Some aspects of Horn's problem

Jean-Bernard Zuber (LPTHE, Sorbonne Université)

Toulouse, 30 June 2023

Some aspects of Horn's problem

Jean-Bernard Zuber (LPTHE, Sorbonne Université)

Toulouse, 30 June 2023

Collaboration with Robert Coquereaux and Colin McSwiggen

What is Horn's problem ?

Given two Hermitian $n \times n$ matrices A and B, of known spectrum

$$\alpha = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$$

and $\beta = \{\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\}$, what can be said on the spectrum $\gamma = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$ of their sum $C = A + B$?

An old problem, with a rich history. . .
Obviously, $\sum_{k=1}^{n}(\gamma_k - \alpha_k - \beta_k) = 0$, thus the stage is in $\mathbb{R}^{n-1}$.
In general, set of *linear* inequalities between the α's, β's, γ's.
For example, $\gamma_1 \leq \alpha_1 + \beta_1$, (Obvious: recall that $\alpha_1 = \sup_\psi \frac{(\psi, A\psi)}{(\psi,\psi)}$, etc)

What is Horn's problem ?

Given two Hermitian $n \times n$ matrices A and B, of known spectrum

$$\alpha = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$$

and $\beta = \{\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\}$, what can be said on the spectrum $\gamma = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$ of their sum $C = A + B$?

An old problem, with a rich history. . .

Obviously, $\sum_{k=1}^n \gamma_k - \alpha_k - \beta_k = 0$, thus the stage is in $\mathbb{R}^{n-1}$.

In general, set of *linear* inequalities between the α's, β's, γ's.

For example, $\gamma_1 \leq \alpha_1 + \beta_1$,

or Weyl's inequality (1912)

$i + j - 1 \leq n \;\Rightarrow\; \gamma_{i+j-1} \leq \alpha_i + \beta_j$, etc.

H. Weyl

What is Horn's problem ?

Given two Hermitian $n \times n$ matrices A and B, of known spectrum

$$\alpha = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$$

and $\beta = \{\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\}$, what can be said on the spectrum $\gamma = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$ of their sum $C = A + B$?

An old problem, with a rich history. . .

Obviously, $\sum_{k=1}^n \gamma_k - \alpha_k - \beta_k = 0$, thus the stage is in $\mathbb{R}^{n-1}$.

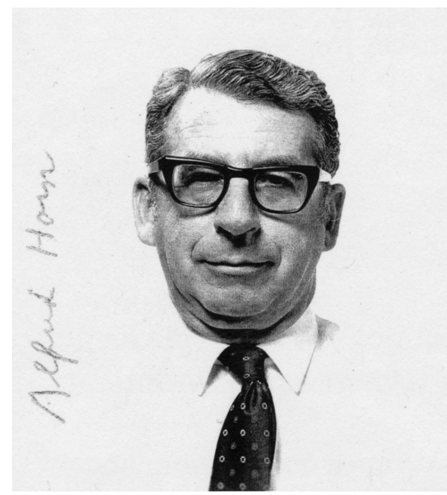
In general, set of *linear* inequalities between the α's, β's, γ's.

For example, $\gamma_1 \leq \alpha_1 + \beta_1$ or $i + j - 1 \leq n \ \Rightarrow \ \gamma_{i+j-1} \leq \alpha_i + \beta_j$, etc.

Horn (1962) conjectured the form of a (necessary and sufficient) set of inequalities

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

for some triplets $\{I, J, K\}$ of subsets of $\{1, \cdots, n\}$, $|I| = |J| = |K|$, determined recursively.



A. Horn

Thus the γ's belong to a *convex polytope* in $\mathbb{R}^{n-1}$.

Horn's inequalities

For example, for $n = 3$

$$\begin{aligned}
\gamma_{3min} := \alpha_3 + \beta_3 \leq \quad & \gamma_3 \quad \leq \min(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1) =: \gamma_{3max} \\
\gamma_{2min} := \max(\alpha_2 + \beta_3, \alpha_3 + \beta_2) \leq \quad & \gamma_2 \quad \leq \min(\alpha_1 + \beta_2, \alpha_2 + \beta_1) =: \gamma_{2max} \\
\gamma_{1min} := \max(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1) \leq \quad & \gamma_1 \quad \leq \alpha_1 + \beta_1 =: \gamma_{1max} \,.
\end{aligned}$$

in addition to

$$\gamma_3 \leq \gamma_2 \leq \gamma_1$$

and

$$\gamma_1 + \gamma_2 + \gamma_3 = \sum_i (\alpha_i + \beta_i)$$

What is Horn's problem ?

Given two Hermitian $n \times n$ matrices A and B, of known spectrum

$$\alpha = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n\}$$

and $\beta = \{\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\}$, what can be said on the spectrum $\gamma = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n\}$ of their sum $C = A + B$?

An old problem, with a rich history...
Obviously, $\sum_{k=1}^{n} \gamma_k - \alpha_k - \beta_k = 0$, thus the stage is in $\mathbb{R}^{n-1}$.
In general, set of *linear* inequalities between the α's, β's, γ's.
For example, $\gamma_1 \leq \alpha_1 + \beta_1$.

Horn (1962) conjectured the form of a (necessary and sufficient) set of inequalities

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

for some triplets $\{I, J, K\}$ of subsets of $\{1, \cdots, n\}$, $|I| = |J| = |K|$, determined recursively.

Thus the γ's belong to a *convex polytope* in $\mathbb{R}^{n-1}$.

⋮

Klyachko (1998) and Knutson and Tao (1999) prove Horn's conjecture.

A. Klyachko

A. Knutson

T. Tao

Problem interesting by its many facets and ramifications, in symplectic geometry (Atiyah–Guillemin-Sternberg convexity theorem), in algebraic geometry, representation theory & combinatorics, etc . . .

See a beautiful introduction by A. Knutson and T. Tao (Notices of the AMS, 2001)

and a comprehensive review by W. Fulton (Bull. Am. Math. Soc. 2000)

Outline of this talk

1. The classical Horn's problem revisited

2. Explicit results for SU(n) orbits, $n = 2, 3$

3.? Extension and generalizations. SO(n) orbits of real symmetric matrices

4. Connection with representation theory and combinatorics

5. Summary and open issues

1. The classical Horn's problem revisited

Rephrase the problem as follows:
Let $\mathcal{O}_\alpha$ be the *orbit* of $\mathrm{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n)$ under action by conjugation of $\mathsf{U}(n)$,

$$\mathcal{O}_\alpha = \{U \mathrm{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n) U^* \,|\, U \in \mathsf{U}(n)\}$$

and likewise $\mathcal{O}_\beta$.
Which orbits $\mathcal{O}_\gamma$ appear in the "sum of orbits" $\mathcal{O}_\alpha \boxplus \mathcal{O}_\beta$?

Two possible generalizations:

Up to a factor i, Hermitian matrices live in the Lie algebra $\mathfrak{su}(n)$. Orbits are “coadjoint orbits” of SU(n). This suggests two natural generalizations of Horn’s original problem.

– **Coadjoint orbits** of other (simple, connected, compact) Lie groups and algebras. Symplectic geometry, piecewise polynomiality of measure, convexity theorems, etc [Heckman ’82, Knutson ’01, . . .]

