Graph approximation of nonlocal interaction equations

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Motivation: why evolution PDEs on graphs?

- Social networks: polarisation and formation of echo chambers

  F. Baumann, P. Lorenz-Spreen, I. M. Sokolov, M. Starnini., *Phys. Rev. Lett*, 2020

  A. Benatti, H. F. de Arruda, F. N. Silva, C. H. Comin, L. da Fontoura Costa, *Journal of Statistical Mechanics: Theory and Experiment*, 2020

- Transportation Newtorks: gravity interactions

  K. Tamura, H. Takayasu, M. Takayasu, *Scientific Reports*, 2018

  H. Koike, H. Takayasu, and M. Takayasu, *Journal of Statistical Physics*, 2022

- Data Science/Machine Learning: data representation as point clouds for clustering and classification

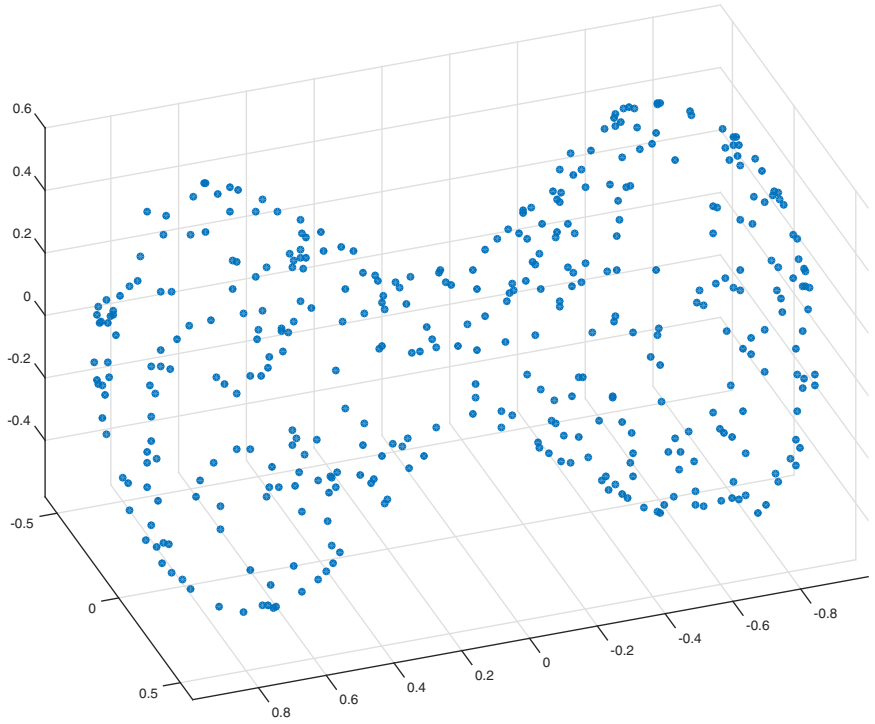
  M. Belkin, P. Niyogi, *Neural Comput.*, 2002

  R. R. Coifman, S. Lafon, *Appl. Comput. Harmon. Anal.*, 2006

  K. Craig, N. Garcia-Trillos, N. Garcia, D. Slepcev, *Springer International Publishing*, 2022.

Notation

- $X = \{x_1, x_2, ..., x_n\}$ random sample i.i.d. according to $\mu \in \mathcal{M}^+(\mathbb{R}^d)$
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$



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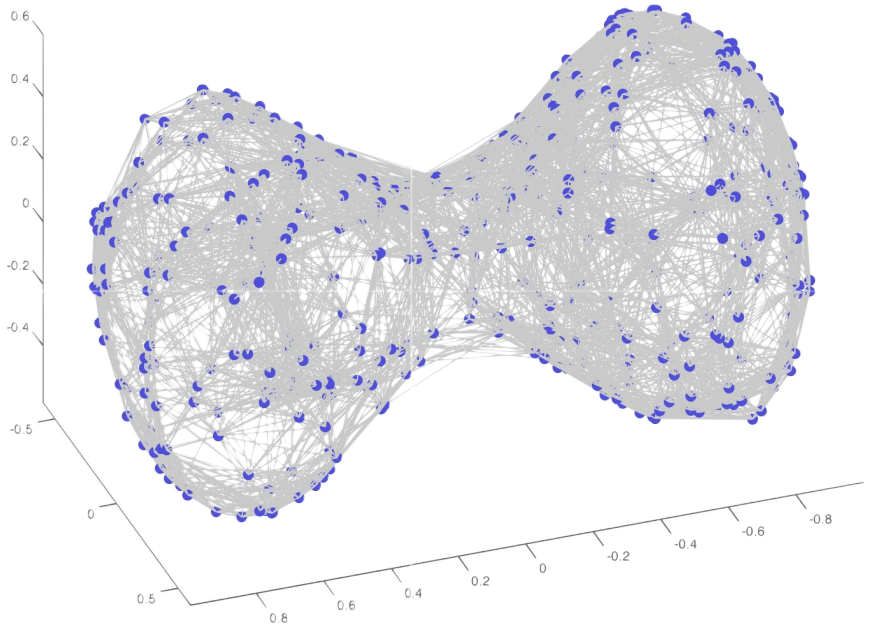
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$
- a symmetric weight function $\eta : D \to [0, \infty)$ with
  $D := (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{x = y\}$
  $\Rightarrow (\mu^n, \eta)$ defines an undirected discrete weighted graph



