Toroflux in non-centrosymmetric superconductors

Vortex magnetic field inversion and applications

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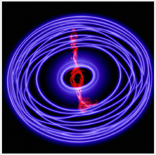
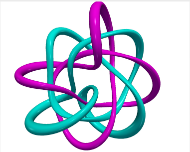
with M. Chernodub, D. Kharzeev
and also A. Samoilenka, E. Babaev, A. Korneev, A. Molochkov

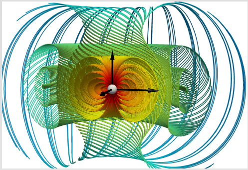
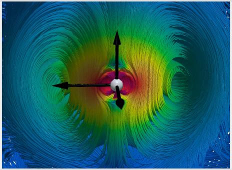
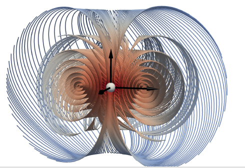
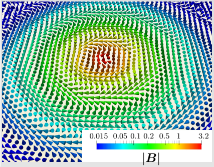
Institut Denis-Poisson, CNRS/UMR 7013, Université de Tours, France

March 21, 2024





Topological defects in non-centrosymmetric superconductors

based on

- **JG**, A. Korneev, A. Samoilenka, A. Molochkov, E. Babaev, and M. Chernodub
  Toroflux: A counterpart of the Chandrasekhar-Kendall state in noncentrosymmetric superconductors, *Phys. Rev. B* **108**, 014504 (2023). arXiv:2208.08180 [cond-mat]
- **JG**, M. N. Chernodub and D. E. Kharzeev
  Vortices with magnetic field inversion in non-centrosymmetric superconductors, *Phys. Rev. B* **102**, 184516 (2020). arXiv:2003.10917 [cond-mat]
- M. N. Chernodub, **JG** and D. E. Kharzeev
  Chiral Magnetic Josephson junction: a base for low-noise superconducting qubits?, *Universe* **8**,12:657 (2022). arXiv:1908.00392 [cond-mat]

Outline

1. **Introduction**
   - Parity breaking and non-centrosymmetricity
   - Superconductivity
   - Non-centrosymmetric superconductors
2. **Toroflux in non-centrosymmetric superconductors**
   - Chandrasekhar-Kendall states
   - Derivation of toroflux solution sourced by magnetic dipole
   - Toroflux properties and observation
3. **Vortices and applications in non-centrosymmetric superconductors**
   - Vortex solutions, field inversion
   - Applications: Chiral Magnetic Josephson junctions
   - Conclusion

Parity-breaking phenomena in physics

Parity under space inversion is a fundamental symmetry: $P(\boldsymbol{x}) \to -\boldsymbol{x}$

- **conserved**: gravitational, electromagnetic, strong interaction, ...
- **broken**: weak interactions, chirality of molecules, toplogical materials, ...

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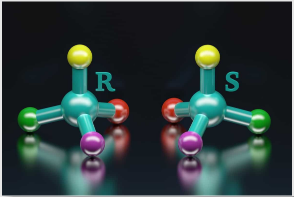
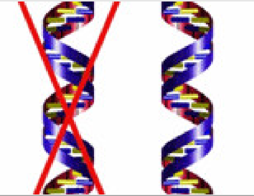
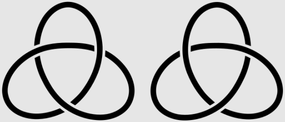


DNA



Knots



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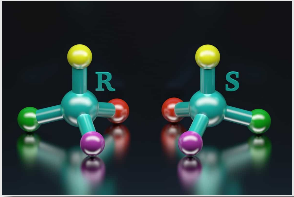

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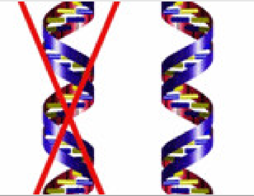

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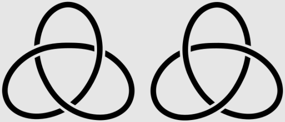


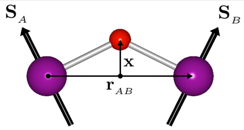
Non-centrosymmetricity in materials

Crystals or molecules lacking an inversion center
$\Rightarrow$ piezoelectricity, ferroelectricity, nonlinear optical effects, Weyl semimetals, ...
$\Rightarrow$ chiral magnets, non-centrosymmetric superconductors

Non-centrosymmetricity in (chiral) magnets

Antisymmetric exchange [Dzyaloshinskii 1958; Moriya 1960]

⇒ relativistic spin-orbit coupling effect, which couples the magnetic moments with the crystal lattice



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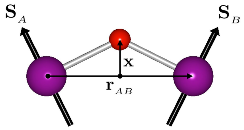

Heisenberg exchange ⇒ Landau-Lifshitz

$H \propto -J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \qquad \propto \sum_i (\boldsymbol{\nabla}_i \boldsymbol{m})^2$

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Dzyaloshinskii-Moriya ⇒ Lifshitz invariants

$H_{\mathrm{DM}} \propto \sum_{\langle ij \rangle} \boldsymbol{D}_{ij} \cdot \boldsymbol{S}_i \times \boldsymbol{S}_j \qquad \propto \mathcal{L}_{ij}^{(k)} = m_i \partial_k m_j$

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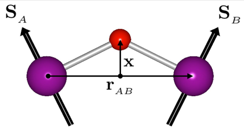

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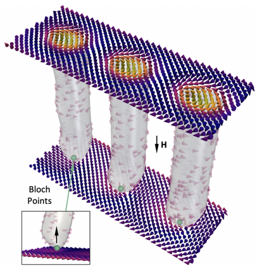
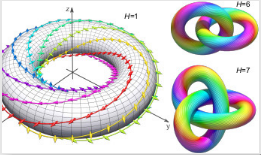
Stabilizes chiral spin textures (e.g. in MnSi or FeGe)

can have important consequences on domain-wall motion, skyrmions, and spin-orbit torques, ...

Skyrmions



Hopfions



Non-centrosymmetric superconductors

Novel effects in non-centrosymmetric superconductors

Earlier theoretical works: [Bulaevskii, Guseinov, Rusinov 1976; Levitov, Nazarov, Éliashberg 1985; Mineev, Samokhin 1994]

- unusual magnetoelectric transport, helical states, exotic vortex lattices, unconventional Josephson effect, ...

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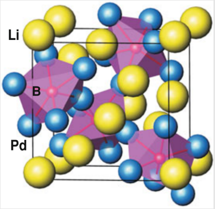
Various known noncentrosymmetric superconductors

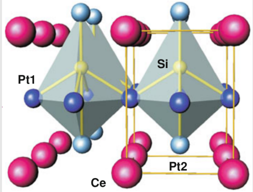
- With cubic O-point group symmetry: Li_2Pd_3B [Yuan, Agterberg, et al. 2006; Badica, et al. 2005], Mo_3Al_2C [Karki, et al. 2010; Bauer, et al. 2010], PtSbS [Mitzutani, et al. 2019]
- With tetragonal $C_{4\nu}$-point group symmetry: $CePt_3Si$ [Bauer, et al. 2004] $CeRhSi_3$ [Kimura, et al. 2005] , $CeIrSi_3$ [Tateiwa, et al. 2007]
- With tetrahedral T_d-point group symmetry: Y_2C_3 [Amano, et al. 2004], KOs_2O_6 [Schuck et al. 2006]

Reviews: [Bauer, Sigrist 2003], [Smidan, et al. 2017], [Yip 2014],...