– Other **“self-adjoint”** $n \times n$ matrices: $A = (A^T)^*$

Orbits of	Real Symmetric	Complex Hermitian	Quaternionic self-dual
Conjugation by	SO(n)	SU(n)	USp(n)

More specific questions

unique invariant normalized measure $d\mu(U) = d\mu(VU) = d\mu(UV)$

• Suppose we take A *uniformly distributed* on $\mathcal{O}_\alpha$ (for the Haar measure), and likewise B on $\mathcal{O}_\beta$, and *independent* of A, can one determine the PDF (probability distribution function) of γ ?

• Compute this PDF for the coadjoint orbits of various Lie algebras, see below.

• What about orbits of self-adjoint matrices?
Compare real symmetric, complex Hermitian and quaternionic self-dual matrices.

A general result by Fulton: Horn's inequalities on the γ's are the same for these three cases. Hence the γ's lie in the same polytope (for given n and α, β).

What about their distribution ?

More specific questions

• Suppose we take A *uniformly distributed* on $\mathcal{O}_\alpha$ (for the Haar measure), and likewise B on $\mathcal{O}_\beta$, and *independent* of A, can one determine the PDF (probability distribution function) of γ ?

• Compute this PDF for the coadjoint orbits of various Lie algebras.

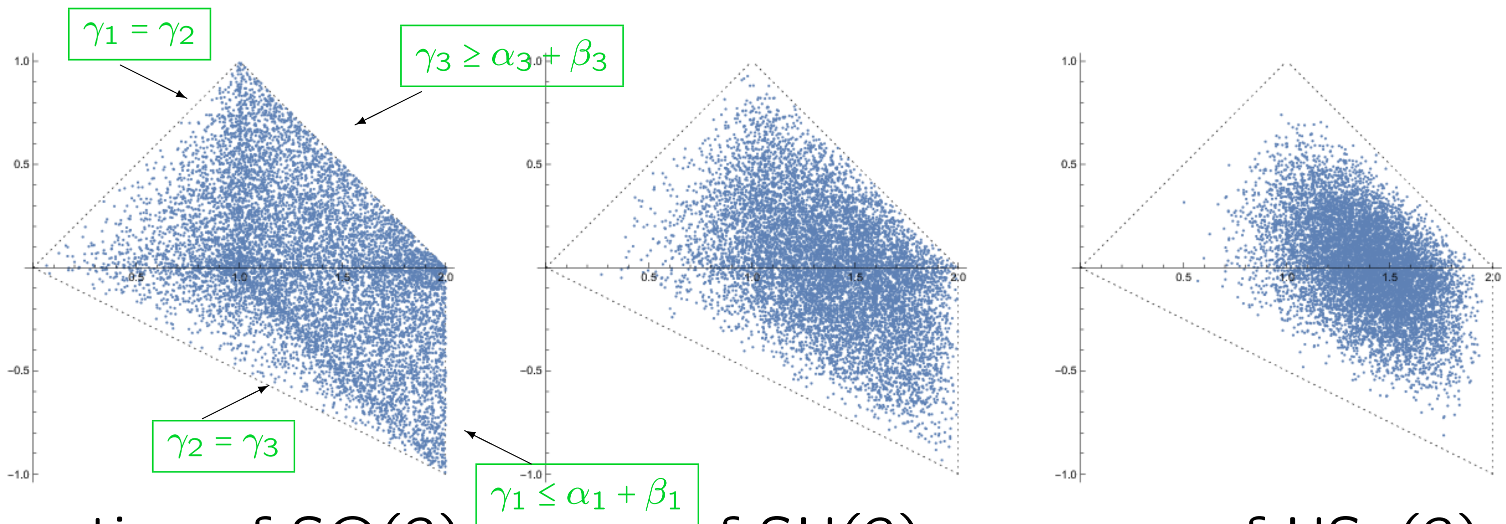
• What about orbits of self-adjoint matrices?
Compare real symmetric, complex Hermitian and quaternionic self-dual matrices.

A general result by Fulton: Horn's inequalities on the γ's are the same. Hence the γ's lie in the same polytope (for given α, β).

What about their distribution ?

Make a (numerical) experiment ! Take $n = 3$, $\alpha = \beta = (1, 0, -1)$, generate big samples of $C = \mathrm{diag}\,(\alpha) + V \cdot \mathrm{diag}\,(\beta) \cdot V^{-1}$, diagonalize them and plot (γ_1, γ_2). Recall by convention $\gamma_1 \geq \gamma_2 \geq \gamma_3 = -\gamma_1 - \gamma_2$.

$n = 3 \quad \alpha = \beta = (1, 0, -1)$. Plot of (γ_1, γ_2)



action of SO(3) of SU(3) of USp(3)

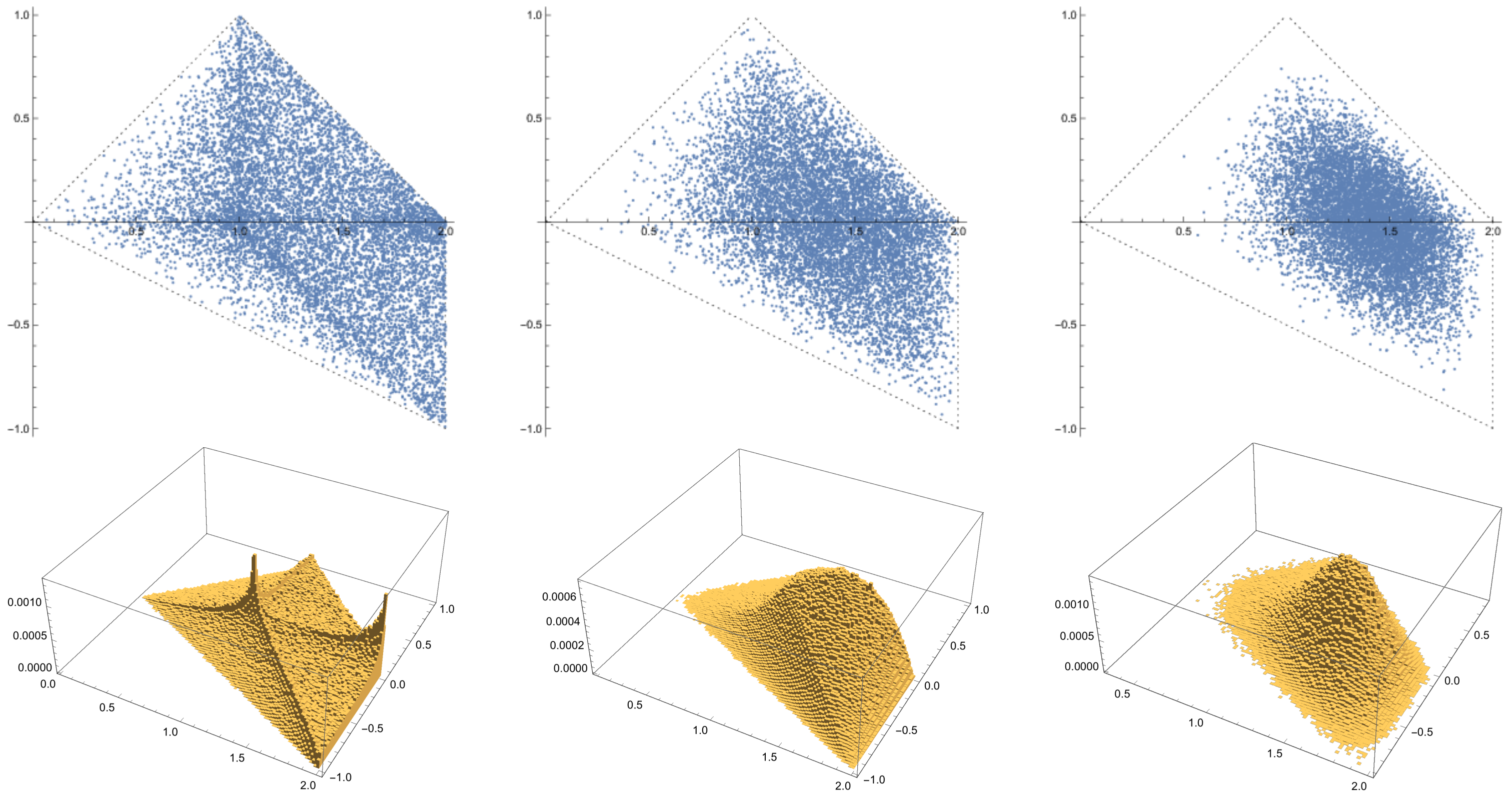
Observe

Same polygon of support (as expected)