Dynamics driven by interaction energies on graphs

$$\mathcal{E}_X(\rho) = \frac{1}{2}\sum_{x\in X}\sum_{y\in X} K_{x,y}\rho_x\rho_y \tag{1}$$

On $\mathbb{R}^d$

$$\dot{x}_i = -\sum_{j=1}^{n} \rho_j \nabla_x K(x_i, x_j) \tag{2}$$

On finite graphs

$$\frac{d\rho_x}{dt} = -\sum_{y\in X} j_{x,y}\eta(x,y) \tag{3}$$

$$j_{x,y} = I\big(\rho_x, \rho_y\big) v_{x,y} \tag{4}$$

A.E., F. S. Patacchini, A. Schlichting - EJAM '23

Goals

- Define gradient flow of interaction energy on graph (μ, η)
- Dynamics stable under graph limit $n \to \infty$ (discrete-to-continuum)
- Dynamics stable for local limit: $\mu = \mathrm{Leb}(\mathbb{R}^d)$, $\eta^\varepsilon(x,y) = \varepsilon^{-d}\eta\left(\frac{x-y}{\varepsilon}\right)$
  $\Rightarrow$ limit $\varepsilon \to 0$ should give $\partial_t \rho = \nabla \cdot (\rho \nabla K * \rho)$

Dynamics driven by interaction energies on graphs

General framework

- $\mathbb{R}^d$ set of possible vertices, $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ set of possible edges
- $\eta : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to [0, \infty)$ symmetric weight function
- $G := \{\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} | \eta(x, y) > 0\}$ set of edges
- $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ set of vertices
- $\rho \in \mathcal{P}(\mathbb{R}^d)$ distribution of mass

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Evolution of interest

Gradient descent of the energy $\mathcal{E} : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\mathcal{E}(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y) \, d\rho(x) \, d\rho(y),$$

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Continuum (local) setting: NLIE

$\partial_t \rho = \nabla \cdot (\rho \nabla K * \rho)$ is a Wasserstein gradient flow for $\mathcal{E}$[a]

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What is the analogue of the NLIE on a graph?

Related Literature (not exhaustive!)

- [Maas '11] / [Mielke '11] / [Chow, Huang, Li, Zhou '12]
  Diffusion on graphs as gradient flows of the entropy
  $\Rightarrow$ Wassertein metric on a finite graph
- [Erbar '14] Jump processes $-(-\Delta)^{\alpha/2}$ for $\alpha \in (0,2)$
- [Erbar, Fathi, Laschos, Schlichting '16] Gradient flow structure for McKean-Vlasov on discrete spaces
- [Heinze, Schmidtchen, Pietschmann '22, '23] Systems on graphs
- [D. Slepčev, A. Warren '22] nonlocal wasserstein distance
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Gradient flows for free energies/(relative) entropies:

$$\mathcal{F}^\sigma(\rho) = \sigma \int \rho(x) \log \rho(x)\, dx + \frac{1}{2} \iint K(x,y)\, d\rho(x)\, d\rho(y)$$

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What if $\sigma = 0$?

$\sigma \to 0$: nonlocal metrics above do not have a clear/well-defined limit!

What is a suitable metric for gradient structure of interaction energies?

