Institut Camille Jordan - Université Claude Bernard - Lyon 1

New Lipschitz estimates and long-time asymptotic behavior for porous medium and fast diffusion equations

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Joint work with Filippo Santambrogio

Gradient flows face to face

Lyon
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Introduction

$$\frac{\partial n}{\partial t} = \nabla \cdot (n\nabla(p+V)), \quad \text{in } (0,\infty)\times\Omega, \quad d\geq 2,$$

$$n(0,x) = n_0(x) \geq 0, n_0 \in L^1(\Omega)$$

- Ω is either $\mathbb{R}^d$ or a convex bounded set (hom. Neumann boundary conditions)
- Constitutive law of the pressure:

$$p = P(n) := sign(\gamma)n^\gamma,$$

- $V : \mathbb{R}^d \to \mathbb{R}$ is either: $V = 0$, $V = |x|^2/2$, $|\nabla V|, D^2V \in L^\infty(\mathbb{R}^d)$

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Goal: study the quantity

$$u(t) := \max_x |p(t,x)|^b |\nabla p(t,x) + \nabla V(x)|^2$$

Strategy: exploit the pressure equation

$$\frac{\partial p}{\partial t} = \gamma p \Delta q + \nabla p \cdot \nabla q, \quad \text{with } q := p + V.$$

Some remarks

$$\frac{\partial n}{\partial t} = \frac{|\gamma|}{\gamma + 1} \Delta n^{\gamma+1}$$

- for $\gamma > 0$: slow diffusion or **porous medium equation**
- for $-2/d < \gamma < 0$: **fast diffusion equation**

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Aronson-Bénilan estimate:

$$\Delta p \geq -\frac{1}{\left(\gamma + \frac{2}{d}\right) t}, \quad \text{for all } t > 0.$$

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Barenblatt solution:

$$\mathcal{B}(t,x) = t^{-\alpha d} F\left(x t^{-\alpha}\right), \qquad F(\xi) = \left(C - k|\xi|^2\right)_+^{\frac{1}{\gamma}},$$

$$\alpha = \frac{1}{d\gamma + 2}, \qquad k = \frac{\alpha\gamma}{2(\gamma+1)}$$

Regularity:

- Hölder continuity of $n(t,x)$ in space and time
- the pressure $p(t,x)$ is actually Lipschitz after a *waiting time* (for $d \geq 2$)

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Bénilan's lecture notes: for $1-\gamma^2(d-1)>0$

$$\left|\nabla n^{\gamma+1}\right|^2 \leq \frac{nK_1}{t}, \qquad K_1 = K_1(\gamma, d, \|n_0\|_\infty),$$
$$u(t) = \max_x n\left|\nabla n^{\gamma}\right|^2 \leq \frac{K_2}{t}, \qquad K_2 = K_2(\gamma, d, \|n_0\|_\infty)$$

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Convergence to the self-similar profile:

$$\lim_{t\to\infty} \|n(t) - \mathcal{B}(t)\|_{L^1(\mathbb{R}^d)} = 0,$$
$$\lim_{t\to\infty} t^{\alpha d}\|n(t) - \mathcal{B}(t)\|_{L^\infty(\mathbb{R}^d)} = 0,$$

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Some literature: Aronson, Bénilan, Caffarelli, Friedman, Gil, Herrero, Vázquez, Wolanski (1970-1987), Carrillo, Toscani (2000), Carrillo, Jüngel, Markowich, Toscani, Unterreiter (2001), Vázquez's book (2006)

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- see Matteo Bonforte's talk

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- wait Nikita Simonov's talk

Main results

Goal: study the quantity $u(t) := \max_x |p(t)|^b |\nabla q(t)|^2$, where $q = p + V$

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- **Trivial V:** there exists a positive constant C such that

$$\max_x |p(t)|^b |\nabla p(t)|^2 \le Ct^{-1-\gamma d(b+1)\alpha}, \text{ for all } t > 0,$$

where $\alpha = \dfrac{1}{d\gamma + 2}$. Moreover, the exponent is sharp.