Distribution more condensed for USp(3)

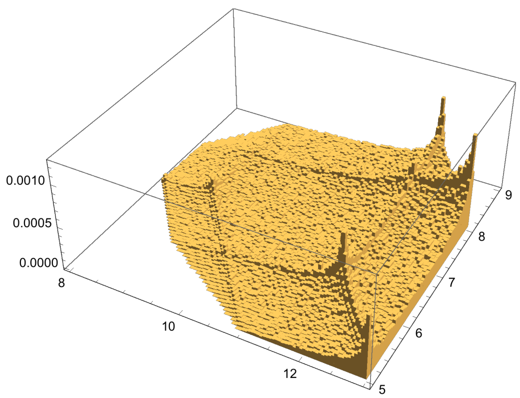
Lines of enhancement in the SO(3) case ??

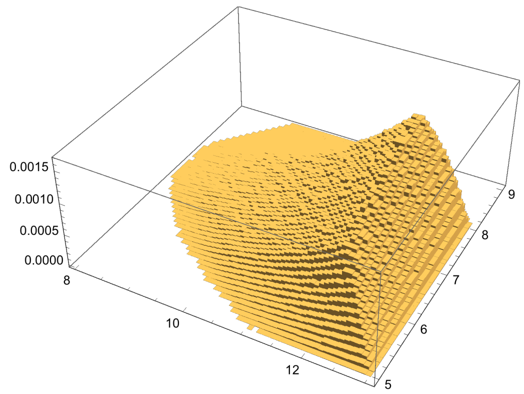
$n = 3 \quad \alpha = \beta = (1, 0, -1)$. Plot and histogram of (γ_1, γ_2)

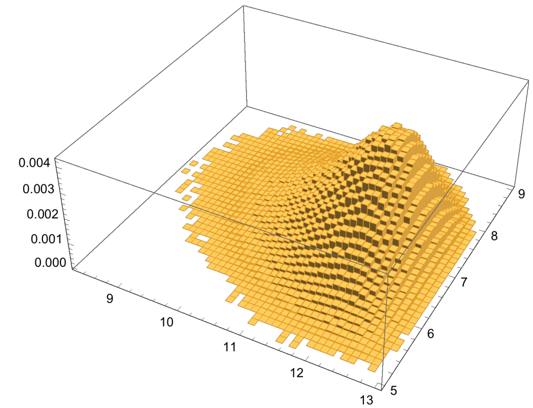


action of SO(3) | of SU(3) | of USp(3)

Another example: $\alpha = (7, 3, 0)$, $\beta = (6, 5, 0)$







Question:

Can one compute the PDF for the three cases and understand the origin, location and nature of the singularities in the orthogonal case ?

The locus of singularities

Compare the three "self-adjoint cases" , of real symmetric, complex Hermitian or quaternionic self-dual, $n \times n$ (traceless) matrices.

For given n and α, β, not only the support of the γ's is the same ([Fulton]) but also the *locus* of non-differentiability (although of quite different nature)

Proposition 1 *[C-MS-Z]The PDF is a piecewise real analytic function of γ. Non analyticities occur only when γ lies on hyperplanes of the form*

$$\sum_{k \in K} \gamma_k = \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

with $I, J, K \subset \{1, \cdots, n\}$, $|I| = |J| = |K|$, independently on the pair $(G, \mathcal{M}_n)$

Hint of proof: look at points where the differential of the map $\Phi : G \times G \to \mathcal{M}_n^0$, $(g_1, g_2) \mapsto C = A + B = g_1 \cdot \alpha + g_2 \cdot \beta$ is not surjective.

Remarks:
– includes boundaries of Horn's domain other than the hyperplanes $\gamma_i = \gamma_{i+i}$
– a necessary, not a sufficient condition ! Which singularities do occur ?

Computing the PDF

A central role is played by the *orbital integral* (aka generalized or multivariate Bessel function)

$$\mathcal{H}_\theta(A,X) = \int_{G_\theta} \exp(\mathrm{tr}\,(VAV^{-1}X))\,dV$$

normalized Haar measure

where $\theta = \frac{1}{2}, 1, 2$ (half Dyson index) and

θ	A, X	G_θ
$\frac{1}{2}$	Real Symmetric	$\mathrm{SO}(n)$
1	Complex Hermitian	$\mathrm{SU}(n)$
2	Quaternionic Self-Dual	$\mathrm{USp}(n)$

Likewise for coadjoint orbits $\mathcal{H}_{\mathfrak{g}}(A,X) = \int_G \exp\langle gAg^{-1}, X\rangle\,dg$.

Note

- $\mathcal{H}(A, \mathrm{i}\,X)$ = Fourier transform of the orbital measure.
- $\mathcal{H}(A,X)$ only function of e-values α and x of A and X. Denote it also $\mathcal{H}(\alpha, x)$.

Proposition 2 . *For self-adjoint matrices A and B, independently and uniformly distributed on their G_θ-orbits $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, PDF of γ is*

$$\mathsf{p}(\gamma|\alpha,\beta) = \mathrm{const}(\theta,n)\,|\Delta(\gamma)|^{2\theta}\int_{\mathbb{R}^n} d^n x\,|\Delta(x)|^{2\theta}\,\mathcal{H}_\theta(\alpha,\mathrm{i}\,x)\mathcal{H}_\theta(\beta,\mathrm{i}\,x)\mathcal{H}_\theta(\gamma,\mathrm{i}\,x)^*\,.$$

where $\Delta(x)=\prod_{i<j}(x_i-x_j)$ is the Vandermonde determinant. For coadjoint orbits, similar formula with $x\in\mathfrak{t}$ and $|\Delta(x)|^{2\theta}$ changed to $\Delta^2_{\mathfrak{g}}(x):=\prod_{\boldsymbol{\alpha}>0}\langle\boldsymbol{\alpha},x\rangle^2$ *($\mathfrak{t}$ a Cartan subalgebra, $\boldsymbol{\alpha}$ the +ve roots).*

Elementary proof: $\mathcal{H}(A,\mathrm{i}\,X)$ is the *characteristic function* of the random variable $A\in\mathcal{O}_\alpha$. Characteristic function of $C=A+B$ is the product $\mathcal{H}(A,\mathrm{i}\,X)\mathcal{H}(B,\mathrm{i}\,X)$. The PDF of C then obtained by inverse Fourier transform. The Δ's come from Jacobians. $\square$.

