The Scharfetter-Gummel scheme for the aggregation-diffusion equation and vanishing diffusion limit

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joint work with **André Schlichting** and **Oliver Tse**

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Gradient flows face-to-face 3, Lyon, 14 September 2023

Aggregation-diffusion equation

$$\partial_t \rho = \varepsilon \Delta \rho + \mathrm{div}\big(\rho \nabla (\Lambda * \rho)\big) \qquad \text{in } (0, T) \times \Omega \qquad \text{(ADE)}$$

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- Population dynamics $\Lambda(x) = w(|x|)$. Attractive forces $w'(r) > 0$, repulsive forces $w'(r) < 0$.

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- The Morse potential

$$\Lambda(x) = C_r e^{-|x|/\ell_r} - C_a e^{-|x|/\ell_a}$$

with $C_r \geq C_a > 0$ and $\ell_a > \ell_r$.

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- a diffusion coefficient $\varepsilon > 0$;
- an interaction potential $\Lambda \in \mathrm{Lip}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d \backslash \{0\})$ (pointy).
- no-flux boundary condition

$$\varepsilon \partial_\nu \rho + \rho \partial_\nu (\Lambda * \rho) = 0 \qquad \text{on } \partial\Omega,$$

ν denotes the outer normal vector on $\partial\Omega$.

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Gradient flow in $(\mathcal{P}(\Omega), W_2)$ with respect to the driving energy:

$$\mathcal{E}_\varepsilon(\rho) = \varepsilon \int_\Omega \log \frac{\mathrm{d}\rho}{\mathrm{d}\mathcal{L}^d}(x) \rho(\mathrm{d}x) + \frac{1}{2} \int_\Omega \int_\Omega \Lambda(x - y) \rho(\mathrm{d}x) \rho(\mathrm{d}y)$$

Zoo of numerical schemes

- Based on the JKO scheme

$$\rho_k^\tau = \arg\min_{\rho\in\mathcal{P}(\Omega)} \left\{ \mathcal{E}_\varepsilon(\rho) + \frac{1}{2\tau} W_2^2\left(\rho, \rho_{k-1}^\tau\right) \right\}$$

purely continuous [Benamou-Brenier '00]; semi-discrete [Benamou-Carlier-Merigot-Oudet '16], [Kitagawa-Mérigot-Thibert '19]; purely discrete [Cuturi '13]; other [Carrillo-Ranetbauer-'16]

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- Discretize the equation directly

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$$\rho^N = \sum_{i=1}^N m_i \delta_{x_i(t)}$$

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Structure-preserving properties

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\mathrm{d}x) &= \int_\Omega \mathrm{div}(\varepsilon\nabla\rho + \rho(\nabla\Lambda * \rho))\,\mathrm{d}x \\ &= \int_{\partial\Omega}(\varepsilon\nabla\rho + \rho(\nabla\Lambda * \rho))\cdot\nu\,\mathrm{d}x = 0.\end{aligned}$$

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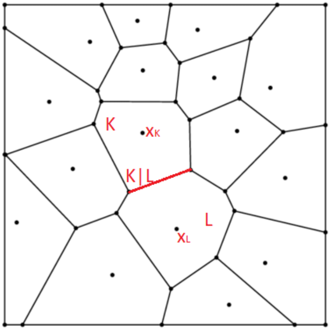
3. Dissipation of the driving energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\rho_t) = \int_\Omega \mathcal{E}'(x)\,\partial_t \rho(x)\,\mathrm{d}x = -\int_\Omega \left|\nabla \mathcal{E}'(x)\right|^2 \rho_t(\mathrm{d}x) \leq 0.$$

The idea of finite-volume schemes



$$\partial_t \rho + \mathrm{div}\, j = 0 \qquad \text{on } (0, T) \times \Omega$$