Li_2Pd_3B/Li_2Pd_3B



$CePt_3Si$



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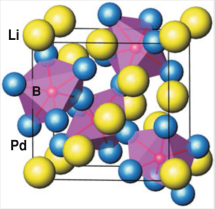

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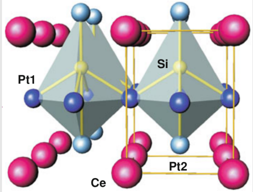

Here: O-point group

Topological defects and vortices

Topological defects are ubiquitous in modern physics

superfluid, superconductors, cold atoms BEC, (chiral) magnets, ferroelectric, (liquid) crystals, spin ices,. . .
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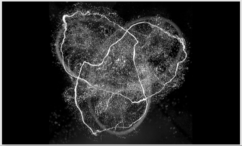
(quantum) vortices (superfluid, superconductors)

- [Onsager 1949; Feynman 1955]: circulation of the superflow is quantized;
- [London 1948, Abrikosov 1957]: magnetic vortices in superconductors
- [Onsager 1949; Peskin 1978; Dasgupta, Halperin 1981]: phase transitions: thermal proliferation of vortex loops
- [Berezinskiĭ 1971; Kosterlitz, Thouless 1972]: in 2d (V/AV)

Vortex [Helmholtz 1858]

Vortex knot



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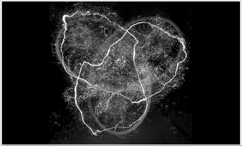


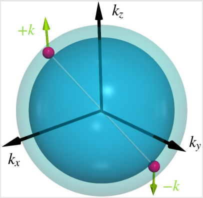
Overall, vortices are topological defects that control the thermal, rotational/magnetic responses of superfluids and superconductors

Superconductivity – Generalities

Conventional mechanism [Bardeen, Cooper, Schrieffer 1957]

- in a metal Fermi sphere of occupied states
- states near Fermi surface can interact via phonons

Fermi sphere



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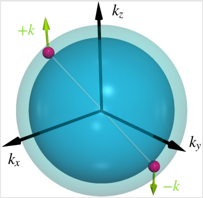

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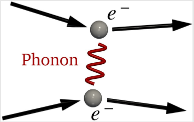

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- they can undergo Bose-Einstein condensation
- macroscopic occupation of the zero momenta states

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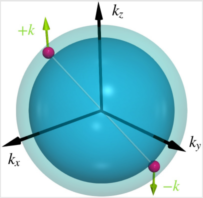

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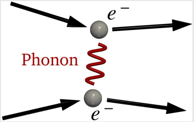

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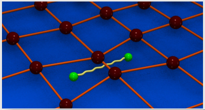

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At mean field, one single macroscopic wave function

Ginzburg-Landau: effective classical mean field theory near T_c

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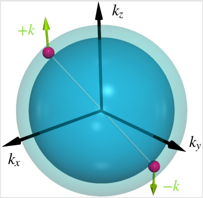

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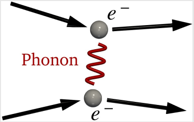

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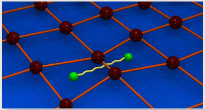


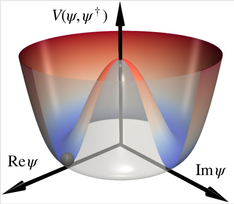
Superconductivity – Properties & Ginzburg-Landau

$$E = \int_{\mathbb{R}^3} |\boldsymbol{\nabla} \times \boldsymbol{A}|^2 + D_\mu \Psi^* D^\mu \Psi + \kappa(|\Psi|^2 - 1)^2 \,, \text{ with } D_\mu = \nabla_\mu - iA_\mu$$

Classical field theory

- at the mean field level, one macroscopic wave function (density of Cooper pairs), the gap function $\Psi = |\Psi| e^{i\varphi}$
- Ψ: charged bosonic scalar field; $\boldsymbol{A}$ gauge field (photon)
- longitudinal component of the photon becomes massive
- Anderson-Higgs mechanism [Anderson 1962; Higgs 1964]

Broken $U(1)$



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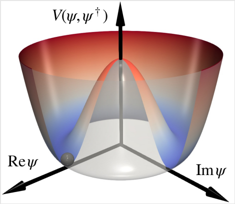
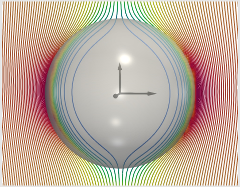


Properties of superconductors

- dissipationless current
- perfect diamagnetism (Meissner effect): $\boldsymbol{B}$ is screened by the superflow of Cooper pairs $\boldsymbol{J} = 2e|\Psi|^2(\boldsymbol{\nabla}\varphi + \boldsymbol{A})$
- Massive photon $\Rightarrow$ London eq.: $\lambda\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{B}$ (Proca)
- Quantized flux$= \frac{\Phi_0}{2\pi}\oint\boldsymbol{\nabla}\varphi\cdot\boldsymbol{d}\ell = n\Phi_0$ and $n\in\pi_1(S^1)=\mathbb{Z}$
- $\Rightarrow$ vortices [London 1948; Onsager 1949; Abrikosov 1957]

Meissner effect

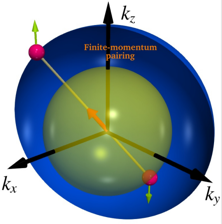


Ginzburg-Landau theory for NCS [Agterberg 2003; Samoilenka, Babaev 2020]

Origin of parity-odd terms in microscopic single-particle Hamiltonian

antisymmetric SO couplings $\boldsymbol{g_k} \cdot \boldsymbol{\sigma}$ with $\boldsymbol{g_k} = -\boldsymbol{g_{-k}}$ and $\boldsymbol{\sigma}$ acting on the spin space

Finite-*k* pairing



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Parity-odd superconductor with O point group symmetry

$$\mathcal{F} = \frac{\boldsymbol{B}^2}{8\pi} + \frac{k}{2}\sum_{a=\pm}\left|\mathcal{D}_a\psi\right|^2 + \frac{\beta}{2}(|\psi|^2 - \psi_0^2)^2$$

- where $\mathcal{D}_\pm \equiv \boldsymbol{\nabla} - ie\boldsymbol{A} + ie\varkappa_\pm \boldsymbol{B}$ $\quad\quad \boldsymbol{j} = e\,\mathrm{Im}(\psi^* \boldsymbol{D}\psi)$
- $\psi = |\psi|e^{i\varphi}$ cplx (bosonic) scalar field; $\boldsymbol{A}$ gauge field

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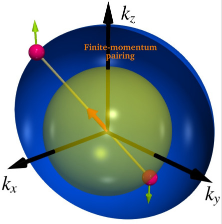

$$\kappa_\pm = \chi \pm \nu$$

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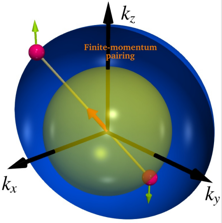
- $\psi = |\psi|e^{i\varphi}$ cplx (bosonic) scalar field; $\boldsymbol{A}$ gauge field

Parity-odd superconductor with O point group

$$\frac{1}{2}\sum_{a=\pm} |\mathcal{D}_a\psi|^2 = |\boldsymbol{D}\psi|^2 + \chi \boldsymbol{j} \cdot \boldsymbol{B} + e^2(\chi^2 + \nu^2)|\psi|^2 \boldsymbol{B}^2$$

Finite-k pairing



$$\kappa_\pm = \chi \pm \nu$$

Lifshitz invariants

$$\propto \gamma_{\mu\nu} B_\mu \mathrm{Im}(\psi^* D_\nu \psi)$$

Ginzburg-Landau theory for NCS [Agterberg 2003; Samoilenka, Babaev 2020]