Nonlocal continuity equation

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Continuity equation

$$\partial_t \rho_t + \nabla \cdot j_t = 0 \qquad \text{where} \qquad j_t(x) := \rho_t(x) v_t(x)$$

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On Graphs

$$\partial_t \rho_t(x) + (\overline{\nabla} \cdot j_t)(x) = \partial_t \rho_t(x) + \int_{\mathbb{R}^d} j_t(x,y)\, \eta(x,y)\, dy = 0$$

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Upwind interpolation: density along edges = density at the source

$$j_t(x,y) = \rho(x) v_t(x,y)_+ - \rho(y) v_t(x,y)_-$$

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Nonlocal continuity equation

For $\rho_t \ll \mu$

$$\partial_t \rho_t(x) + \int_{\mathbb{R}^d} \left(\rho_t(x) v_t(x,y)_+ - \rho_t(y) v_t(x,y)_-\right) \eta(x,y)\, d\mu(y) = 0 \qquad \text{(NCE)}$$

Nonlocal interaction equation on graphs: NL^{2}IE

(NCE) with $v_t^{\mathcal{E}} := -\overline{\nabla} \frac{\delta \mathcal{E}}{\delta \rho} = -\overline{\nabla} K * \rho_t$

Upwind transportation “metric”

Upwind transportation "metric"

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Benamou-Brenier

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |v_t(x)|^2 \rho_t(x)\, dx\, dt \mid (\rho_t, v_t) \in \mathrm{CE}(\rho_0, \rho_1) \right\}$$

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Upwind nonlocal transportation "metric": Benamou-Brenier

$$\inf_{(\rho,v)\in \mathrm{NCE}} \left\{ \frac{1}{2} \int_0^1 \iint_G \left(|v_t(x,y)_+|^2 \rho_t(x) + |v_t(x,y)_-|^2 \rho_t(y)\right) \eta(x,y)\, d\mu(x)\, d\mu(y)\, dt \right\}$$

Note that:

- ρ might contain atoms, even if μ is Lebesgue!
  $\Rightarrow$ measure valued framework
- Benamou-Brenier functional is not jointly convex in (ρ_t, v_t)
  $\Rightarrow$ flux variables

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Definition (Nonlocal upwind transportation quasi-metric)

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ satisfying moment bound and local blow-up control, and $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$, the nonlocal upwind transportation cost between ρ_0 and ρ_1 is defined by

$$\mathcal{T}_\mu(\rho_0,\rho_1)^2 = \inf \left\{ \int_0^1 \mathcal{A}(\mu; \rho_t, \boldsymbol{j}_t)\, dt : (\rho, \boldsymbol{j}) \in \mathrm{NCE}(\rho_0,\rho_1) \right\}. \qquad (1)$$

Gradient descent in Finsler geometry [Ohta-Sturm '09, '12] - [Agueh '12]

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Inner product

$\boldsymbol{j} \in T_\rho \mathcal{P}_2(\mathbb{R}^d)$, we define an inner product $g_{\rho,\boldsymbol{j}} \colon T_\rho \mathcal{P}_2(\mathbb{R}^d) \times T_\rho \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

$$g_{\rho,\boldsymbol{j}}(\boldsymbol{j}_1, \boldsymbol{j}_2) = \frac{1}{2} \iint_G j_1(x,y)\, j_2(x,y) \eta(x,y) \left(\frac{\chi_{\{j>0\}}(x,y)}{\rho(x)} + \frac{\chi_{\{j<0\}}(x,y)}{\rho(y)} \right) d\mu(x) d\mu(y)$$

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Goal: direction of steepest discent from ρ!

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Gradient flows in $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{T})$: $\partial_t \rho_t = \overline{\nabla} \cdot \mathrm{grad}^- \mathcal{E}(\rho)$

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Nonlocal interaction energy

$$\mathrm{grad}^-\,\mathcal{E}(\rho)(x,y) = -\overline{\nabla}(K*\rho)(x,y) \left(\rho(x) \chi_{\{-\overline{\nabla} K*\rho>0\}}(x,y) + \rho(y) \chi_{\{-\overline{\nabla} K*\rho<0\}}(x,y) \right)$$

Variational characterisation of (NL^{2}IE)

Theorem

A curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$ is a weak solution to (NL^{2}IE) if and only if ρ belongs to $\mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$ and is a curve of maximal slope for $\mathcal{E}$ with respect to $\sqrt{\mathcal{D}}$, that is, satisfies

$$\mathcal{G}_T(\rho) = 0.$$

Local slope & De Giorgi Functional

For any $\rho \in \mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$, the De Giorgi functional at ρ is defined as

$$\mathcal{G}_T(\rho) := \mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \left(\mathcal{D}(\rho_\tau) + |\rho'_\tau|^2\right) d\tau \geq 0,$$

Variational characterisation of (NL^{2}IE)

