle\int_\Omega H'(X_2(m) - X_1(m'))\, dm' - \sigma V_2 - A_2'(X_2)
\end{cases}
\qquad \text{(Newtonian system)}
$$

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\partial_t V_2 = -\displaystyle\int_\Omega \mathrm{sign}(X_2(m) - X_2(m'))\, dm' \\
\qquad\qquad - \displaystyle\int_\Omega H'(X_2(m) - X_1(m'))\, dm' - \sigma V_2 - A_2'(X_2)
\end{cases}
\qquad \text{(Newtonian system)}
$$

Stationary solutions are $(\rho_1^s, \rho_2^s) = (\delta_0, \delta_0)$, where δ is the Dirac measure, which corresponds to $(X_1^s, X_2^s) = (0, 0)$ in terms of the Lagrangian description.

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Existence result

Proposition

*Assume H as in (**Adm**). Assume $A_1,\ A_2 \in C^2(\mathbb{R})$. Then, for every initial data $(\overline{X}_1, \overline{X}_2, \overline{V}_1, \overline{V}_2)$ there exists a generalised Lagrangian solution to (Newtonian system) with initial data $(\overline{X}_1, \overline{X}_2, \overline{V}_1, \overline{V}_2)$.*

Notice that the external potentials A_1 and A_2 do not affect the study of existence of solutions, but are only required in the study of asymptotic behaviour.

Large-time collapse

Theorem

*Let H be an attractive potential as in (**Adm**). Assume $A_1,\ A_2 \in C^2(\mathbb{R})$ as in (**Coer**). Let $(X_1, X_2) \in Lip_{loc}([0,\infty), \mathcal{K})^2$ be a generalised Lagrangian solution to (Newtonian system). Assume that the initial positions $(\overline{X}_1, \overline{X}_2) \in \mathcal{K}^2$ and velocities $(\overline{V}_1, \overline{V}_2) \in (L^2(\Omega))^2$ satisfy*

$$\|\overline{X}_1\|_{L^2} + \|\overline{X}_2\|_{L^2} + \|\overline{V}_1\|_{L^2} + \|\overline{V}_2\|_{L^2} < \infty,$$

then

$$\lim_{t\to\infty} \left(\|X_1\|_{L^2} + \|X_2\|_{L^2} + \|V_1\|_{L^2} + \|V_2\|_{L^2} \right) = 0,$$

that is

$$\lim_{t\to\infty} \mathcal{W}_2^2((\rho_1, \rho_2), (\rho_1^s, \rho_2^s)) = 0,$$

with $\rho_i(t,\cdot) = \Psi^{-1}(X_i(t,\cdot))$.

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Sketch of the proof

- We associate to (Newtonian system) the functional

$$\mathcal{F}(X_1, X_2) = \frac{1}{2}\int_\Omega \int_\Omega |X_1(m) - X_1(m')|\, dm'\, dm + \frac{1}{2}\int_\Omega \int_\Omega |X_2(m) - X_2(m')|\, dm'\, dm$$
$$+ \int_\Omega \int_\Omega H(X_2(m) - X_1(m'))\, dm'\, dm + \int_\Omega A_1(X_1(m))\, dm + \int_\Omega A_2(X_2(m))\, dm.$$

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- We estimate the total energy of the system, proving that it is non-increasing, i.e.,

$$\sup_{t\geq 0}\left(\mathcal{F}(X_1, X_2) + \frac{1}{2}\|V_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|V_2\|^2_{L^2(\Omega)}\right) \leq \mathcal{F}(\overline{X}_1, \overline{X}_2) + \frac{1}{2}\|\overline{V}_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|\overline{V}_2\|^2_{L^2(\Omega)}.$$

- By using the coercive assumption on the external potentials, we estimate the (time derivative of the) L^2-distance between (X_1, X_2) and (X_1^s, X_2^s).
- We end up with

$$\int_0^\infty \int_\Omega (|X_1| + |X_2| + |V_1| + |V_2|)\, dm\, dt < +\infty.$$

- By using a compactness argument and the monotonicity and continuity of $\mathcal{F}$, we have the assertion.