Origin of parity-odd terms in microscopic single-particle Hamiltonian

antisymmetric SO couplings $\boldsymbol{g_k} \cdot \boldsymbol{\sigma}$ with $\boldsymbol{g_k} = -\boldsymbol{g_{-k}}$ and $\boldsymbol{\sigma}$ acting on the spin space

Parity-odd superconductor with O point group symmetry

$$\mathcal{F} = \frac{\boldsymbol{B}^2}{8\pi} + \frac{k}{2}\sum_{a=\pm}|\mathcal{D}_a\psi|^2 + \frac{\beta}{2}(|\psi|^2 - \psi_0^2)^2$$

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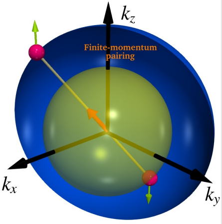

$$\kappa_\pm = \chi \pm \nu$$

Parity inversion, $P(\boldsymbol{x}) = -\boldsymbol{x}$, $\boldsymbol{B}$ is parity-even: $P(\boldsymbol{B}) = \boldsymbol{B}$

Energy is not invariant under the parity inversion:

$$P(\mathcal{F}) = \mathcal{F} - 2ek(\varkappa_+ + \varkappa_-)\boldsymbol{B}\cdot\mathrm{Im}(\psi^*\boldsymbol{D}\psi)$$

Lifshitz invariants

$$\propto \gamma_{\mu\nu}B_\mu\mathrm{Im}(\psi^* D_\nu\psi)$$

Ginzburg-Landau theory for noncentrosymmetric superconductors

Ginzburg-Landau equation $\boldsymbol{J}_a = \mathrm{Im}(\psi^* \boldsymbol{\mathcal{D}}_a \psi)$

$$\frac{\delta \mathcal{F}}{\delta \psi^*} \Rightarrow \qquad k \sum_{a=\pm} \boldsymbol{\mathcal{D}}_a \boldsymbol{\mathcal{D}}_a \psi = 2\beta(|\psi^2| - \psi_0^2)\psi$$

$$\frac{\delta \mathcal{F}}{\delta \boldsymbol{A}} \Rightarrow \qquad \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{B}}{4\pi} + ke \sum_{a=\pm} \varkappa_a \boldsymbol{J}_a \right) = ke \sum_{a=\pm} \boldsymbol{J}_a$$

Length-scales

$$\lambda_L = \lambda_0 \sqrt{1 + \frac{\varkappa_+^2 + \varkappa_-^2}{2\lambda_0^2}} ,$$

$$\lambda_0^2 = \frac{1}{8\pi k e^2 \psi_0^2} , \quad \xi^2 = \frac{k}{2\beta\psi_0^2}$$

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London limit of the free energy $|\psi| = \psi_0$

$$\tilde{\mathcal{F}}_L = \left\{ \boldsymbol{B}^2 + \hat{\boldsymbol{\jmath}}^2 + 2\Gamma \hat{\boldsymbol{\jmath}} \cdot \boldsymbol{B} \right\}$$

Dimensionless

$$0 \leqslant \Gamma = \frac{\chi}{\lambda_L} \leqslant 1$$

$$\hat{\boldsymbol{\jmath}} = \frac{\boldsymbol{j}}{2\lambda_L e^2 \psi_0^2}$$

$$\tilde{\boldsymbol{x}} = \boldsymbol{x}/\lambda_L \,, \quad \tilde{\boldsymbol{\nabla}} = \lambda_L \boldsymbol{\nabla}$$

$$\tilde{\mathcal{F}}_L := \frac{\mathcal{F}_L}{k\lambda_L^2 e^2 \psi_0^2}$$

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$$\tilde{\boldsymbol{\nabla}} \times \boldsymbol{H} = \tilde{\boldsymbol{\nabla}} \times (\boldsymbol{B} - 4\pi \boldsymbol{M}) = \hat{\mathbf{J}} \,,$$

where $\boldsymbol{H} = \boldsymbol{B} + \Gamma \hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{J}} = \hat{\boldsymbol{\jmath}} + \Gamma \boldsymbol{B}$, and $\boldsymbol{M} = -\frac{\Gamma \hat{\boldsymbol{\jmath}}}{4\pi}$,

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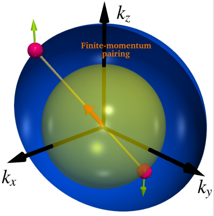
$$\tilde{\mathcal{F}}_L := \frac{\mathcal{F}_L}{k \lambda_L^2 e^2 \psi_0^2}$$

Now everything depends on the single (dimensionless) parity-breaking parameter Γ
$\Rightarrow$ Γ is expected to be small (theor. and exper.)

London theory for noncentrosymmetric superconductors

the Ampère-Maxwell eq.: $\tilde{\nabla}\times(\boldsymbol{B}+\Gamma\hat{\jmath})=\hat{\jmath}+\Gamma\boldsymbol{B}$

Finite-k pairing



Due to antisymmetric spin-orbit coupling

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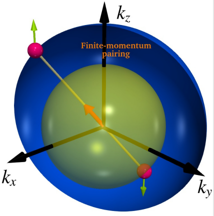
2nd London eq. obtained from current $\boldsymbol{j}=2e\psi_0^2(\nabla\varphi-e\boldsymbol{A})$

$\boldsymbol{B}=\Phi_0\boldsymbol{v}-\tilde{\nabla}\times\hat{\jmath}$, where $\boldsymbol{v}=\frac{1}{2\pi}\nabla\times\nabla\varphi$ is the density of vortex field, which accounts for the phase singularities

Finite-k pairing



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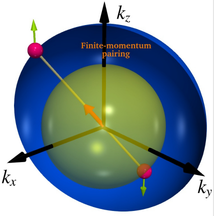
London eq. for noncentrosymmetric superconductor

$$\tilde{\boldsymbol{\nabla}}\times\tilde{\boldsymbol{\nabla}}\times\hat{\boldsymbol{\jmath}}-2\Gamma\tilde{\boldsymbol{\nabla}}\times\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{\jmath}}=\Phi_0\left(\tilde{\boldsymbol{\nabla}}\times\boldsymbol{v}-\Gamma\boldsymbol{v}\right)$$

Finite-*k* pairing



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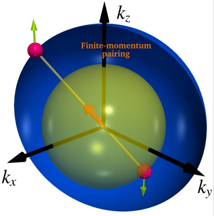
Shorthand notation, with the operator: $\mathcal{L}\hat{\jmath} = \tilde{\nabla} \times \hat{\jmath} - \eta\hat{\jmath}$

$\mathcal{L}\mathcal{L}^*\hat{\jmath} = \Phi_0 \mathrm{Re}\left[\mathcal{L}^* \boldsymbol{v}\right]$ and $\tilde{\mathcal{F}}_L = (\mathcal{L}^*\hat{\jmath} - \Phi_0 \boldsymbol{v}) \cdot (\mathcal{L}\hat{\jmath} - \Phi_0 \boldsymbol{v})$

Finite-k pairing



Due to antisymmetric spin-orbit coupling

$\mathbb{C}$ parameter

$\eta = \Gamma + i\sqrt{1 - \Gamma^2}$

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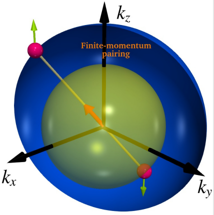

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$\eta=\Gamma+i\sqrt{1-\Gamma^2}$

The London eq. allows for new analytic solutions in parity-breaking medium

- Toroflux: knotted configs. of $\boldsymbol{B}$, analogous to Chandrasekhar-Kendall states
- Vortices, which feature inversion of the magnetic field...