Theorem

A curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$ is a weak solution to (NL^{2}IE) if and only if ρ belongs to $\mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$ and is a curve of maximal slope for $\mathcal{E}$ with respect to $\sqrt{\mathcal{D}}$, that is, satisfies

$$\mathcal{G}_T(\rho) = 0.$$

Local slope & De Giorgi Functional

For any $\rho \in \mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$, the De Giorgi functional at ρ is defined as

$$\mathcal{G}_T(\rho) := \mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \left(\mathcal{D}(\rho_\tau) + |\rho'_\tau|^2\right) d\tau \geq 0,$$

$$\begin{aligned}\mathcal{D}(\rho) :&= \widehat{g}_{\rho,-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}}\left(-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right)\\ &= -\iint_G \left|\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(x,y)_-\right|^2 \eta(x,y)\, d\rho(x)\, d\mu(y)\end{aligned}$$

Stability with respect to graph approximations

Stability of gradient flows

Let $(\mu^n)_n \subset \mathcal{M}^+(\mathbb{R}^d)$ and suppose that $(\mu^n)_n$ narrowly converges to μ. Suppose that ρ^n is a gradient flow of $\mathcal{E}$ with respect to μ^n for all $n \in \mathbb{N}$, that is,

$$\mathcal{G}_T(\mu^n; \rho^n) = 0 \quad \text{for all } n \in \mathbb{N},$$

such that $(\rho_0^n)_n$ satisfies $\sup_{n\in\mathbb{N}} M_2(\rho_0^n) < \infty$ and $\rho_t^n \rightharpoonup \rho_t$ as $n \to \infty$ for all $t \in [0, T]$ for some curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$. Then, $\rho \in \mathrm{AC}([0, T]; (\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}_\mu))$ and ρ is a gradient flow of $\mathcal{E}$ with respect to μ, that is,

$$\mathcal{G}_T(\mu; \rho) = 0.$$

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$$\mathcal{G}_T(\mu; \rho) = 0.$$

Corollary

Existence of weak solution to (NL^{2}IE) via finite-dimensional approximation.

Graph-to-local limit

Consider a localising graph (μ, η^ε), for

$$\eta^\varepsilon(x,y) := \frac{1}{\varepsilon^{d+2}}\vartheta\Big(\frac{x+y}{2}, \frac{x-y}{\varepsilon}\Big) \tag{η}$$

$$\begin{aligned}\partial_t \rho_t^\varepsilon(x) + \int_{\mathbb{R}^d} \overline{\nabla}(K * \rho_t^\varepsilon)(x,y)_- \eta^\varepsilon(x,y)\rho_t^\varepsilon(x)\mathrm{d}\mu(y) \\ - \int_{\mathbb{R}^d} \overline{\nabla}(K * \rho_t^\varepsilon)(x,y)_+ \eta^\varepsilon(x,y)\mathrm{d}\rho_t^\varepsilon(y) = 0\end{aligned} \tag{NL^{2}IE$_\varepsilon$}$$

$$\Big\downarrow{\scriptstyle \varepsilon \to 0} \tag{NLIE$_\mathbb{T}$}$$

$$\partial_t \rho_t = \mathrm{div}(\rho_t \mathbb{T}(\nabla K * \rho_t + \nabla P))$$

The tensor $\mathbb{T} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is of the form

$$\mathbb{T}(x) := \frac{1}{2}\widetilde{\mu}(x) \int_{\mathbb{R}^d \setminus \{0\}} w \otimes w\, \vartheta(x,w)\mathrm{d}w. \tag{$\mathbb{T}$}$$

- S. Lisini - ESAIM Control Optim. Calc. Var. (2009) diffusion
- D. Forkert, J. Maas, and L. Portinale - SIMA (2022) Evolutionary Γ-convergence for FP
- A. Hraivoronska, O.Tse - SIMA (2023) limiting behaviour of random walks on tessellations

Linking nonlocal and local continuity equation

Proposition (Local flux)

Let $j \in \mathcal{M}(\mathbb{R}^{2d}_{\diagup})$ satisfy the integrability condition $\iint_{\mathbb{R}^{2d}_{\diagup}} |x-y|\eta(x,y)|j|(x,y) < \infty$. Then there exists $\hat{\jmath} \in \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$\frac{1}{2}\iint_{\mathbb{R}^{2d}_{\diagup}} \overline{\nabla}\varphi\,\eta dj = \int_{\mathbb{R}^d} \nabla\varphi \cdot d\hat{\jmath}, \qquad \text{for all } \varphi \in C^1_c(\mathbb{R}^d). \tag{2}$$

In particular, if $(\boldsymbol{\rho},\boldsymbol{j}) \in \mathrm{NCE}_T$ such that $\mathcal{A}(\mu,\eta;\boldsymbol{\rho},\boldsymbol{j}) < \infty$, then there exists $(\hat{\jmath}_t)_{t\in[0,T]} \subset \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ such that $(\boldsymbol{\rho},\hat{\boldsymbol{\jmath}}) \in \mathrm{CE}_T$.