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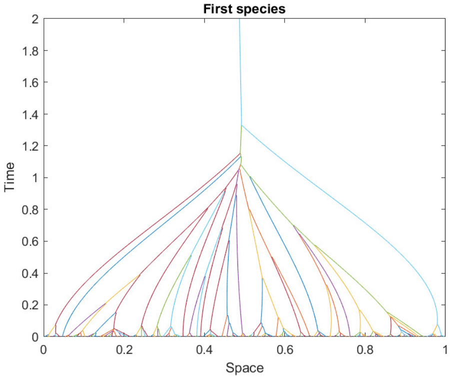
Numerical simulations



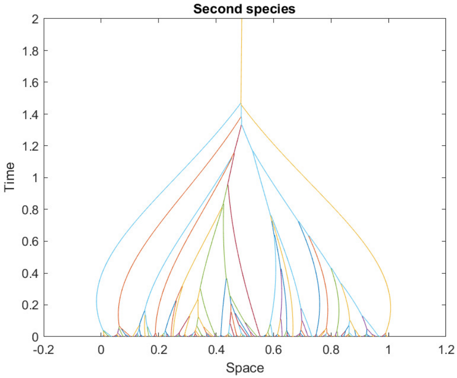


Figure: In this first example, we fix $N = 160$ and $M = 150$. All the potentials are attractive. In particular we set $K_{11}(x) = -e^{-|x|^3}$, $K_{22}(x) = -e^{-|x|^4}$, $K_{12}(x) = K_{21}(x) = -e^{-|x|^2}$.

Numerical simulations

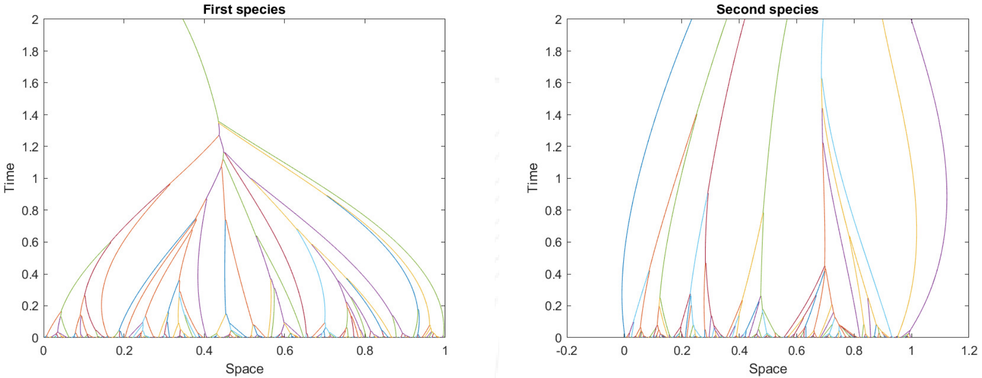


Figure: Evolution under the action of attractive self potentials given by $K_{11}(x) = -3e^{-|x|^2}$, and $K_{22}(x) = -2e^{-2|x|^3}$, and repulsive cross-potentials $K_{12}(x) = -|x|^2$, $K_{21}(x) = e^{-|x|^2}$. Here, $N = 180$, and $M = 200$.