Outline

1. Introduction
   - Parity breaking and non-centrosymmetricity
   - Superconductivity
   - Non-centrosymmetric superconductors
2. **Toroflux in non-centrosymmetric superconductors**
   - Chandrasekhar-Kendall states
   - Derivation of toroflux solution sourced by magnetic dipole
   - Toroflux properties and observation
3. Vortices and applications in non-centrosymmetric superconductors
   - Vortex solutions, field inversion
   - Applications: Chiral Magnetic Josephson junctions
   - Conclusion

Chandrasekhar-Kendall states [Chandrasekhar, Kendall 1957]

Force-free magnetic field equation

- the electric current is parallel to the magnetic field

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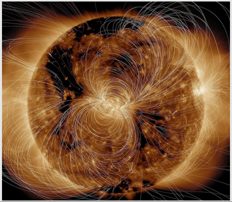
[Chandrasekhar, Kendall 1957]

Force-free magnetic field equation

- the electric current is parallel to the magnetic field
- astrophysical plasma, solar corona

Force-free plasma



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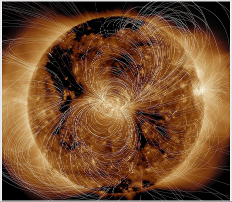
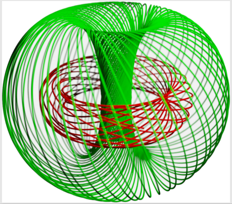


Spheromak



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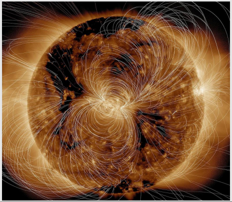
Divergence-free eigenfunctions of the curl operator

$$\nabla \times \boldsymbol{H} = \lambda \boldsymbol{H}\,, \quad \text{and} \quad \nabla \cdot \boldsymbol{H} = 0$$

$$\Rightarrow \quad \nabla \times \nabla \times \boldsymbol{H} = \lambda^2 \boldsymbol{H}$$

Force-free plasma



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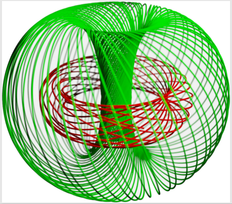

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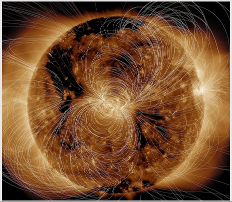
Decomposition in toroidal-poloidal fields [CK 1957]

$$\boldsymbol{H}=\frac{1}{\lambda}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\psi\boldsymbol{n})+\boldsymbol{\nabla}\times\psi\boldsymbol{n}$$

- $\boldsymbol{n}$ is a unit vector
- ψ solves Helmoltz equation: $\nabla^2\psi+\lambda^2\psi=0$

Force-free plasma



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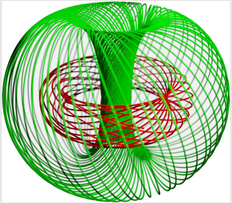

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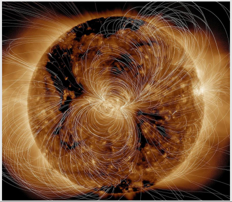

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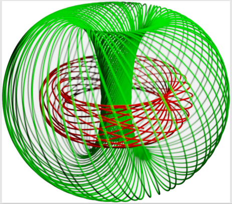

Toroflux are analogous to CK-states in NCS (thus for massive vector field)

Localized force-free solutions

[JG, et al. 2022]

Source-free ($v = 0$) London equation $\eta = \Gamma + i\sqrt{1-\Gamma^2}$

$$\mathcal{L}\mathcal{L}^*\hat{\boldsymbol{\jmath}} = 0\,, \quad \text{where} \quad \mathcal{L}\hat{\boldsymbol{\jmath}} = \tilde{\boldsymbol{\nabla}}\times\hat{\boldsymbol{\jmath}} - \eta\hat{\boldsymbol{\jmath}}$$

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Complex force-free vector field $\mathcal{Q}$

$\mathcal{L}\mathcal{Q} = 0$, then London eq. $\mathcal{L}^*\hat{\boldsymbol{\jmath}} = i\mathrm{Im}(\eta)\mathcal{Q}$

Physical fields

$$\hat{\boldsymbol{\jmath}} = \mathrm{Re}\mathcal{Q}\,,\ \boldsymbol{J} = \sqrt{1-\Gamma^2}\mathrm{Im}(\eta\mathcal{Q})\,,$$

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$$\mathcal{Q}(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(\sum_{\boldsymbol{Z}=\boldsymbol{Y},\boldsymbol{\Psi},\boldsymbol{\Phi}} Q_{lm}^{\boldsymbol{Z}}(r)\,\boldsymbol{Z}_{lm}(\hat{\boldsymbol{r}})\right)$$

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Vector spherical harmonics

$$\boldsymbol{Y}_{lm}(\hat{\boldsymbol{r}}) = Y_{lm}(\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,, \quad (\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r)$$

$$\boldsymbol{\Psi}_{lm}(\hat{\boldsymbol{r}}) = r\boldsymbol{\nabla}Y_{lm}(\hat{\boldsymbol{r}})\,,$$

$$\boldsymbol{\Phi}_{lm}(\hat{\boldsymbol{r}}) = \boldsymbol{r}\times\boldsymbol{\nabla}Y_{lm}(\hat{\boldsymbol{r}})\,.$$

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The $\mathcal{L}\mathcal{Q}=0$ yields solutions bounded at ∞

$$Q^{\boldsymbol{\Phi}}_{lm}=c_{lm}h^{(1)}_l(\eta r)\,,\quad Q^{\boldsymbol{Y}}_{lm}=-c_{lm}\frac{l(l+1)}{\eta r}h^{(1)}_l(\eta r)\,,$$

$$Q^{\boldsymbol{\Psi}}_{lm}=-c_{lm}\left(\frac{l+1}{\eta r}h^{(1)}_l(\eta r)-h^{(1)}_{l+1}(\eta r)\right)\,.$$

Physical fields

$$\hat{\jmath}=\mathrm{Re}\mathcal{Q}\,,\ \boldsymbol{J}=\sqrt{1-\Gamma^2}\mathrm{Im}(\eta\mathcal{Q})\,,$$

$$\boldsymbol{B}=-\mathrm{Re}(\eta\mathcal{Q})\,,\ \boldsymbol{H}=\sqrt{1-\Gamma^2}\mathrm{Im}(\mathcal{Q})$$

Vector spherical harmonics

$$\boldsymbol{Y}_{lm}(\hat{\boldsymbol{r}})=Y_{lm}(\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,,\quad(\hat{\boldsymbol{r}}\equiv\boldsymbol{r}/r)$$

$$\boldsymbol{\Psi}_{lm}(\hat{\boldsymbol{r}})=r\boldsymbol{\nabla}Y_{lm}(\hat{\boldsymbol{r}})\,,$$

$$\boldsymbol{\Phi}_{lm}(\hat{\boldsymbol{r}})=\boldsymbol{r}\times\boldsymbol{\nabla}Y_{lm}(\hat{\boldsymbol{r}})\,.$$

Spherical Hankel functions

$$h^{(1)}_l(z)=j_l(z)+iy_l(z)$$

j_l, y_l: spherical Bessel functions

Toroflux: force-free magnetic solutions induced by dipoles

All modes are singular $\Rightarrow$ need regularization

$$Q^{\Phi}_{lm} \sim r^{-(l+2)}\,, \quad Q^{\mathbf{Y}}_{lm} \sim r^{-(l+2)}\,, \quad Q^{\Psi}_{lm} \sim r^{-(l+1)}\,.$$