Proposition (Compactness)

Let $(\mu^\varepsilon)_{\varepsilon>0} \subset \mathcal{M}^+(\mathbb{R}^d)$ and $(\eta^\varepsilon)_{\varepsilon>0}$ identify localising graphs, uniformly in ε. Let $(\boldsymbol{\rho}^\varepsilon,\boldsymbol{j}^\varepsilon)_{\varepsilon>0} \subset \mathrm{NCE}_T$ be such that $\sup_{\varepsilon>0}\mathcal{A}(\mu^\varepsilon,\eta^\varepsilon;\boldsymbol{\rho}^\varepsilon,\boldsymbol{j}^\varepsilon) < \infty$ and let $\hat{\boldsymbol{\jmath}}^\varepsilon$ be associated to $\boldsymbol{j}^\varepsilon$ as in Proposition above. Then there exists a (not relabeled) subsequence of pairs $(\boldsymbol{\rho}^\varepsilon,\hat{\boldsymbol{\jmath}}^\varepsilon) \in \mathrm{CE}_T$ and a pair $(\boldsymbol{\rho},\hat{\boldsymbol{\jmath}}) \in \mathrm{CE}_T$ such that $\rho^\varepsilon_t \rightharpoonup \rho_t$ narrowly in $\mathcal{P}(\mathbb{R}^d)$ for a.e. $t \in [0,T]$ and such that $\int_\cdot \hat{\jmath}^\varepsilon_t dt \overset{*}{\rightharpoonup} \int_\cdot \hat{\jmath} dt$ weakly-$*$ in $\mathcal{M}((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$.

Limiting tensor structure

Space of tangent velocities

$$\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) := \left\{ v : G^{\varepsilon} \to \mathbb{R} : v_+ \mathrm{d}(\rho \otimes \mu) - v_- \mathrm{d}(\mu \otimes \rho) \in T^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) \right\} \tag{3}$$

$\{\overline{\nabla}\varphi : \varphi \in C^{\infty}_c(\mathbb{R}^d)\}$ is dense in $\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)$ wrt "L^2-norm"

Tangent-to-cotangent mapping

$\widetilde{I}^{\varepsilon}_{\rho} : \widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) \to \left(\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)\right)^*$, for a fixed $v \in \widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)$

$$\widetilde{I}^{\varepsilon}_{\rho}(v)[w] := \frac{1}{2}\iint_{G^{\varepsilon}} w\eta^{\varepsilon}[v_+ \mathrm{d}(\rho \otimes \mu) - v_- \mathrm{d}(\mu \otimes \rho)] \tag{4}$$

$$\begin{aligned}
\widetilde{I}^{\varepsilon}_{\rho}(\overline{\nabla}\varphi)[\overline{\nabla}\psi] &= \iint_{G^{\varepsilon}} (\overline{\nabla}\varphi)_+(x,y)\overline{\nabla}\psi(x,y)\eta^{\varepsilon}(x,y)\mathrm{d}\rho^{\varepsilon}(x)\mathrm{d}\mu(y) \\
&= \frac{1}{2}\iint_{G^{\varepsilon}} \overline{\nabla}\varphi(x,y)\overline{\nabla}\psi(x,y)\eta^{\varepsilon}(x,y)\mathrm{d}\rho(x)\mathrm{d}\mu(y) + o(1) \\
&= \int_{\mathbb{R}^d} \nabla\varphi(x)\cdot\mathbb{T}^{\varepsilon}(x)\nabla\psi(x)\mathrm{d}\rho(x) + o(1)
\end{aligned}$$

$$\mathbb{T}^{\varepsilon}(x) := \frac{1}{2}\int_{\mathbb{R}^d\setminus\{x\}} (x-y)\otimes(x-y)\,\eta^{\varepsilon}(x,y)\mathrm{d}\mu(y).$$

Limiting tensor structure

Theorem (Limiting inner product)