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Numerical simulations

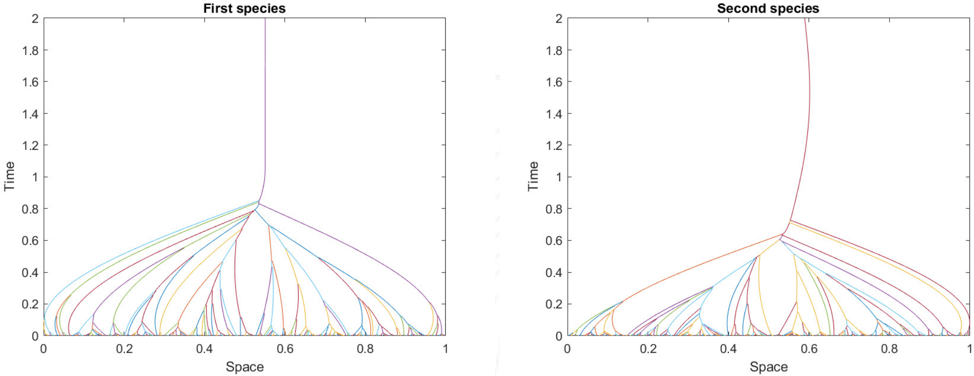

Figure: Evolution under the action of attractive Newtonian self-potentials and attractive Gaussian cross-potentials given by $H = -e^{-|x|^2}$. The external potentials are $A_1(x) = |x - \frac{1}{2}|^2$ and $A_2(x) = 2|x - \frac{1}{2}|^2$. Here, $N = 200$ and $M = 210$.

Numerical simulations

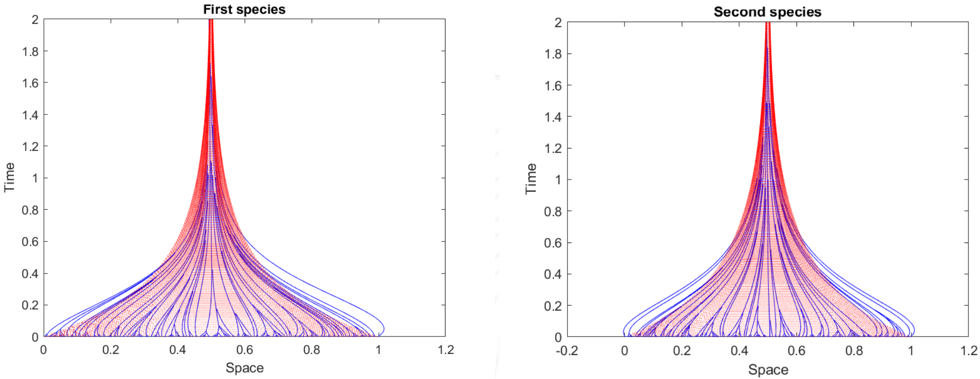

Figure: Solutions to the second-order system (in blue) and solutions to first-order system (in red) under the action of the following potentials are $K_{11}(x) = -e^{-|x|^3}$, $K_{22}(x) = -e^{-|x|^4}$, $K_{12}(x) = K_{21}(x) = -e^{-|x|^2}$. Here, $N = 160$, $M = 150$ and $\sigma = 10$.

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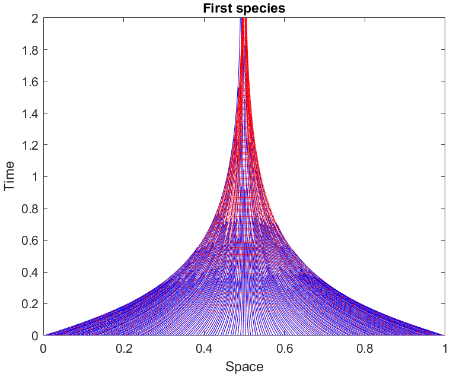
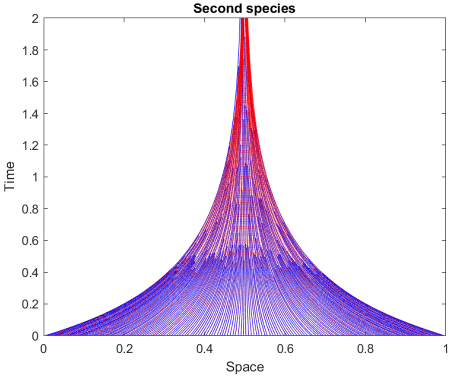
Numerical simulations





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Thank you for your kind attention!