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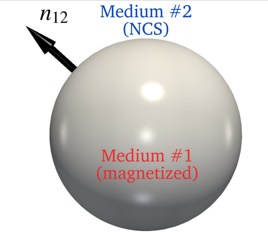
Magnetized domain

$$\tilde{\nabla}\times \boldsymbol{H} = 0\,, \quad \tilde{\nabla}\cdot \boldsymbol{B} = 0\,, \quad \text{where} \quad \boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M}\,.$$

- Decompose inner solutions on VSH: $\boldsymbol{Y}, \boldsymbol{\Phi}, \boldsymbol{\Psi}$
- Matching: $0 = \boldsymbol{J}\cdot\boldsymbol{n}_{12}\,\big|_{r=r_0}$ and $0 = \boldsymbol{n}_{12}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1)\,\big|_{r=r_0}$

Magnetized inclusion



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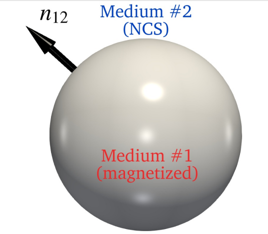

Matching conditions imply that

$$c_{lm} = \frac{4\pi r_0 \check{M}^{\boldsymbol{Y}}_{lm}(r_0)}{l(2l+1)h^{(1)}_l(\eta r_0)} \quad \text{for } l > 0$$

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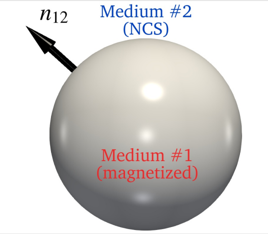

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Ferromagnetic inclusion

$$\check{\boldsymbol{M}} = M_0\hat{\boldsymbol{z}} = M_0\left(\hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)$$
$$= \sqrt{\frac{4\pi}{3}}M_0\left(\boldsymbol{Y}_{10}+\boldsymbol{\Psi}_{10}\right)\,.$$
$$\Rightarrow c_{10} = \frac{r_0 M_0}{h^{(1)}_1(\eta r_0)}\left(\frac{4\pi}{3}\right)^{3/2}$$

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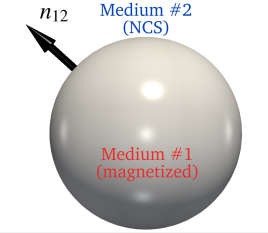

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For a pointlike magnetic dipole ($r_0 \to 0$)

$$c_{10} = i\sqrt{\frac{4\pi}{3}}\eta^2 M^d_0$$

Ferromagnetic inclusion

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Toroflux: force-free magnetic solutions induced by dipoles

The force-free field $\mathcal{Q}$ accosiated to the $(l, m) = (1, 0)$ harmonics

$$\mathcal{Q}_{10} = -M_0^d \frac{e^{i\eta r}}{\eta r^3}\Big[(1 - i\eta r)\big(2\cos\theta \boldsymbol{e}_r + \eta r \sin\theta \boldsymbol{e}_\varphi\big) + \big(1 - i\eta r(1 - i\eta r)\big)\sin\theta \boldsymbol{e}_\theta\Big]$$

Parity-breaking param.

$$\eta = \Gamma + i\sqrt{1 - \Gamma^2}$$

Toroflux size

$$L_{\text{toroflux}} = \frac{\lambda_L}{\sqrt{1-\Gamma^2}}$$

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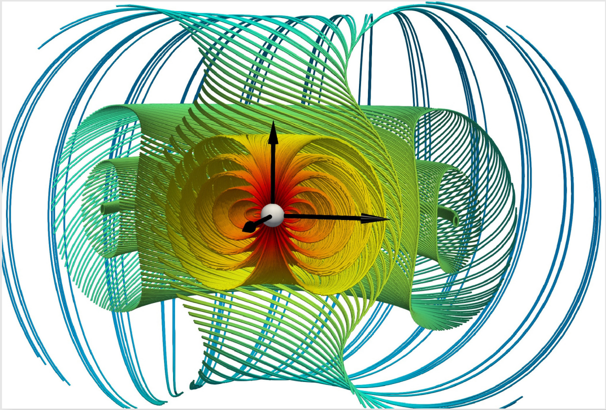
Toroflux solution ($\Gamma = 0.5$) Streamlines of $\boldsymbol{H}$



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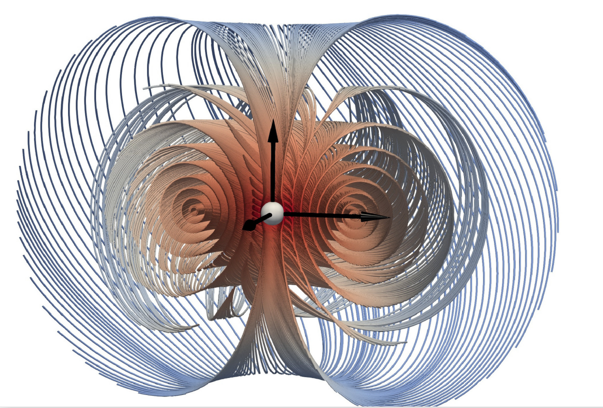
Toroflux solution ($\Gamma = 0.5$) Streamlines of $\boldsymbol{J}$



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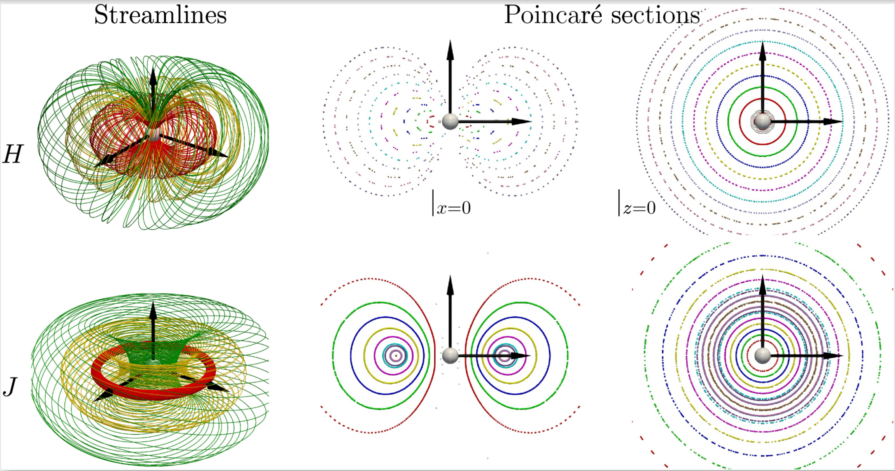
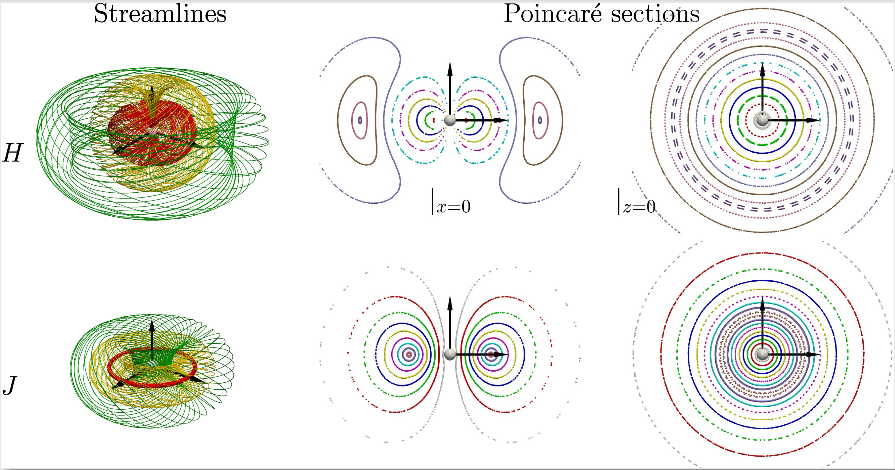
Knotted nature of the toroflux