See also [Dooley–Repka–Wildberger 1993; Frumkin&Goldberger 2006; Suzuki 2013; Kuijlaars & Roman 2016]

The orbital integrals, self-adjoint and coadjoint cases

In the unitary ($\theta = 1$) case, explicit formula known for long

[Harish-Chandra 1957, Itzykson–Z 1980] (for A and X "regular", *i.e.*, $\alpha_i \neq \alpha_j$ and $x_i \neq x_j$),

$$\mathcal{H}_2(\alpha, \mathrm{i}\, x) = \int_{\mathsf{SU}(n)} e^{\mathrm{i}\,\mathrm{tr}\,(XVAV^*)}\, dV = \prod_{p=1}^{n-1} p!\, \frac{(\det e^{\mathrm{i}\, x_i \alpha_j})_{1 \le i,j \le n}}{\Delta(ix)\Delta(\alpha)},$$

i.e., semi-classical approximation is exact ! [Duistermaat-Heckman 1982].

Generalizes to other coadjoint orbits. [Harish-Chandra]

In the symplectic ($\theta = 2$) case,

[Brézin–Hikami 2002]

$$\mathcal{H}_4(\alpha, \mathrm{i}\, x) = \text{const.} \sum_{P \in S_n} \frac{e^{\mathrm{i} \sum_j x_j \alpha_{Pj}}}{\Delta^3(ix)\Delta^3(\alpha_P)} f_n(x, \alpha_P),$$

f_n a polynomial in the variables $\tau_{i,j} := (x_i - x_j)(\alpha_{Pi} - \alpha_{Pj})$, $\deg(f_2) = 1$, $\deg(f_3) = 3$, etc. (Recursive formula for higher $f_n \ldots$)

In the orthogonal ($\theta = \frac{1}{2}$) case, ???

Explicit computation of the PDF $\mathrm{p}(\gamma)$ in the $\mathrm{SU}(n)$ case.

Make use of HCIZ integral

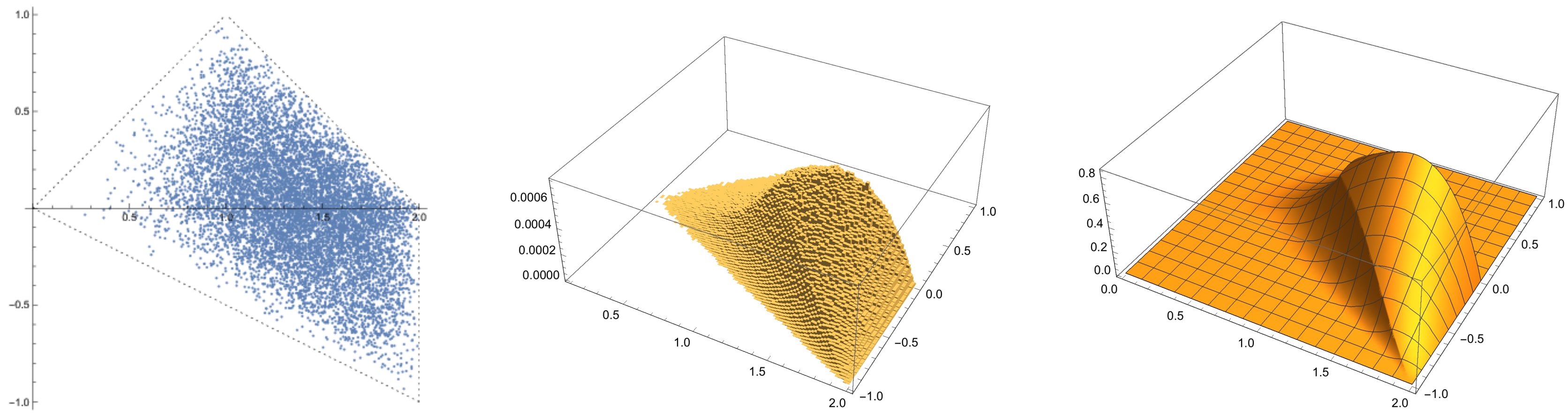
$$\begin{aligned}\mathrm{p}(\gamma|\alpha,\beta) &= \text{const.}\,\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\int\frac{d^n x}{\Delta(x)}\det e^{\mathrm{i}\,x_i\alpha_j}\det e^{\mathrm{i}\,x_i\beta_j}\det e^{-\mathrm{i}\,x_i\gamma_j}\\ &= \frac{\prod_1^{n-1}p!}{n!}\,\delta\Big(\sum_k(\gamma_k-\alpha_k-\beta_k)\Big)\,\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\mathcal{J}_n(\alpha,\beta;\gamma)\end{aligned}$$

"volume function"

A priori, $\mathcal{J}_n(\gamma)$ is a distribution (generalized function), in fact
– a piece-wise polynomial of degree $(n-1)(n-2)/2$
(also a general result in symplectic geometry, [Heckman],[Duistermaat–Heckman],. . .)
and
– a function of differentiability class C^{n-3}, for $n>2$
(a consequence of Riemann–Lebesgue theorem; also [Guillemin–Lerman–Sternberg])

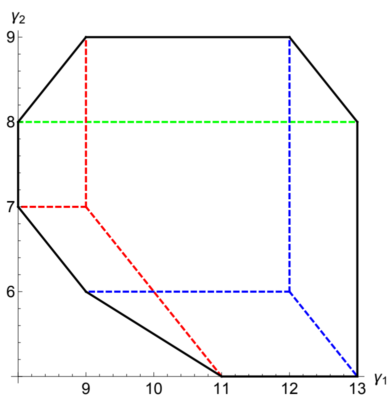
Group theoretic and geometric interpretations of $\mathcal{J}_n$ . . . yet to come

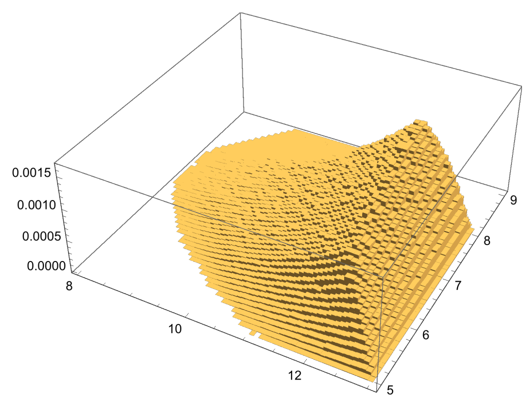
Example: $n = 3$, $\alpha = \beta = (1, 0, -1)$.

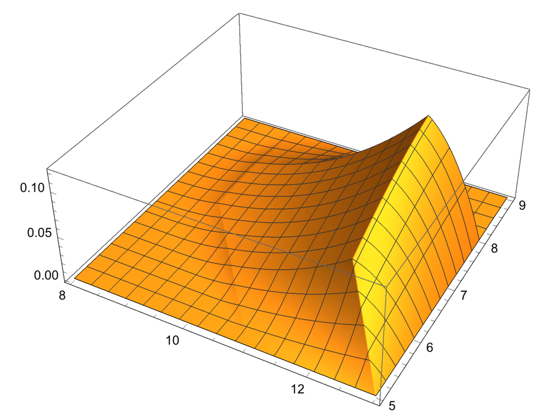


left: distribution of 10,000 eigenvalues in the γ_1, γ_2 plane; middle: histogram of 5×10^6 eigenvalues; right: plot of the PDF as computed above