Tessellation $(\mathcal{T}^h, \Sigma^h)$

$$h = \max_{K \in \mathcal{T}^h} \mathrm{diam}(K)$$

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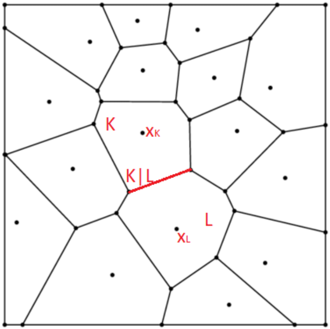

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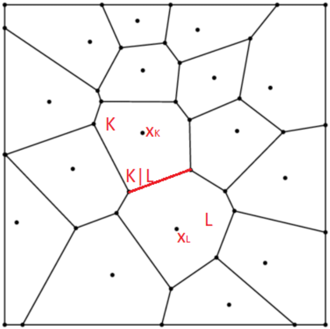

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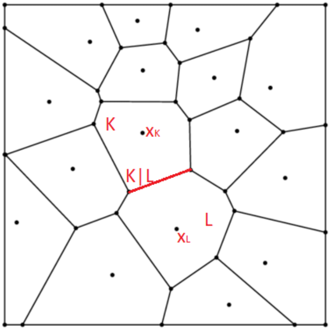

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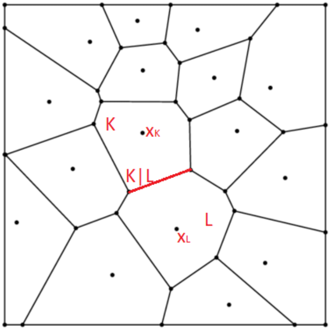

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The Scharfetter-Gummel flux

$$\begin{aligned}\partial_t \rho + \operatorname{div} j &= 0 \\ j &= \varepsilon \nabla \rho + \rho \nabla (\Lambda * \rho)\end{aligned}$$

Finite-volume approximation:

$$\partial_t \rho_K^h + \sum_{L \sim K} \mathcal{J}_{K|L}^{\rho} = 0, \qquad K \in \mathcal{T}^h.$$

Scharfetter-Gummel '69, Farrell-Gartland Jr. '91, Eymard-Fuhrmann-Gärtner '06, Bessemoulin-Chatard '12, Schlichting-Seis '22

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The idea of the Scharfetter-Gummel is **solving a cell problem.**
$u \in C^2([x_K, x_L])$

$$\begin{cases} -\partial_x \big(\varepsilon \partial_x u + u\, q_{K|L}^h \big) = 0 \quad \text{on } [x_K, x_L] \\ u(x_K) = \rho_K^h / |K|, \;\; u(x_L) = \rho_L^h / |L| \end{cases} \tag{Cell-Pr}$$

Define $\mathcal{J}_{K|L}^{\rho} := \varepsilon \partial_x u + u\, q_{K|L}^h$.

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Scharfetter-Gummel flux

The solution of (Cell-Pr) is

$$\mathcal{J}^{\rho}_{K|L} = \varepsilon \tau_{K|L} \left(\beta(q_{K|L}/\varepsilon) u^h_K - \beta(-q_{K|L}/\varepsilon) u^h_L \right),$$

where

- u^h is the density $u^h_K := \dfrac{\rho^K}{|K|}$ for all $K \in \mathcal{T}^h$;
- the transmission coefficient $\tau_{K|L} := \dfrac{|(K|L)|}{|x_L - x_K|}$ for all $(K, L) \in \Sigma^h$;
- β is the Bernoulli function $\beta(s) = \dfrac{s}{e^s - 1}$
- $q_{K|L}$ is a discrete approximation for $\nabla(\Lambda * \rho)$:

$$q_{K|L} = \sum_{M \in \mathcal{T}^h} \big(\Lambda(x_M - x_L) - \Lambda(x_M - x_K) \big) \rho^h_M$$

Properties of the Scharfetter-Gummel flux

1. If $\Lambda \equiv 0$, then
$$\mathcal{J}^{\rho}_{K|L} = \varepsilon \tau_{K|L}(u^h_K - u^h_L).$$
2. In the vanishing diffusion limit $\varepsilon \to 0$, the flux converges to the upwind scheme:
$$\mathcal{J}^{\rho}_{K|L} = \tau_{K|L}\big(q^{+}_{K|L} u^h_K - q^{-}_{K|L} u^h_L\big)$$

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3. [Schlichting-Seis '22] The Scharfetter-Gummel scheme is:
   + Positivety preserving;
   + Mass conservative;
   + Energy-dissipative;
   ? Does it have a gradient structure?