Assuming $n_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a positive constant C such that

$$\max_x |p(t)|^b |\nabla p(t)|^2 \le Ct^{-1}, \text{ for all } t > 0.$$

- **Quadratic V:** let $C_0 := \sup_x |p_0|^b |\nabla p_0 + x|^2 < \infty$, there exists a positive constant C such that

$$\max_x |p(t)|^b |\nabla q(t)|^2 \le C_0 e^{-Ct}, \text{ for all } t > 0.$$

Assumptions on γ and b

For V=0, for $\gamma, b > 0$, *resp.* $\gamma < 0, b < -1$, we assume

(i) $\gamma < \frac{1}{\sqrt{d-1}}$, *resp.* $|\gamma| < \frac{2}{d}$,

(ii) $1 - \sqrt{1 - \gamma^2(d-1)} < \gamma b < 1 + \sqrt{1 - \gamma^2(d-1)}$

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$\gamma > 0$ γ small

$\gamma < 0$ $-\frac{2}{d} < \gamma < 0$

$b \in [c^1(\gamma, d), c^2(\gamma, d)]$

$\frac{1}{\gamma}$

Let $V = 0$, the pressure equation is

$$\frac{\partial p}{\partial t} = \gamma p \Delta p + |\nabla p|^2.$$

We will study $\max_x e^{\alpha(p(t,x))}|\nabla p(t,x)|^2$ and then choose $\alpha(p) = ln|p|^b$.
Since we are at a maximum point

$$\nabla \left(e^{\alpha(p)} \frac{|\nabla p|^2}{2} \right) = 0, \qquad \Delta \left(e^{\alpha(p)} \frac{|\nabla p|^2}{2} \right) \leq 0.$$

The first condition gives

$$D^2 p \nabla p = -\frac{\alpha'}{2} |\nabla p|^2 \nabla p,$$

while from the second we get

$$\Delta \left(\frac{|\nabla p|^2}{2} \right) \leq \frac{|\nabla p|^4}{2} \left(|\alpha'|^2 - \alpha'' \right) - \alpha' \frac{|\nabla p|^2}{2} \Delta p.$$

Ideas of proof

$$\frac{\partial}{\partial t}\left(e^{\alpha}\frac{|\nabla p|^2}{2}\right)$$

$$=e^{\alpha}\left(\alpha'\frac{|\nabla p|^2}{2}\left(\gamma p\Delta p+|\nabla p|^2\right)+\nabla p\cdot\nabla\left(\gamma p\Delta p+|\nabla p|^2\right)\right)$$

$$=e^{\alpha}\left(\gamma p\frac{\alpha'}{2}|\nabla p|^2\Delta p+\frac{\alpha'}{2}|\nabla p|^4+\gamma|\nabla p|^2\Delta p+\gamma p\nabla p\cdot\nabla\Delta p+2\nabla p D^2 p\nabla p\right).$$

Now we use $2\nabla p\cdot\nabla\Delta p=\Delta(|\nabla p|^2)-2|D^2p|^2$, where $|D^2p|^2:=\sum_{i,j=1}^{d}(\partial_{i,j}p)^2$

$$\gamma p\nabla p\cdot\nabla\Delta p\le\gamma p\left(\frac{|\nabla p|^4}{2}\left(|\alpha'|^2-\alpha''\right)-\alpha'\frac{|\nabla p|^2}{2}\Delta p\right)-\gamma p|D^2p|^2$$

Therefore we have

$$\frac{\partial}{\partial t}\left(e^{\alpha}\frac{|\nabla p|^2}{2}\right)\le e^{\alpha}\left(\frac{|\nabla p|^4}{2}(\gamma p|\alpha'|^2-\gamma p\alpha''-\alpha')+\gamma|\nabla p|^2\Delta p-\gamma p|D^2p|^2\right)$$

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Ideas of proof

Let us denote $\lambda := -\frac{\alpha'}{2}|\nabla p|^2, \quad \delta_i = (D^2 q)_{i,i}$ for $i = 2, \dots, d, \quad \delta := \sum_{i=2}^{d} \delta_i.$