Another example: $\alpha = (7,3,0)$, $\beta = (6,5,0)$

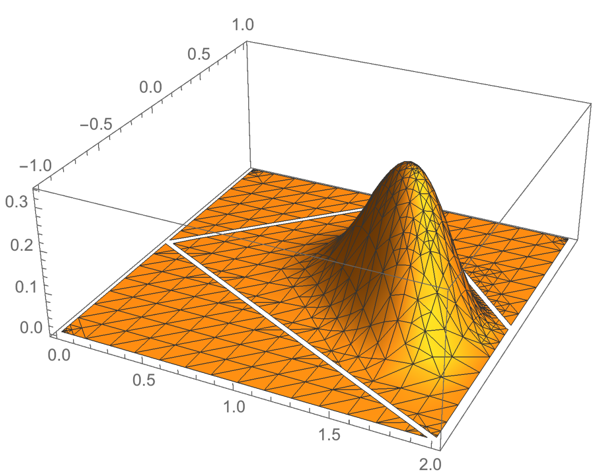


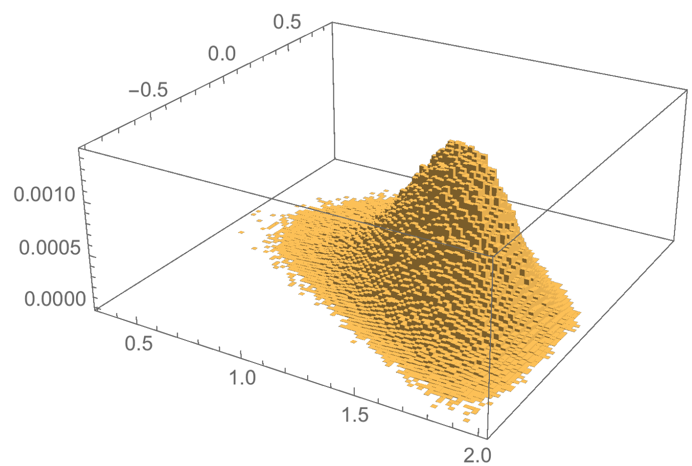




3. Extensions and generalizations

* USp(n) orbits of *quaternionic self-dual* matrices: generalized H-C formula for $n = 2, 3, 4$ [Brézin-Hikami]. PDF of differentiability class $C^{2(n-2)}$. ✓





$n = 3$, $\alpha = \beta = \{1, 0, -1\}$

3. Extensions and generalizations

* USp(n) orbits of *quaternionic self-dual* matrices: generalized H-C formula for $n = 2, 3, 4$ [Brézin-Hikami]. PDF of differentiability class $C^{2(n-2)}$. ✓

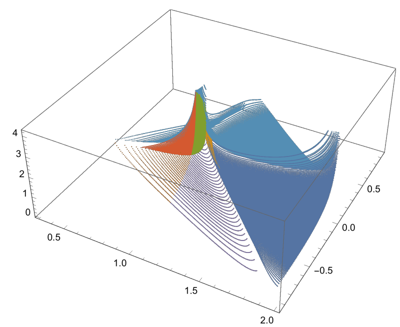
⋆ O(n) or SO(n) orbits of *real symmetric matrices* ?? (No Harish-Chandra formula for $n > 2$!!).

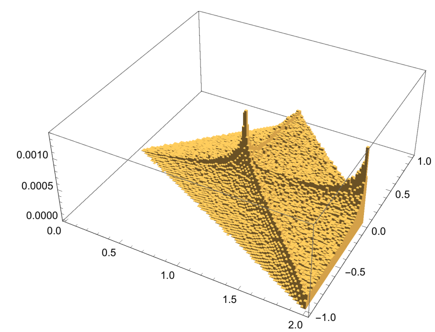
3. Extensions and generalizations

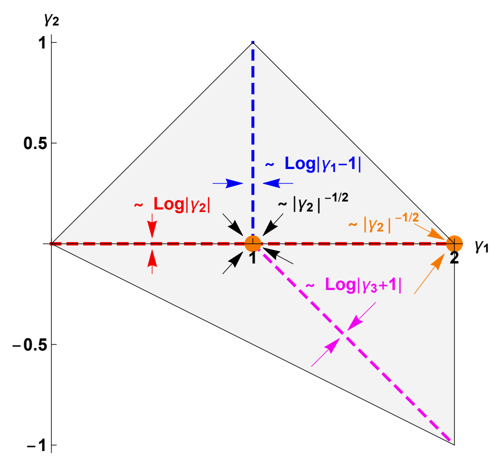
* USp(n) orbits of *quaternionic self-dual* matrices: generalized H-C formula for $n = 2, 3, 4$ [Brézin-Hikami]. PDF of differentiability class $C^{2(n-2)}$. ✓

⋆ O(n) or SO(n) orbits of *real symmetric matrices* ?? (No Harish-Chandra formula for $n > 2$!!).

Plot and Singularities of $\mathrm{p}(\gamma_1, \gamma_2)$ **for** $n = 3$, $\alpha = \beta = \{1, 0, -1\}$ [Coquereaux-Z]







4. Connection with representation theory

G a simple simply-connected compact Lie group, V_α its irreducible representations (irreps) labelled by a highest weight (h.w.) α (a vector in the r-dim space $\mathfrak{t}^*$, r the *rank*).

Decompose the tensor product into irreps

$$V_\alpha \otimes V_\beta = \oplus_\gamma C_{\alpha\,\beta}^{\gamma} V_\gamma \tag{1}$$

with "Littlewood–Richardson multiplicities" $C_{\alpha\,\beta}^{\gamma}$ (aka "Clebsch–Gordan decomposition").

Q: Which γ in (1)? How to compute the $C_{\alpha\,\beta}^{\gamma}$?

Using orthonormal characters $\chi_\alpha(g)$, one may write

$$C_{\alpha\,\beta}^{\gamma} = \int_G dg\, \chi_\alpha(g)\chi_\beta(g)\chi_\gamma^*(g)\,.$$

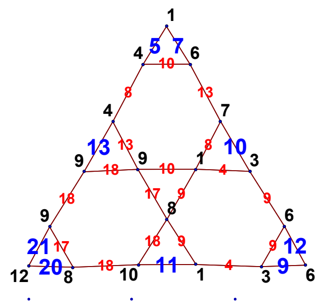
Littlewood–Richardson algorithm (for SU(n)): Young diagrams. . .

Also Kostant–Steinberg formulae, Brauer, Racah–Speiser, Klimyk rules. . .

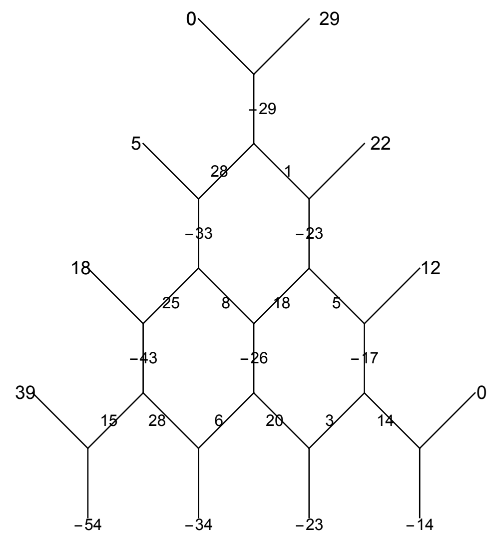
Also various combinatorial models ("pictographs") that count the $C_{\alpha\,\beta}^{\gamma}$:

Berenstein–Zelevinsky triangles, Knutson–Tao honeycombs/hives, Ocneanu blades, etc