Convergence and asymptotic limits

$$\partial_t \rho_K^h + \sum_{L\sim K} \mathcal{J}_{K|L}^\rho = 0$$

$$\mathcal{J}_{K|L}^\rho = \varepsilon \tau_{K|L} \left(\beta(q_{K|L}/\varepsilon) u_K^h - \beta(-q_{K|L}/\varepsilon) u_L^h\right)$$

$$\partial_t \rho = \mathrm{div}\big(\varepsilon \nabla \rho + \rho \nabla(\Lambda * \rho)\big)$$

Scharfetter–Gummel scheme $\xrightarrow{h\to 0}$ **Aggregation-diffusion equation**

$\downarrow \epsilon\to 0$ $\qquad\qquad$ $\downarrow \epsilon\to 0$

Upwind scheme $\xrightarrow{h\to 0}$ **Aggregation equation**

$$\partial_t \rho_K^h + \sum_{L\sim K} \mathcal{J}_{K|L}^\rho = 0$$

$$\mathcal{J}_{K|L}^\rho = \tau_{K|L} \left(q_{K|L}^+ u_K^h - q_{K|L}^- u_L^h\right)$$

$$\partial_t \rho = \mathrm{div}\big(\rho \nabla(\Lambda * \rho)\big)$$

A definition of GF solutions via energy-dissipation balance

$$\mathcal{I}(\rho, j) := \int_0^T \left\{ \mathcal{R}(\rho_t, j_t) + \mathcal{R}^*\big(\rho_t, -\nabla \mathcal{E}'(\rho_t)\big) \right\} \mathrm{d}t + \mathcal{E}(\rho_T) - \mathcal{E}(\rho_0).$$

Definition

A measure-flux pair (ρ, j) is called a gradient flow solution if

$$\partial_t \rho + \operatorname{div} j = 0 \qquad \text{and} \qquad \mathcal{I}(\rho, j) = 0.$$

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W_2-gradient structure:

$$\mathcal{R}(\rho, j) = \frac{1}{2} \int_\Omega \left| \frac{\mathrm{d}j}{\mathrm{d}\rho} \right|^2 \mathrm{d}\rho$$

$$\mathcal{D}(\rho) := \mathcal{R}^*\big(\rho, -\nabla \mathcal{E}'(\rho)\big) = \frac{1}{2} \int_\Omega \left| \nabla \mathcal{E}'(\rho) \right|^2 \mathrm{d}\rho$$

Gradient structure for the S-G scheme

Remember the finite-volume approximation (the discrete continuity equation):

$$\partial_t \rho_K^h + \sum_{L\sim K} \mathcal{J}_{K|L}^{\rho} = 0, \qquad K \in \mathcal{T}^h.$$

Maas '11, Mielke '11, Chow-Huang-Li-Zhou '12, Peletier-Rossi-Savaré-Tse '22, Peletier-Schlichting '23

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We need to choose a driving energy $\mathcal{E}_h$ and a dissipation potential $\mathcal{R}_h$ such that

$$\mathcal{J}_{K|L}^\rho = D_2 \mathcal{R}_h^*(\rho^h, -\overline{\nabla}\mathcal{E}_h'(\rho^h)). \tag{KRh}$$

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$$\mathcal{E}_h(\rho^h) = \varepsilon \sum_{K \in \mathcal{T}^h} \log(u_K^h) \rho_K^h + \frac{1}{2} \sum_{(K,L) \in \mathcal{T}^h \times \mathcal{T}^h} \Lambda_{KL}^h \rho_K^h \rho_L^h, \quad u_K^h := \frac{\rho_K^h}{|K|}$$