Since $\gamma p > 0$, we have

$$\begin{aligned}
\gamma|\nabla p|^2 \Delta p - \gamma p |D^2 p|^2 &\leq \gamma|\nabla p|^2(\lambda + \delta) - \gamma p \left(\lambda^2 + \sum_{i=2}^{d} \delta_i^2\right) \\
&\leq \gamma|\nabla p|^2(\lambda + \delta) - \gamma p \left(\lambda^2 + \frac{\delta^2}{d-1}\right) \\
&= -\gamma \frac{\alpha'}{2}|\nabla p|^4 + \gamma\delta|\nabla p|^2 - \gamma p \frac{|\alpha'|^2}{4}|\nabla p|^4 - \gamma p \frac{\delta^2}{d-1}.
\end{aligned}$$

Using Young's inequality, we have

$$\gamma\delta|\nabla p|^2 \leq \gamma p \frac{\delta^2}{d-1} + \gamma \frac{|\nabla p|^4}{4p}(d-1),$$

and we finally find

$$\gamma|\nabla p|^2 \Delta p - \gamma p |D^2 p|^2 \leq \left(-\gamma\frac{\alpha'}{2} + \frac{\gamma(d-1)}{4p} - \gamma p \frac{|\alpha'|^2}{4}\right)|\nabla p|^4.$$

Let's come back to the main estimate

$$\frac{\partial}{\partial t}\left(e^{\alpha}\frac{|\nabla p|^2}{2}\right) \leq e^{\alpha}\left(|\nabla p|^4\left(-\frac{\alpha'}{2}+\gamma p\frac{|\alpha'|^2}{4}-\gamma p\frac{\alpha''}{2}-\gamma\frac{\alpha'}{2}+\frac{\gamma(d-1)}{4p}\right)\right).$$

Choosing $\alpha(p) = b\ln|p|$ we find

$$\frac{\partial}{\partial t}\left(|p|^b\frac{|\nabla p|^2}{2}\right) \leq c_0|p|^{b-1}|\nabla p|^4,$$

where c_0 is defined as

$$c_0 = -\frac{|b|}{2}+\frac{|\gamma|b^2}{4}+\frac{|\gamma|(d-1)}{4},$$

and is **negative**.

Let's come back to the main estimate

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla p|^2}{2}\right) \leq |p|^b \left(|\nabla p|^4 \left(-\frac{\alpha'}{2} + \gamma p \frac{|\alpha'|^2}{4} - \gamma p \frac{\alpha''}{2} - \gamma \frac{\alpha'}{2} + \frac{\gamma(d-1)}{4p}\right)\right).$$

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Ideas of proof

$$\frac{\partial}{\partial t}\left(|p|^{b}\frac{|\nabla p|^{2}}{2}\right) \leq c_0 |p|^{-b-1}|p|^{2b}|\nabla p|^{4},$$

Let us first prove the claimed result for $n_0 \in L^{\infty}(\mathbb{R}^d)$.

For $\gamma, b > 0$, we have $|p(t,x)|^{-b-1} \geq |\overline{p}|^{-b-1}$.
For $\gamma < 0$, $b \leq -1$, we have $|p(t,x)|^{-b-1} \geq |\overline{p}|^{-b-1}$.

Therefore, in both cases we find

$$\frac{\partial}{\partial t}\left(|p|^{b}\frac{|\nabla p|^{2}}{2}\right) \leq c_0 |\overline{p}|^{-b-1}|p|^{2b}|\nabla p|^{4},$$

From this we get

$$u'(t) \leq -Cu^{2}(t),$$

$$u(t) \leq Ct^{-1}.$$

Ideas of the proof

Without asking $n_0 \in L^\infty(\mathbb{R}^d)$ we exploit the Aronson-Bénilan estimate to find

$$\|n(t)\|_\infty \leq C t^{-\frac{d}{d\gamma+2}},$$

which is equivalent to

- $\max_x |p(t)| \leq C t^{-\frac{d\gamma}{d\gamma+2}}$, for $\gamma > 0$,
- $\min_x |p(t)| \geq C t^{-\frac{d\gamma}{d\gamma+2}}$, for $\gamma < 0$.