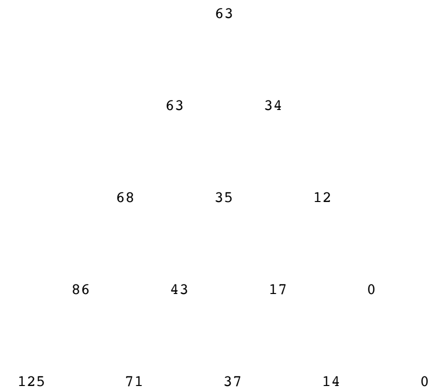
An example in $su(4)$: $\alpha = (21, 13, 5)$, $\beta = (7, 10, 12)$, $\gamma = (20, 11, 9)$, $C^{\gamma}_{\alpha\beta} = 367$



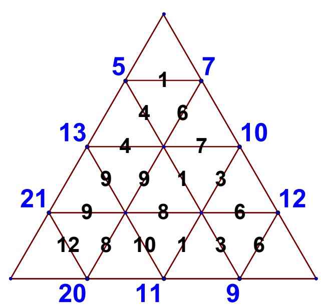
BZ triangle

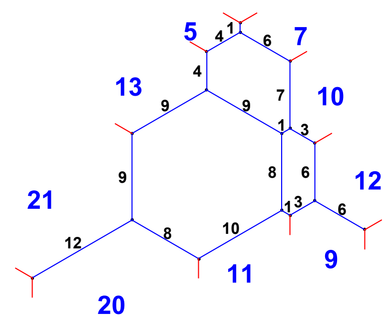


KT honeycomb . . .



and hive





Ocneanu blade and its metric dual

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = semi-classical approximation of L–R problem

or conversely

L–R problem = "quantum" Horn's problem [Knutson–Tao]

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = semi-classical approximation of L–R problem

or conversely

L–R problem = "quantum" Horn's problem [Knutson–Tao]

– Semi-classical version of multiplicity problems [Heckman 1982]

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = semi-classical approximation of L–R problem

or conversely

L–R problem = "quantum" Horn's problem [Knutson–Tao]

– Semi-classical version of multiplicity problems [Heckman 1982]

– It was noticed that (in SU(n)) the γ that appear in $\alpha \otimes \beta$ satisfy Horn's inequalities !

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = semi-classical approximation of L–R problem

or conversely

L–R problem = "quantum" Horn's problem [Knutson–Tao]

– Semi-classical version of multiplicity problems [Heckman 1982]

– It was noticed that (in SU(n)), the γ that appear in $\alpha\otimes\beta$ satisfy Horn's inequalities !

$\star$ Similarity between $C_{\alpha\,\beta}^{\gamma} = \int_G dg\,\chi_\alpha(g)\chi_\beta(g)\chi_\gamma^*(g)$ and expression of $\mathcal{J}(\alpha,\beta;\gamma) \propto \int_{\mathfrak{t}} dx|\Delta_{\mathfrak{g}}(x)|^2\,\mathcal{H}(\alpha,\mathrm{i}\,x)\mathcal{H}(\beta,\mathrm{i}\,x)\mathcal{H}(\gamma,\mathrm{i}\,x)^*$

$\mathfrak{t}$ =Cartan subalgebra. Recall $\Delta_{\mathfrak{g}}(x) := \prod_{\boldsymbol{\alpha}>0}\langle\boldsymbol{\alpha},x\rangle$, $\boldsymbol{\alpha}$ the roots of $\mathfrak{g}$.

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = semi-classical approximation of L–R problem

or conversely

L–R problem = "quantum" Horn's problem [Knutson–Tao]

– Semi-classical version of multiplicity problems [Heckman 1982]

– It was noticed that (in SU(n)), the γ that appear in $\alpha \otimes \beta$ satisfy Horn's inequalities !

⋆ Similarity between $C_{\alpha\,\beta}^{\gamma} = \int_G dg\, \chi_\alpha(g)\chi_\beta(g)\chi_\gamma^*(g)$ and expression of $\mathcal{J}(\alpha,\beta;\gamma) \propto \int_{\mathfrak{t}} dx\, |\Delta_{\mathfrak{g}}(x)|^2\, \mathcal{H}(\alpha,\mathrm{i}\,x)\mathcal{H}(\beta,\mathrm{i}\,x)\mathcal{H}(\gamma,\mathrm{i}\,x)^*$

$\mathfrak{t}$ =Cartan subalgebra. Recall $\Delta_{\mathfrak{g}}(x) := \prod_{\boldsymbol{\alpha}>0}\langle\boldsymbol{\alpha},x\rangle$, $\boldsymbol{\alpha}$ the roots of $\mathfrak{g}$.

⋆ Kirillov orbit theory: orbit ↔ irrep.

Weyl character ↗ $$\frac{\chi_\alpha(e^{\mathrm{i}\,x})}{\dim V_\alpha} = \frac{\Delta_{\mathfrak{g}}(\mathrm{i}\,x)}{\hat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x})}\mathcal{H}(\alpha+\rho,\mathrm{i}\,x) \qquad x \in \mathfrak{t}$$

where $\rho = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}$ and $\hat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x}) := \prod_{\boldsymbol{\alpha}>0}\left(e^{\frac{\mathrm{i}}{2}\langle\boldsymbol{\alpha},x\rangle} - e^{-\frac{\mathrm{i}}{2}\langle\boldsymbol{\alpha},x\rangle}\right)$ is "Weyl denominator".

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = a semi-classical approximation of L–R problem
– Semi-classical version of multiplicity problems [Heckman 1982], [Guillemin–Lerman–Sternberg]

– It was noticed that (in SU(n)), the γ that appear in $\alpha\otimes\beta$ satisfy Horn's inequalities !

⋆ Similarity between $C_{\alpha\,\beta}^{\gamma}=\int_G dg\,\chi_\alpha(g)\chi_\beta(g)\chi_\gamma^*(g)$ and
expression of $\mathcal{J}(\alpha,\beta;\gamma)\propto\int_{\mathfrak{t}}dx|\Delta_{\mathfrak{g}}(x)|^2\,\mathcal{H}(\alpha,\mathrm{i}\,x)\mathcal{H}(\beta,\mathrm{i}\,x)\mathcal{H}(\gamma,\mathrm{i}\,x)^*$ $\mathfrak{t}$ =Cartan subalgebra

⋆ Kirillov orbit theory: orbit ↔ irrep.

Weyl character ↗ $$\frac{\chi_\alpha(e^{\mathrm{i}\,x})}{\dim V_\alpha}=\frac{\Delta_{\mathfrak{g}}(\mathrm{i}\,x)}{\hat\Delta_{\mathfrak{g}}(e^{\mathrm{i}\,x})}\mathcal{H}(\alpha+\rho,\mathrm{i}\,x)\qquad x\in\mathfrak{t}$$

where $\rho=\frac{1}{2}\sum_{\boldsymbol{\alpha}>0}\boldsymbol{\alpha}$ and $\hat\Delta_{\mathfrak{g}}(e^{\mathrm{i}\,x}):=\prod_{\boldsymbol{\alpha}>0}\left(e^{\frac{\mathrm{i}}{2}\langle\boldsymbol{\alpha},x\rangle}-e^{-\frac{\mathrm{i}}{2}\langle\boldsymbol{\alpha},x\rangle}\right)$ is "Weyl denominator".