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$$\mathcal{R}^*_h(\rho^h, \xi^h) = 2 \sum_{(K,L)\in\Sigma^h} \tau_{K|L}\, \alpha^*_\varepsilon\left(u^h_K, u^h_L, \frac{\xi^h_{KL}}{2}\right)$$

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The convergence result with diffusion

Let $\{(\mathcal{T}^h, \Sigma^h)\}_{h>0}$ satisfy the inner ball and orthogonality assumptions. Assume that $(\rho^h, j^h)_{h>0}$ are discrete gradient flow solutions of the Scharfetter-Gummel scheme with $\sup_{h>0} \mathcal{E}_h(\rho_0^h) < \infty$.

Then there exists a subsequence of $\{(\hat{\rho}^h, \hat{\jmath}^h)\}_{h>0}$ and a pair (ρ, j) such that

1. (ρ, j) satisfies (CE)
   - $\mathrm{d}\hat{\rho}_t^h / \mathrm{d}\mathcal{L}^d \to u_t$ in $L^1(\Omega)$ for every $t \in [0, T]$;
   - $\int_\cdot \hat{\jmath}_t^h \,\mathrm{d}t \rightharpoonup^* \int_\cdot j_t \,\mathrm{d}t$ weakly-$*$.
2. The following estimate holds:

$$\liminf_{h\to 0} \mathcal{I}_h(\rho^h, j^h) \geq \mathcal{I}(\rho, j).$$

3. The limit pair (ρ, j) is a minimizer of the Otto energy-dissipation functional $\mathcal{I}(\rho, j)$ and, consequently, is the gradient flow solution of

$$\partial_t \rho = \operatorname{div}\left(\varepsilon \nabla \rho + \rho \nabla (\Lambda * \rho)\right).$$

Γ-convergence of the Fisher information

Recall the energy-dissipation functional:

$$\mathcal{I}(\rho, j) := \int_0^T \left\{\mathcal{R}(\rho_t, j_t) + \mathcal{R}^*\big(\rho_t, -\nabla\mathcal{E}'(\rho_t)\big)\right\} \mathrm{d}t + \mathcal{E}(\rho_T) - \mathcal{E}(\rho_0).$$

The Fisher information is defined as

$$\mathcal{D}(\rho) := \mathcal{R}^*(\rho, -\nabla\mathcal{E}'(\rho))$$

$$\mathcal{D}_\epsilon(\rho) = 2\epsilon^2 \int \left|\nabla\sqrt{u}\right|^2 \mathrm{d}x + \epsilon \int \nabla u \cdot \nabla(\Lambda * \rho)\, \mathrm{d}x + \frac{1}{2} \int \left|\nabla(\Lambda * \rho)\right|^2 \mathrm{d}\rho$$

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The discrete Fisher information:

$$\mathcal{D}_{\epsilon,h}(\rho^h) = \sum_{(K,L)\in\Sigma^h} \left[\beta_\epsilon(u_K^h, u_L^h) + \frac{\epsilon}{2}(u_L^h - u_K^h)\, q_{K|L}^h \tau_{K|L}^h + |q_{K|L}^h|^2\, h_\epsilon(u_K^h, u_L^h, q_{K|L}^h)\right] \tau_{K|L}^h$$

$$\text{Goal: } \Gamma\text{-}\lim_{h\to 0} \mathcal{D}_{\epsilon,h} = \mathcal{D}_\epsilon$$