The tangent-to-cotangent mapping $\widetilde{I}^{\varepsilon}_{\rho} : \widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d) \to \left(\widetilde{T}^{\varepsilon}_{\rho}\mathcal{P}_2(\mathbb{R}^d)\right)^*$ *defined in* (4) *satisfies*

$$\lim_{\varepsilon\to 0} \widetilde{I}^{\varepsilon}_{\rho^\varepsilon}(\overline{\nabla}\varphi)[\overline{\nabla}\psi] = \int_{\mathbb{R}^d} \nabla\varphi \cdot \mathbb{T}\nabla\psi d\rho, \qquad \forall \varphi, \psi \in C^2_c(\mathbb{R}^d),$$

with the tensor $\mathbb{T} \in C(\mathbb{R}^d; \mathbb{R}^{d\times d})$ *obtained as limit of* $(\mathbb{T}^\varepsilon)_{\varepsilon_0 \geq \varepsilon > 0}$. *The limiting tensor, given by*

$$\mathbb{T}(x) := \frac{1}{2}\widetilde{\mu}(x) \int_{\mathbb{R}^d\setminus\{0\}} w \otimes w \, \vartheta(x, w) dw, \tag{$\mathbb{T}$}$$

is bounded and uniformly continuous.
Furthermore, the tensor $\mathbb{T}$ *is uniformly elliptic, i.e. there exist* $c, C > 0$ *such that for any* $x, \xi \in \mathbb{R}^d$ *we have*

$$c|\xi|^2 \leq \xi \cdot \mathbb{T}(x)\xi \leq C|\xi|^2.$$

Finally, for any $x \in \mathbb{R}$ *the matrix* $\mathbb{T}(x)$ *is symmetric.*

Variational graph-to-local limit

Theorem (Graph-to-local limit)

Let (μ, η^ε) be a localising graph. For any $\varepsilon > 0$ suppose that $\boldsymbol{\rho}^\varepsilon$ is a gradient flow of $\mathcal{E}$ in $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}_\varepsilon))$, that is,

$$\mathcal{E}(\rho_T^\varepsilon) - \mathcal{E}(\rho_0^\varepsilon) + \frac{1}{2}\int_0^T \left(\mathcal{D}_\varepsilon(\rho_\tau^\varepsilon) + |\rho_\tau'|_\varepsilon^2\right) d\tau = 0 \quad \textit{for any } \varepsilon > 0,$$

with $(\rho_0^\varepsilon)_\varepsilon \subset \mathcal{P}_2(\mathbb{R}^d)$ be such that $\sup_{\varepsilon>0} M_2(\rho_0^\varepsilon) < \infty$. Then there exists $\boldsymbol{\rho} \in \mathrm{AC}^2([0,T]; (\mathcal{P}_2(\mathbb{R}^d_{\mathbb{T}}), W_{\mathbb{T}}))$ such that $\rho_t^\varepsilon \rightharpoonup \rho_t$ as $\varepsilon \to 0$ for all $t \in [0,T]$ and $\boldsymbol{\rho}$ is a gradient flow of $\mathcal{E}$ in $(\mathcal{P}_2(\mathbb{R}^d_{\mathbb{T}}), W_{\mathbb{T}}))$, that is,

$$\mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \left(\mathcal{D}_{\mathbb{T}}(\rho_\tau) + |\rho_\tau'|_{\mathbb{T}}^2\right) d\tau = 0,$$

where the metric slope is

$$\mathcal{D}_{\mathbb{T}}(\rho) = \int_{\mathbb{R}^d} \left\langle \nabla \frac{\delta \mathcal{E}}{\delta \rho}, \mathbb{T}\nabla \frac{\delta \mathcal{E}}{\delta \rho} \right\rangle d\rho.$$

$$W_{\mathbb{T}}^2(\varrho_0, \varrho_1) = \inf\left\{ \int_0^1 \int_{\mathbb{R}^d} \left\langle \mathbb{T}^{-1}(x)\frac{\mathrm{d}j}{\mathrm{d}\rho}(x), \frac{\mathrm{d}j}{\mathrm{d}\rho}(x) \right\rangle \mathrm{d}\rho(x)\mathrm{d}t : (\boldsymbol{\rho}, \boldsymbol{j}) \in \mathrm{CE}(\varrho_0, \varrho_1) \right\}$$

Take-home messages

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- **A. E., G. Heinze, A. Schlichting**, *Graph-to-local limit for the nonlocal interaction equation*, preprint arXiv:2306.03475.
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Thank you for your attention!