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Therefore, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla p|^2}{2}\right) &\le c_0 |p|^{2b} |p|^{-(b+1)} |\nabla p|^4 \\ &\le -Cp^{2b} |\nabla p|^4 t^{(b+1)\frac{d\gamma}{d\gamma+2}},\end{aligned}$$

from which we find

$$u(t) \le Ct^{-1-\gamma d(b+1)\alpha}.$$

Let us consider the quadratic potential $V = |x|^2/2$, then

$$\frac{\partial p}{\partial t} = \gamma p \Delta q + \nabla p \cdot \nabla q, \text{ with } q = p + V.$$

With the same argument, we can find

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) \leq |p|^{b-1}\left(c_1 |\nabla q|^2 |\nabla p|^2 + c_2 |\nabla q \cdot \nabla p|^2\right) + c_3 |p|^b |\nabla q|^2,$$

where

$$c_3 = \frac{\gamma b d}{2} - 1.$$

The assumptions ensure $c_0 := c_1 + c_2 \leq 0$, and $c_1, c_3 < 0$. Therefore, we establish

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) \leq c_0 |p|^b |\nabla q \cdot \nabla p|^2 + c_3 |p|^b |\nabla q|^2$$

$$\leq c_3 |p|^b |\nabla q|^2,$$

and finally

$$u(t) \leq C_0 e^{-Ct}, \quad \forall t > 0.$$

Time-dependent scaling:

$$\hat{n}(t,x) := \varphi(t)^d n(\psi(t), \varphi(t)x), \qquad \varphi(t) = e^t,\ \psi(t) = e^{(d\gamma+2)t}.$$

If $n(t,x)$ is a solution of

$$\frac{\partial n}{\partial t} = \frac{|\gamma|}{\gamma+1}\Delta n^{\gamma+1},$$

then $\hat{n}(t,x)$ satisfies

$$\frac{\partial \hat{n}}{\partial t} = \frac{|\gamma|}{\gamma+1}\Delta \hat{n}^{\gamma+1} + \nabla\cdot(\hat{n}x).$$

By our result:

$$\max_x |\hat{p}(t,x)|^b \left|\nabla \hat{p}(t,x) + x\right|^2 \le C_0 e^{-Ct},$$

which translate into

$$\max_x |p(t,x)|^b \ \left|\nabla p(t,x) + xt^{-1}\right|^2 \le C_0 t^\beta t^{-\frac{C}{d\gamma+2}},$$

with $\beta = -\alpha\gamma db - 2 + 2\alpha$.

Implications: rate of convergence to the Barenblatt

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For $\gamma < 0$, the pressure of the Barenblatt solution is

$$\mathcal{P}(t,x) = t^{-\alpha\gamma d}\left(C - k|x|^2 t^{-2\alpha}\right).$$

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Let us take $x \in C_{a,b}(t) := \{x;\ |x|t^{-\alpha} \in [a,b]\}$ for some constants $0 < a < b$.

For $n_0 \in \mathcal{X} \setminus \{0\}$ (see Nikita's talk)

$$n^{\gamma b}(t,x) \gtrsim t^{-\alpha\gamma d b}, \qquad |\nabla \mathcal{P}(t,x)|^2 \simeq t^{-2+2\alpha}.$$

Finally, we find

$$\frac{\left\|\nabla p(t) - \nabla \mathcal{P}(t)\right\|^2_{L^\infty(C_{a,b}(t))}}{\left\|\nabla \mathcal{P}(t)\right\|^2_{L^\infty(C_{a,b}(t))}} \lesssim t^{-\frac{C}{d\gamma+2}}, \quad \forall t > 0.$$

Implications: rate of convergence to the Barenblatt

$$\max_x |p(t,x)|^b \; \left|\nabla p(t,x) + xt^{-1}\right|^2 \leq C_0 t^{\beta} t^{-\frac{c}{d\gamma+2}},$$

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Thank you for the attention!