Γ-convergence of the Fisher information

$$
\begin{aligned}
\mathcal{F}_h(\varphi^h, Q_x) &= \frac{1}{2} \sum_{(K,L)\in\Sigma^h|_{Q_x}} (\varphi^h_L - \varphi^h_K)^2 \tau^h_{K|L} \\
&= \frac{1}{2} \sum_{(K,L)\in\Sigma^h|_{Q_x}} (\xi \cdot x_L - \xi \cdot x_K)^2 \tau^h_{K|L} \\
&= \frac{1}{2} \Big\langle \xi, \sum_{(K,L)\in\Sigma^h|_{Q_x}} \tau^h_{K|L} (x_L - x_K) \otimes (x_L - x_K) \xi \Big\rangle \\
&= \Big\langle \xi, \frac{1}{|Q_x|} \int_{Q_x} \mathbb{T}^h(x)\, \mathrm{d}x\, \xi \Big\rangle
\end{aligned}
$$

$$
\mathbb{T}^h(x) := \frac{1}{2} \sum_{K\in\mathcal{T}^h} \mathbb{I}_K(x) \sum_{L\sim K} \tau_{K|L} (x_L - x_K) \otimes (x_L - x_K)
$$

What about the tensor?

$$\mathbb{T}^h(x) := \frac{1}{2} \sum_{K \in \mathcal{T}^h} \mathbb{I}_K(x) \sum_{L \sim K} \tau_{K|L}(x_L - x_K) \otimes (x_L - x_K)$$

For an admissible tessellation $\mathbb{T}^h \rightharpoonup^* \mathbb{T}$ in weakly-$*$ in $\sigma(L^\infty, L^1)$.

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Proposition HST '23

Let a family of tessellations $\left\{(\mathcal{T}^h, \Sigma^h)\right\}_{h>0}$ satisfy the orthogonality assumption $x_L - x_K \perp (K|L)$, then the family of tensors $\{\mathbb{T}^h\}_{h>0}$ is such that $\mathbb{T}^h \rightharpoonup^* \mathrm{Id}$ weakly-$*$ in $\sigma(L^\infty, L^1)$.

Convergence and asymptotic limits

$$\partial_t \rho_K^h + \sum_{L\sim K} \mathcal{J}_{K|L}^{\rho} = 0$$

$$\mathcal{J}_{K|L}^{\rho} = \varepsilon \tau_{K|L} \left(\beta(q_{K|L}/\varepsilon) u_K^h - \beta(-q_{K|L}/\varepsilon) u_L^h\right)$$

$$\partial_t \rho = \mathrm{div}\big(\varepsilon \nabla \rho + \rho \nabla (\Lambda * \rho)\big)$$

Scharfetter–Gummel scheme $\xrightarrow{h\to 0}$ **Aggregation-diffusion equation**

$\big\downarrow \epsilon\to 0$ $\qquad\qquad$ $\big\downarrow \epsilon\to 0$

Upwind scheme $\xrightarrow{h\to 0}$ **Aggregation equation**

$$\partial_t \rho_K^h + \sum_{L\sim K} \mathcal{J}_{K|L}^{\rho} = 0$$

$$\mathcal{J}_{K|L}^{\rho} = \tau_{K|L} \left(q_{K|L}^{+} u_K^h - q_{K|L}^{-} u_L^h\right)$$

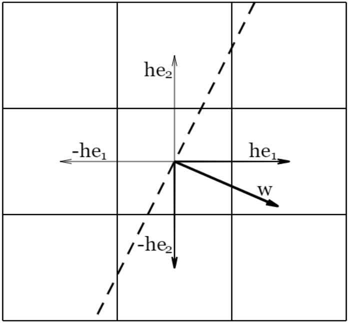
$$\partial_t \rho = \mathrm{div}\big(\rho \nabla (\Lambda * \rho)\big)$$

[Lagoutière-Santambrogio-Tran Tien '23]

Tensor for the upwind scheme

$$\mathbb{T}_\varphi^h(x) := \frac{1}{2} \sum_{K \in \mathcal{T}^h} \mathbb{I}_K(x) \sum_{L \sim K} \tau_{K|L}(x_L - x_K) \otimes (x_L - x_K) \imath_K^\varphi$$

$$\imath_K^\varphi = \mathbb{I}_{\{M \in \mathcal{T}^h : \nabla\varphi(x_K)\cdot(x_M - x_K) > 0\}} + \frac{1}{2}\mathbb{I}_{\{M \in \mathcal{T}^h : \nabla\varphi(x_K)\cdot(x_M - x_K) = 0\}}$$