⋆ In fact, if $C_{\alpha\,\beta}^{\gamma}\neq 0$ [Coquereaux–McSwiggen–Z]

$$\mathcal{J}(\alpha+\rho,\beta+\rho;\gamma+\rho)=\sum_{\kappa\in K,\tau}r_\kappa\,C_{\alpha\,\beta}^{\tau}C_{\tau\,\kappa}^{\gamma}$$

K a finite, G-dependent but α, β-independent, set of weights, $r_\kappa>0$.
[Coquereaux–Z, Etingof–Rains]
For example in $\mathfrak{g}=\mathfrak{su}(3)$, $K=\{0\}$; $\mathcal{J}(\alpha+\rho,\beta+\rho;\gamma+\rho)=C_{\alpha\,\beta}^{\gamma}$.
⋆ Can one invert and express the LR coefficients in terms of the volumes $\mathcal{J}$?
("Box spline deconvolution") [McSwiggen]

In fact, LR problem is closely connected to Horn's problem!

Horn's problem = semi-classical approximation of L–R problem
or conversely
L–R problem = "quantum" Horn's problem [Knutson–Tao]
– Semi-classical version of multiplicity problems [Heckman 1982]

– It was noticed that (in SU(n)), the γ that appear in $\alpha \otimes \beta$ satisfy Horn's inequalities !
⋆ Similarity between $C_{\alpha\,\beta}^{\gamma} = \int_G dg\, \chi_\alpha(g)\chi_\beta(g)\chi_\gamma^*(g)$ and
expression of $\mathcal{J} \propto \int_{\mathfrak{t}} dx |\Delta_{\mathfrak{g}}(x)|^2\, \mathcal{H}(\alpha, \mathrm{i}\,x)\mathcal{H}(\beta, \mathrm{i}\,x)\mathcal{H}(\gamma, \mathrm{i}\,x)^*$ $\mathfrak{t}$ =Cartan subalgebra

⋆ Kirillov orbit theory: orbit ↔ irrep.

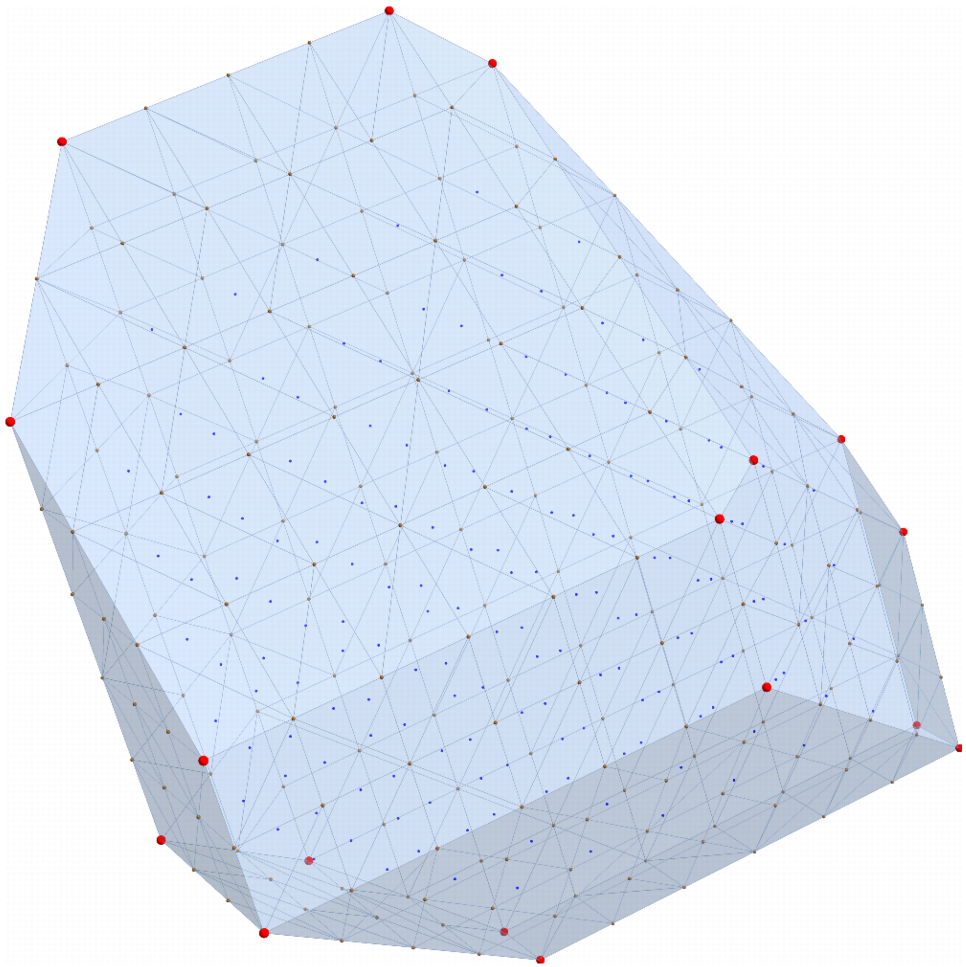
$$\frac{\chi_\alpha(e^{\mathrm{i}\,x})}{\dim V_\alpha} = \frac{\Delta_{\mathfrak{g}}(\mathrm{i}\,x)}{\hat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x})}\mathcal{H}(\alpha+\rho, \mathrm{i}\,x) \qquad x \in \mathfrak{t}$$

where $\rho = \frac{1}{2}\sum_{\boldsymbol{\alpha}>0} \boldsymbol{\alpha}$ and $\hat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x}) := \prod_{\boldsymbol{\alpha}>0}\left(e^{\frac{\mathrm{i}}{2}\langle\boldsymbol{\alpha},x\rangle} - e^{-\frac{\mathrm{i}}{2}\langle\boldsymbol{\alpha},x\rangle}\right)$ is "Weyl denominator".

⋆ Combinatorial viewpoint: $C_{\alpha\,\beta}^{\gamma}$ =number of integer points in a certain polytope of volume $\propto \mathcal{J}$ [Berenstein–Zelevinsky '90s, Knutson–Tao '99]
hence expect for "large weights", Volume $= \mathcal{J}(\alpha,\beta;\gamma) \approx$ #points $= C_{\alpha\,\beta}^{\gamma}$ . . .

rescaling by $s \in \mathbb{N}$ $\qquad C_{s\alpha\, s\beta}^{s\gamma} = P_{\alpha\,\beta}^{\gamma}(s) = s^d \mathcal{J}(\alpha,\beta;\gamma) + \cdots$

$P_{\alpha\,\beta}^{\gamma}(s)$: Ehrhart (quasi-)polynomial $\qquad d \le (n-1)(n-2)/2$ for $su(n)$



The BZ polytope for $su(4)$, and $\alpha = (21,13,5)$, $\beta = (7,10,12)$, $\gamma = (20,11,9)$. It has $C^{\gamma}_{\alpha\beta} = 367$ integer points and a volume $\mathcal{J}(\alpha,\beta;\gamma) = 742/3$

Summary and open issues

PDF in SU(n) cases and other coadjoint orbits ✓
USp(n) orbits of Quaternionic Self-Dual matrices ✓

In the SO(3) case, general formula for PDF $\mathrm{p}(\gamma_1, \gamma_2)$
– which reproduces (in the special case $\alpha = \beta = (1, 0, -1)$) the numerical simulations,
– and enables one to determine the nature of these divergences.

Extend the discussion to similar cases: Schur/Kostka, minor/branching . . . [C–Z,Z],
"quantum marginals" [Collins–McS, McS–Matsumoto]

What is missing
⋆ a better, more systematic approach to ρ, its singularities, etc.
⋆ what happens in SO(n) for $n > 3$? Singularities, but of which type ?
⋆ geometric interpretation of singularities? coordinate singularity [C–McS-Z]. . .

