On Completely Integrable Pfaffian Systems with Normal Crossings

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Abstract

Let $\mathbb{C}$ be the field of complex numbers and $x = (x_1, \dots, x_m)^T \in \mathbb{C}^m$. We denote by $\mathbb{C}[[x]]$ the ring of formal power series in m variables over C and by $\mathbb{C}((x))$ its field of fractions. The completely integrable Pfaffian systems with normal crossings is the class of linear systems of partial differential equations in dimension n and m variables of the form

$$\begin{cases} x_1^{p_1+1} \frac{\partial Y}{\partial x_1} = A^{(1)}(x) Y \\ \vdots \\ x_m^{p_m+1} \frac{\partial Y}{\partial x_m} = A^{(m)}(x) Y \end{cases} \tag{1}$$

where, for $1 \leq i \leq m$, p_i is a nonnegative integer, $(p_1, \dots, p_m)$ is called the rank of (1), and $A^{(i)}(x)$ is $n \times n$- matrix with entries in $\mathbb{C}[[x]]$. For, $1 \leq i, j \leq m$ the latter satisfy the integrability condition:

$$x_i^{p_i+1} \frac{\partial A^{(j)}}{\partial x_i} + A^{(j)} A^{(i)} = x_j^{p_j+1} \frac{\partial A^{(i)}}{\partial x_j} + A^{(i)} A^{(j)}$$

The change of variable $Y = T(x)Z$, where the gauge transformation $T(x) \in GL_n(\mathbb{C}((x)))$, takes the system above into an equivalent system

$$\begin{cases} x_1^{q_1+1} \frac{\partial Y}{\partial x_1} = B^{(1)}(x) Y \\ \vdots \\ x_m^{q_m+1} \frac{\partial Y}{\partial x_m} = B^{(m)}(x) Y \end{cases} \tag{2}$$

In fact,

$$\frac{B^{(i)}}{{x_i}^{q_i+1}} = T^{-1} \frac{A^{(i)}}{{x_i}^{p_i+1}} T - T^{-1} \frac{\partial T}{\partial x_i} \qquad \text{for } i = 1, \dots, m$$

and $B^{(i)}$'s simultaneously satisfy the complete integrability condition.

We say a change of variable $Y = T(x)Z$ is *compatible* with (1) whenever the normal crossing in (2) is preserved, i.e. $B^{(i)}(x) \in \mathbb{C}[[x]]$, and $q_i \leq p_i, 1 \leq i \leq m$.

We call *Poincaré rank* the minimal rank of (1), i.e. the one for which the q_i's are simultaneously and individually the smallest possible. In particular, if (1) is equivalent to a system(2) with $q_1 = 0, \dots, q_m = 0$ then it is called *regular singular*.
A formal fundamental solution of (1) is given in [8, 12] by the form:

$$\Phi(t) \prod_{i=1}^{m} x_i^{\Lambda_i} \prod_{i=1}^{m} e^{P_i(\frac{1}{t_i})}$$

where $t = (t_1, \dots, t_m)$ such that $t_i = {x_i}^{\frac{1}{s_i}}$ for some positive integer s_i, for all $1 \leq i \leq m$; $\Phi(t)$ is an invertible matrix of entries in $\mathbb{C}[[t]]$, Λ_i's are constant diagonal matrices, and $P_i(\frac{1}{t_i})$'s are polynomials in $\frac{1}{t_i}$ with diagonal matrix coefficients.

We are interested in the *formal reduction*, i.e. the algorithmic procedure that constructs the formal solution of (1). Formal normal forms of (1) are given in [6] (case m=2) and [12]. However, the complete formal reduction, in analogy with systems of ordinary differential equations (ODS) [1, 2], is still under investigation.
As a first step, it is natural to tackle the *rank reduction* of (1), i.e. the computation of a compatible gauge transformation which takes (1) to (2) where $(q_1, \dots, q_m)$ is the Poincaré

rank. The regularity of a Pfaffian system is shown in [7, 13] to be equivalent to the regularity of the individual systems $x_i^{p_i+1}\frac{\partial Y}{\partial x_i} = A^{(i)}(x)Y$ considered as ODS. As a consequence, regularity criteria already given for ODS (e.g. [10, 11]) can be applied to the individual systems separately to answer the question for (1). But gauge transformations used for ODS are not generally compatible with (1). Barkatou and LeRoux compute in [5] a compatible transformation based on the existence of stationary sequences of free lattices in the case of two variables (m=2). In this talk, we use results from [3, 4, 9] to give another rank reduction of (1) in the case $m = 2$ and for a generic case whenever $m > 2$. A formal reduction is given under a restrictive condition as well.

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Keywords
Pfaffian systems, Rank reduction, Gauge transformations

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