Upwind-to-aggregation limit

Let $\{(\mathcal{T}^h, \Sigma^h)\}_{h>0}$ be **Cartesian grids and** $\Lambda \in C^1$. Assume that $(\rho^h, j^h)_{h>0}$ are discrete gradient flow solutions of **the upwind scheme**. Then there exists a subsequence of $\{(\hat{\rho}^h, \hat{\jmath}^h)\}_{h>0}$ and a pair (ρ, j) such that

1. (ρ, j) satisfies (CE)
   - $\hat{\rho}^h_t \rightharpoonup^* \rho_t$ **weakly-$*$ in** $\mathcal{P}(\Omega)$ for every $t \in [0, T]$;
   - $\int_\cdot \hat{\jmath}^h_t \, \mathrm{d}t \rightharpoonup^* \int_\cdot j_t \, \mathrm{d}t$ weakly-$*$.
2. The following estimate holds:

$$\liminf_{h \to 0} \mathcal{I}_h(\rho^h, j^h) \geq \mathcal{I}(\rho, j).$$

3. The limit pair (ρ, j) is a minimizer of the Otto energy-dissipation functional $\mathcal{I}(\rho, j)$ and, consequently, is the gradient flow solution of

$$\partial_t \rho = \operatorname{div}\left(\rho \nabla (\Lambda * \rho)\right).$$

Outlook and open problems

1. More singular potentials.
2. Generalization for non-linear mobility or non-linear diffusion.
3. Improve the discrete-to-continuum limit from the upwind scheme to the aggregation equation.
4. Rates of convergence.

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Thank you for your attention!

Hraivoronska, A., Schlichting, A., Tse, O. (2023). Variational convergence of the Scharfetter-Gummel scheme to the aggregation-diffusion equation and vanishing diffusion limit. arXiv preprint arXiv:2306.02226.

'Cosh' dissipation potential:

$$\mathcal{R}^*_h(\rho^h, \overline{\nabla}\varphi^h) = \sum_{(K,L)\in\Sigma^h} \Psi^*\big((\overline{\nabla}\varphi^h)(K,L)\big)\sqrt{u^h_K u^h_L}\,\vartheta^h_{KL}$$

with $\Psi^*(s) = 4\big(\cosh(s/2) - 1\big)$.

Discrete-to-continuum EDP convergence:

$$\begin{aligned} &\partial_t \rho^h_K = \sum_{L\sim K} \big(\rho^h_K \kappa^h_{KL} - \rho^h_L \kappa^h_{LK}\big) \\ &\partial_t \rho^h + \overline{\operatorname{div}}\, j^h = 0 \\ &\mathcal{I}_h(\rho^h, j^h) = 0 \end{aligned} \quad \xrightarrow{h\to 0} \quad \begin{aligned} &\partial_t \rho = \Delta\rho + \nabla(\rho\nabla V) \\ &\partial_t \rho + \operatorname{div}\, j = 0 \\ &\mathcal{I}(\rho, j) = 0 \end{aligned}$$

Gradient structure for the S-G scheme

Lemma [H.-Schlichting-Tse '23]

The Scharfetter-Gummel flux $\mathcal{J}^\rho$ satisfies

$$\mathcal{J}^\rho_{K|L} = D_2\mathcal{R}^*_h(\rho^h, -\overline{\nabla}\mathcal{E}'_h(\rho^h))$$

with the 'cosh' dual dissipation potential $\mathcal{R}^*_h$ with the weights

$$\vartheta_{KL} = \frac{\tau_{K|L}}{\exp(-\Lambda^h_K/\varepsilon)} \frac{2\, q_{K|L}/\varepsilon}{\exp(\Lambda^h_L/\varepsilon) - \exp(\Lambda^h_K/\varepsilon)},$$

where $q_{K|L} = \sum_{M\in\mathcal{T}^h} (\Lambda^h_{ML} - \Lambda^h_{MK})\rho^h_M$ and $\Lambda^h_K = \sum_{M\in\mathcal{T}^h} \Lambda^h_{MK}\rho^h_M$.