Toroflux solution for $\Gamma = 0.15$



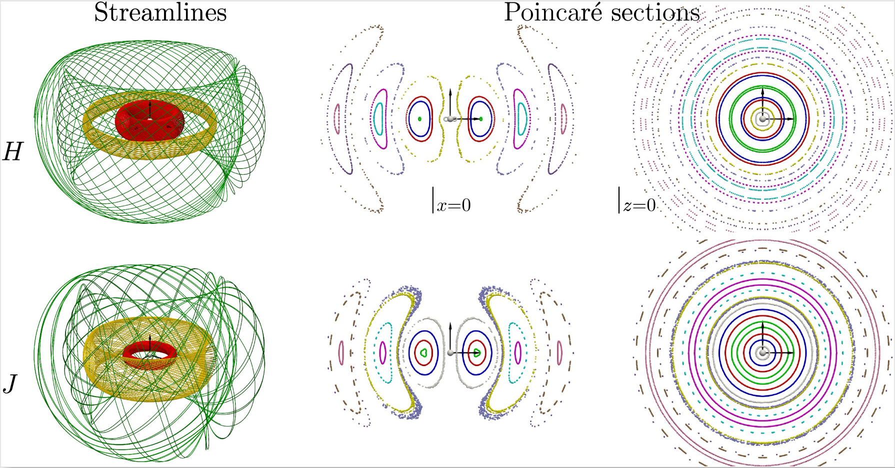
Knotted nature of the toroflux

Toroflux solution for $\Gamma = 0.50$

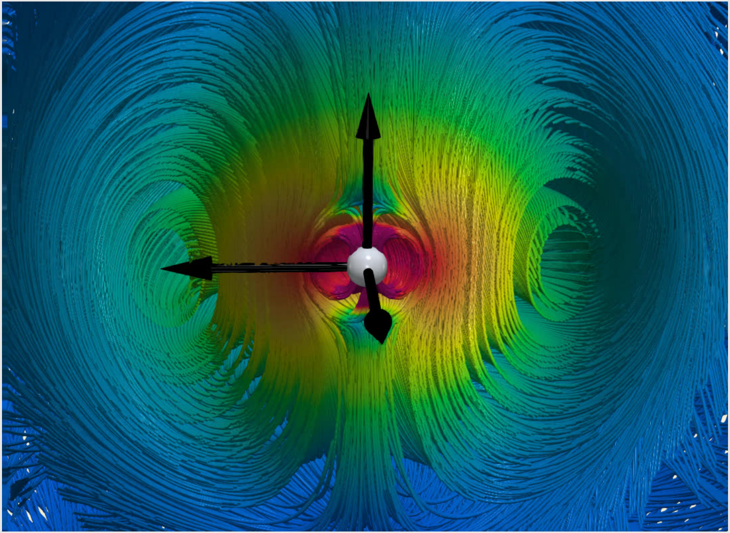


Knotted nature of the toroflux

Toroflux solution for $\Gamma = 0.95$

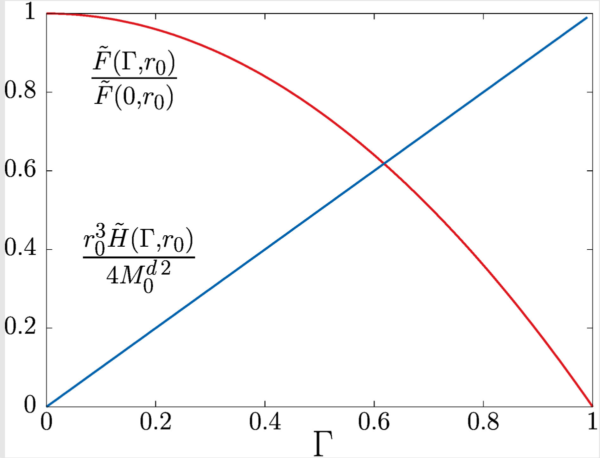


Knotted nature of the toroflux (here $\Gamma = 0.95$)



Obsevrables of the toroflux

Energy and Helicity

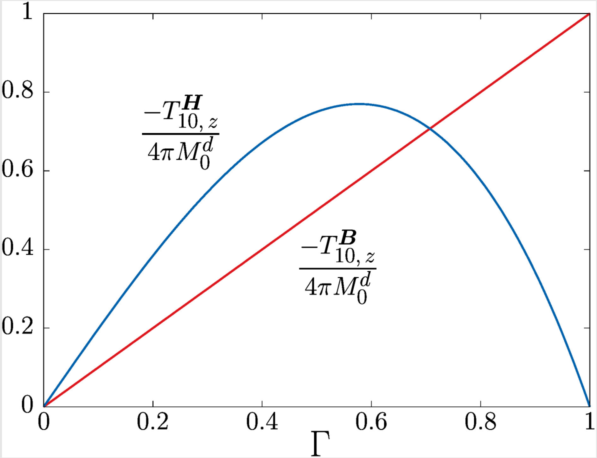


Helicity

$$\mathcal{H} = \int \boldsymbol{A} \cdot \boldsymbol{B}$$

Obsevrables of the toroflux

Toroial dipole moments



Helicity

$$\mathcal{H} = \int \boldsymbol{A} \cdot \boldsymbol{B}$$

Toroial dipole moment $\boldsymbol{T}^{\boldsymbol{H}}$

$$\boldsymbol{T}^{\boldsymbol{H}} = \frac{1}{2}\int \boldsymbol{r} \times \boldsymbol{H}$$
$$= -8\pi M_0^d \Gamma(1-\Gamma^2)\hat{\boldsymbol{z}}$$

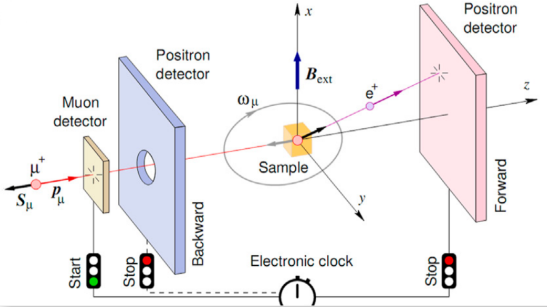
Toroial dipole moment $\boldsymbol{T}^{\boldsymbol{B}}$

$$\boldsymbol{T}^{\boldsymbol{B}} = \frac{1}{2}\int \boldsymbol{r} \times \boldsymbol{B}$$
$$= -4\pi M_0^d \Gamma\hat{\boldsymbol{z}}$$

Obsevrability of the toroflux by μSR

Principle of muon spin spectrosocpy

- implant spin-polarized muons
- spin of implanted muon precess around local ***B***
- decay $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$ in 2.2μs, the e^+ is emitted in the direction of the spin
- allows to probe global and local structure of ***B***



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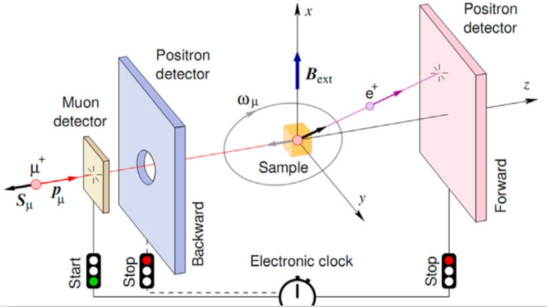

Measured positron beam spectrum should be sensitive to the presence of the toroflux

Magnetic moments of the H of the toroflux

$$a^{Z}_{lm,l'm'} = \int d^3r\, \boldsymbol{H}_{lm}(\boldsymbol{r}) \boldsymbol{Z}_{l'm'}(\boldsymbol{r})$$