Summary and open issues

PDF in SU(n) cases and other coadjoint orbits ✓
USp(n) orbits of Quaternionic Self-Dual matrices ✓

In the SO(3) case, general formula for PDF $\mathrm{p}(\gamma_1, \gamma_2)$
– which reproduces (in the special case $\alpha = \beta = (1, 0, -1)$) the numerical simulations,
– and enables one to determine the nature of these divergences.

Extend the discussion to similar cases: Schur/Kostka, minor/branching ... [C–Z,Z], "quantum marginals" [Collins–McS, McS–Matsumoto]

What is missing
⋆ a better, more systematic approach to ρ, its singularities, etc.
⋆ what happens in SO(n) for $n > 3$? Singularities, but of which type ?
⋆ geometric interpretation of singularities? coordinate singularity [C–McS-Z]...

Combinatorial/probabilistic issues...
⋆⋆ Crossover between finite n and $n \to \infty$ limit (free probability) [Biane]

Summary and open issues

PDF in SU(n) cases and other coadjoint orbits ✓
USp(n) orbits of Quaternionic Self-Dual matrices ✓

In the SO(3) case, general formula for PDF $\rho(p,q)$ or $\mathrm{p}(\gamma_1,\gamma_2)$
– which reproduces (in the special case $\alpha=\beta=(1,0,-1)$) the numerical simulations,
– and enables one to determine the nature of these divergences.

Extend the discussion to similar cases: Schur/Kostka, minor/branching . . . [C–Z,Z], "quantum marginals" [Collins–McS, McS–Matsumoto]

What is missing
⋆ a better, more systematic approach to ρ, its singularities, etc.
⋆ what happens in SO(n) for $n>3$? Singularities, but of which type ?
⋆ geometric interpretation of singularities? coordinate singularity [C–McS-Z]. . .

Combinatorial/probabilistic issues. . .

⋆⋆ Crossover between finite n and $n\to\infty$ limit (free probability) [Biane]
[Narayanan–Sheffield–Tao]

Another challenging question (for the physicist):
⋆⋆ are the enhancements of certain eigenvalues observable in some physical process ?

Thank you !

Appendices

A little calculation. . . Notation $\alpha' = \alpha + \rho$ etc. $\frac{\chi_\alpha(e^{\mathrm{i}\,x})}{\dim V_\alpha} = \frac{\Delta_{\mathfrak{g}}(\mathrm{i}\,x)}{\widehat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x})}\mathcal{H}(\alpha', \mathrm{i}\,x)$

Assume $\alpha + \beta - \gamma \in Q$ (otherwise $C^\gamma_{\alpha\beta} = 0$)

$$\mathcal{J}(\alpha', \beta'; \gamma') := \dim V_\alpha \dim V_\beta \dim V_\gamma \int_{\mathfrak{t} \simeq \mathbb{R}^r} d^r x \, |\Delta_{\mathfrak{g}}(x)|^2 \, \mathcal{H}(\alpha', \mathrm{i}\,x)\mathcal{H}(\beta', \mathrm{i}\,x)(\mathcal{H}(\gamma', \mathrm{i}\,x))^*$$

$$= \int_{\mathfrak{t}} \underbrace{d^r x \, |\widehat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x})|^2}\, \frac{\widehat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,x})}{\Delta_{\mathfrak{g}}(\mathrm{i}\,x)} \chi_\alpha(e^{\mathrm{i}\,x})\chi_\beta(e^{\mathrm{i}\,x})(\chi_\gamma(e^{\mathrm{i}\,x}))^*$$

$$(\mathbb{T} = \mathfrak{t}/P^\vee) \qquad = \int_{\mathbb{T}} dT \sum_{\delta \in P^\vee} \frac{\widehat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,(x+\delta)})}{\Delta_{\mathfrak{g}}(\mathrm{i}\,(x+\delta))} \chi_\alpha(e^{\mathrm{i}\,(x+\delta)})\chi_\beta(e^{\mathrm{i}\,(x+\delta)})(\chi_\gamma(e^{\mathrm{i}\,(x+\delta)}))^*$$

$$= \int_{\mathbb{T}} dT \underbrace{\left(\sum_{\delta \in P^\vee} e^{\mathrm{i}\,\langle \rho, \delta \rangle} \frac{\widehat{\Delta}_{\mathfrak{g}}(e^{\mathrm{i}\,(x+\delta)})}{\Delta_{\mathfrak{g}}(\mathrm{i}\,(x+\delta))} \right)} \chi_\alpha(e^{\mathrm{i}\,x})\chi_\beta(e^{\mathrm{i}\,x})(\chi_\gamma(e^{\mathrm{i}\,x}))^*$$

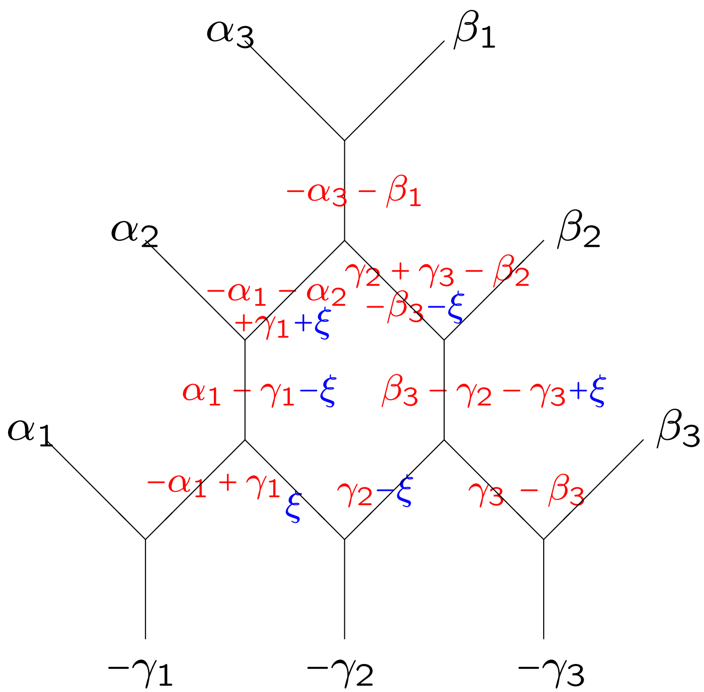
$$= \int_{\mathbb{T}} dT \sum_{\kappa \in K} r_\kappa \chi_\kappa(T) \qquad \chi_\alpha(e^{\mathrm{i}\,x})\chi_\beta(e^{\mathrm{i}\,x})(\chi_\gamma(e^{\mathrm{i}\,x}))^*$$

$$= \sum_{\kappa \in K, \tau} r_\kappa C^\tau_{\alpha\beta} C^\nu_{\tau\kappa} = \sum_{\kappa \in K} r_\kappa N^\nu_{\alpha\beta\kappa} \, .$$

with a finite set of weights K independent of α, β, γ, $r_\kappa \geq 0$, $\sum_\kappa r_\kappa \dim V_\kappa = 1$.

Generalization of $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{u + (2\pi)n} = \frac{1}{2 \sin(u/2)}$

From Knutson–Tao's honeycombs to Horn's inequalities. Example $n = 3$



$$\max(\alpha_1 - \gamma_1 + \gamma_2, \gamma_3 - \beta_3, \alpha_2, -\beta_2 + \gamma_2, \alpha_1 + \alpha_3 + \beta_1 - \gamma_1, \alpha_1 + \alpha_2 + \beta_2 - \gamma_1)$$
$$\le \xi \le \min(\alpha_1, -\beta_3 + \gamma_2, \alpha_1 + \alpha_2 + \beta_1 - \gamma_1)$$

$\Leftrightarrow$ Horn's inequalities (for $n = 3$)