The de-tilting trick

ED functional

$$\mathcal{I}_h(\rho^h, j^h) = \int_0^T \left\{\mathcal{R}_h(\rho_t^h, j_t^h) + \mathcal{R}_h^*\big(\rho_t^h, -\overline{\nabla}\mathcal{E}_h'(\rho_t^h)\big)\right\} \mathrm{d}t + \mathcal{E}_h(\rho_T) - \mathcal{E}_h(\rho_0)$$

Along the solution:

$$\xi_{KL}^h = -\overline{\nabla}\mathcal{E}_h'(\rho^h)(K, L) = -\varepsilon \log \frac{u_L^h}{u_K^h} + q_{K|L}^h$$

$$q_{K|L}^h = \varepsilon \log \frac{u_K^h}{u_L^h} - \xi_{KL}^h = \varepsilon \left(\log\left(u_K^h e^{-\xi_{KL}^h/2\varepsilon}\right) - \log\left(u_L^h e^{\xi_{KL}^h/2\varepsilon}\right)\right)$$

New way to write the flux:

$$\mathcal{J}_{K|L}^{h,\rho} = \varepsilon \sinh\left(\frac{\xi_{KL}^h}{2\varepsilon}\right) \Lambda_H\left(u_K^h e^{-\frac{\xi_{KL}^h}{2\varepsilon}}, u_L^h e^{\frac{\xi_{KL}^h}{2\varepsilon}}\right) |K| \overset{!}{=} D_2\mathcal{R}_{\varepsilon,h}^*(\rho^h, \xi^h)(K, L)$$

Peletier-Schlichting '23

De-tilted structure

De-tilted dual dissipation potential

$$\mathcal{R}_h^*(\rho^h, \xi^h) = 2 \sum_{(K,L)\in\Sigma^h} \tau_{K|L}\, \alpha_\varepsilon^*\left(u_K^h, u_L^h, \frac{\xi_{KL}^h}{2}\right),$$

where

$$\alpha_\varepsilon^*(a, b, \xi) = \varepsilon \int_0^\xi \sinh\left(\frac{x}{\varepsilon}\right) \Lambda_H\left(ae^{-x/\varepsilon}, be^{x/\varepsilon}\right) \mathrm{d}x$$

with the *harmonic-logarithmic mean*

$$\Lambda_H(s,t) = \frac{1}{\Lambda\left(1/s, 1/t\right)} \qquad \text{with} \qquad \Lambda(s,t) = \frac{s-t}{\log s - \log t}. \quad (1)$$

Discrete-to-continuous convergence

1. Compactness. There exists a subsequence such that $(\rho^h, j^h) \to (\rho, j)$ and $\partial_t \rho + \nabla \cdot j = 0$.
2. The limit ED functional:

$$\left.\begin{array}{l} \liminf \mathcal{R}_h(\rho^h, j^h) \geq \mathcal{R}(\rho, j) \\ \liminf \mathcal{D}_h(\rho^h) \geq \mathcal{D}(\rho) \\ \liminf \mathcal{E}_h(\rho^h) \geq \mathcal{E}(\rho) \end{array}\right\} \Longrightarrow \liminf_{h\to 0} \mathcal{I}_h(\rho^h, j^h) \geq \mathcal{I}(\rho, j)$$

3. Prove that $\mathcal{I}$ is proper ED functional (chain rule):

$$0 = \liminf_{h\to 0} \mathcal{I}_h(\rho^h, j^h) \geq \mathcal{I}(\rho, j) \overset{?}{\geq} 0.$$

4. Recover the limit equation.