Non trivial component due to parity-breaking

$$a^{\Phi}_{10} = 8M^d_0 \sqrt{\frac{\pi}{3}} \sin\Gamma + O(r_0)$$

In rotated coordinates

$$a^{\Phi}_{10}[\vartheta] = 4M^d_0 \sqrt{\frac{\pi}{3}} \Gamma \cos\vartheta$$

Outline

1. Introduction
   - Parity breaking and non-centrosymmetricity
   - Superconductivity
   - Non-centrosymmetric superconductors
2. Toroflux in non-centrosymmetric superconductors
   - Chandrasekhar-Kendall states
   - Derivation of toroflux solution sourced by magnetic dipole
   - Toroflux properties and observation
3. Vortices and applications in non-centrosymmetric superconductors
   - Vortex solutions, field inversion
   - Applications: Chiral Magnetic Josephson junctions
   - Conclusion

Vortex solutions of the London theory [JG, Chernodub, Kharzeev 2020]

London eq. for NCS with sources ($\boldsymbol{v} \neq 0$)

$$\tilde{\boldsymbol{\nabla}} \times \tilde{\boldsymbol{\nabla}} \times \hat{\jmath} - 2\Gamma \tilde{\boldsymbol{\nabla}} \times \hat{\jmath} + \hat{\jmath} = \Phi_0 \left(\tilde{\boldsymbol{\nabla}} \times \boldsymbol{v} - \Gamma \boldsymbol{v} \right)$$

Fourier transform

$$\hat{\jmath}(\tilde{\boldsymbol{x}}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \, e^{i\boldsymbol{p}\cdot\tilde{\boldsymbol{x}}} \boldsymbol{j}_{\boldsymbol{p}}$$

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In momentum space

$$-\boldsymbol{p}\times\boldsymbol{p}\times\boldsymbol{j_p} + \boldsymbol{j_p} - 2i\Gamma\boldsymbol{p}\times\boldsymbol{j_p} = \Phi_0\left(i\boldsymbol{p}\times\boldsymbol{v_p} - \Gamma\boldsymbol{v_p}\right)$$

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$$\boldsymbol{v_p} = 2\pi\frac{\delta(p_z)\mathbf{e}_z}{\lambda_L^2}\sum_{a=1}^{N}n_a e^{-i\boldsymbol{p}\cdot\tilde{\boldsymbol{x}}_a}$$

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Back to real space for single vortex

$$B_\theta\left(\frac{\rho}{\lambda_L}\right) = \frac{\Phi_0 \Gamma}{2\pi\lambda_L^2} \int_0^\infty \frac{q^2(1-q^2)dq}{(1+q^2)^2 - 4\Gamma^2 q^2} J_1\left(\frac{q\rho}{\lambda_L}\right)$$

$$B_z\left(\frac{\rho}{\lambda_L}\right) = \frac{\Phi_0}{2\pi\lambda_L^2} \int_0^\infty \frac{q[(1-2\Gamma^2)q^2+1]dq}{(1+q^2)^2 - 4\Gamma^2 q^2} J_0\left(\frac{q\rho}{\lambda_L}\right)$$

and we get similar equations for $\boldsymbol{j}$

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$$\hat{\jmath}(\tilde{\boldsymbol{x}}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\tilde{\boldsymbol{x}}} \boldsymbol{j_p}$$

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$\Rightarrow$ the magnetic field acquires an in-plane component in addition to usual B_z

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In momentum space

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Integrals can be computed

using Hankel transforms

$\Rightarrow$ the magnetic field acquires an in-plane component in addition to usual B_z

Vortex solutions of the London theory

Integrals to be computed

$$G_\nu(x) = \int_0^\infty \frac{P(q)}{(1+q^2)^2 - 4\Gamma^2 q^2} q^{\nu+1} J_\nu(qx)dq$$
$$= 2\mathrm{Re}\left[C\int_0^\infty \frac{q^\nu}{q^2-\eta^2} J_\nu(qx)qdq\right]$$

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Hankel transforms

Integral transforms whose kernel is a Bessel function

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$$f(q) = \frac{q^\nu}{q^2 + a^2} \quad \longleftrightarrow \quad F_\nu(x) = a^\nu K_\nu(ax)$$

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$$\int_0^\infty \frac{q^\nu}{q^2 - \eta^2} J_\nu(qx) q dq = (i\eta)^\nu K_\nu(i\eta x)$$
$$\Rightarrow \quad G_\nu(x) = 2\mathrm{Re}\left[C (i\eta)^\nu K_\nu(i\eta x) \right]$$

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Integrals to be computed

$$G_\nu(x) = \int_0^\infty \frac{P(q)}{(1+q^2)^2 - 4\Gamma^2 q^2} q^{\nu+1} J_\nu(qx) dq$$
$$= 2\mathrm{Re}\left[C \int_0^\infty \frac{q^\nu}{q^2 - \eta^2} J_\nu(qx) q dq\right]$$

Hankel transforms

Integral transforms whose kernel is a Bessel function

$$F_\nu(x) := \int_0^\infty f(q) J_\nu(qx) q dq$$

$$f(q) = \frac{q^\nu}{q^2 + a^2} \quad \longleftrightarrow \quad F_\nu(x) = a^\nu K_\nu(ax)$$

$$\int_0^\infty \frac{q^\nu}{q^2 - \eta^2} J_\nu(qx) q dq = (i\eta)^\nu K_\nu(i\eta x)$$
$$\Rightarrow \quad G_\nu(x) = 2\mathrm{Re}\left[C (i\eta)^\nu K_\nu(i\eta x)\right]$$

Finally: B and j

$$B_\theta\left(\frac{\rho}{\lambda_L}\right) = \frac{\Phi_0}{2\pi\lambda_L^2}\mathrm{Re}\left[i\eta^2 K_1\left(\frac{i\eta\rho}{\lambda_L}\right)\right]$$
$$B_z\left(\frac{\rho}{\lambda_L}\right) = \frac{-\Phi_0}{2\pi\lambda_L^2}\mathrm{Re}\left[\eta^2 K_0\left(\frac{i\eta\rho}{\lambda_L}\right)\right]$$
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where K_0 and K_1 are modified

Bessel functions of the second kind

Vortex solutions of the London theory

Integrals to be computed

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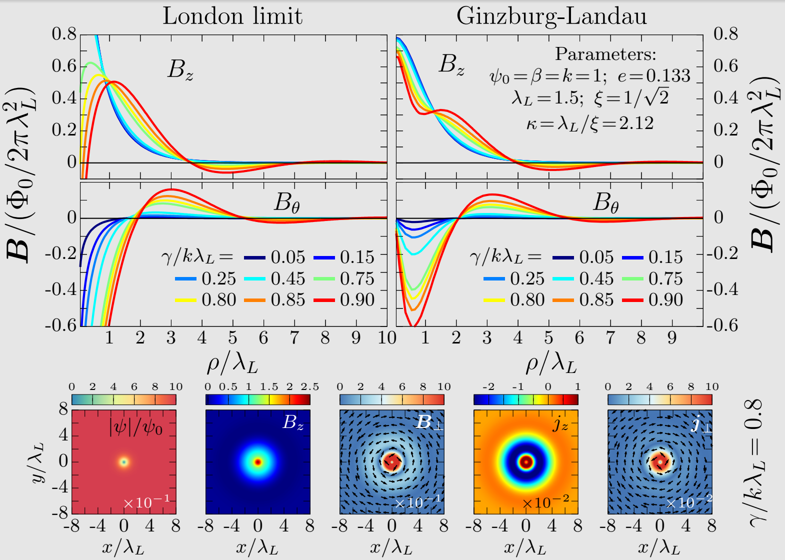
Bessel functions of the second kind

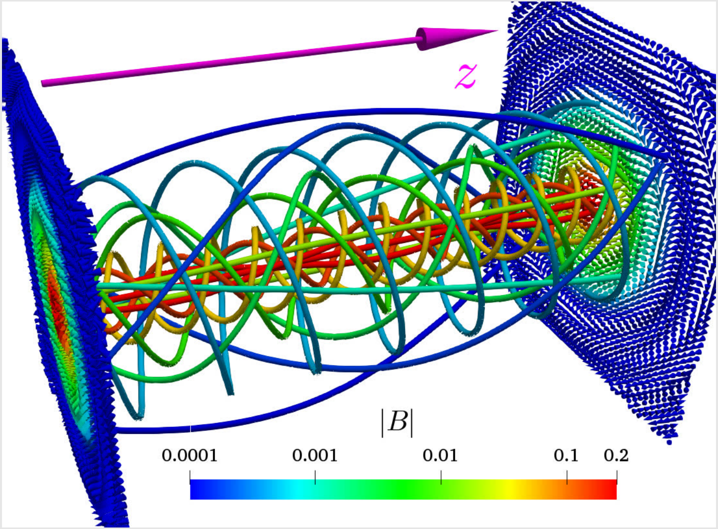
Intervortex forces

$$U(x) = \mathrm{Re}\left[\frac{i\eta}{\sqrt{1-\Gamma^2}} K_0\left(\frac{i\eta x}{\lambda_L}\right)\right]$$

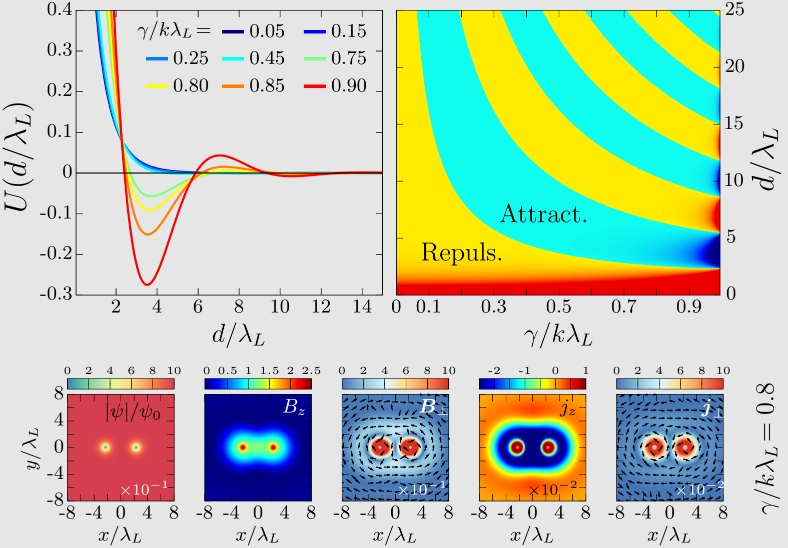
Vortices and field inversion in non-centrosymmetric superconductors



Helicoidal magnetic streamlines around a vortex



Vortex attraction in non-centrosymmetric superconductors

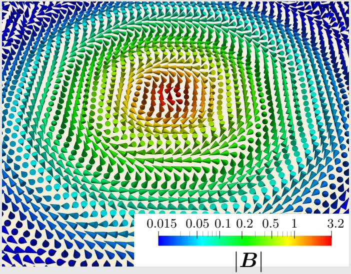


Vortex in noncentrosymm. supercond. [JG, Chernodub, Kharzeev 2020]

Vortex feature an helical magnetic field

- carry both longitudinal and in-plane field
- features field inversion at intermediate Γ
- both London limit analytic calculations and full Ginzburg-Landau simulations agree

Vortex magnetic field



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Non-monotonic intervortex forces

- short-range repulsion and long range attract.
- vortex bound-states: should lead to formation of clusters, superclusters, stripe phases, etc
- metastable vortex/antivortex bound-states possible entropy stabilised V/AV lattice

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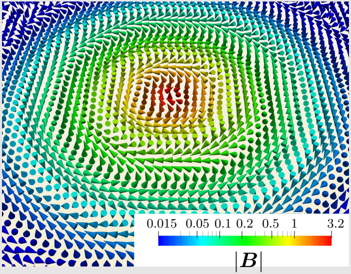
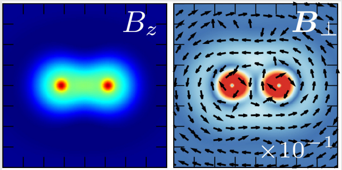


Vortex bound-states



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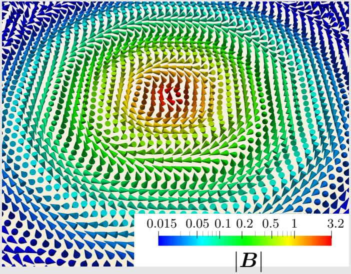
Confirmed by [Samoilenka and Babaev 2020]

method based on Chandrasekhar-Kendall trick

Vortex magnetic field



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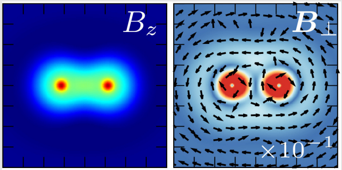
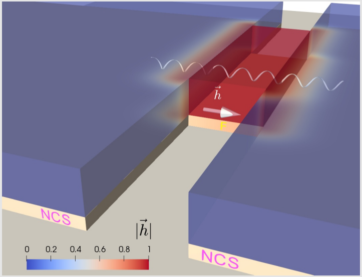


Possible applications of NCS

[Chernodub, JG, Kharzeev 2019]

Chiral Magnetic Josephson Junctions as a base for low noise qubits ?

Chiral magnetic junction



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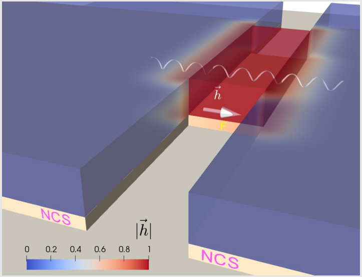
NCS with uniaxial ferromagnetic weak link

Order parameter equation inside the link

$$\left[k\partial^2_{xx} + iek\chi h_x\partial_x - \alpha\right]\psi = 0$$

Chiral magnetic junction



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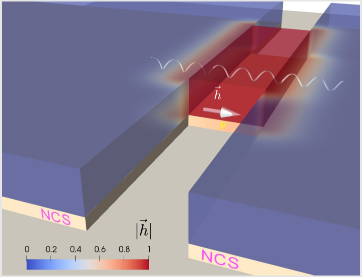
$J = J_c \sin(\varphi - \varphi_g)$, with the bias $\varphi_g \neq 0$

$\Rightarrow$ non-vanishing current across the junction

$\Rightarrow$ the bias $\varphi_g = eh_x\chi L$ plays the role of offset flux

Chiral magnetic junction



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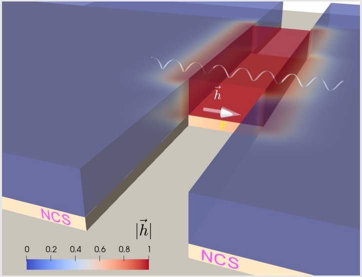
Chiral junction can be used for qubit design

$$E_Q(\varphi, \varphi_g) = E_J[1 - \cos(\varphi - \varphi_g)] + E_L\varphi^2.$$

Qubit Junction Inductive energy

Chiral magnetic junction



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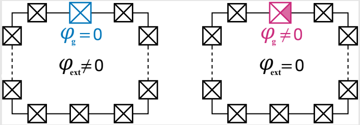


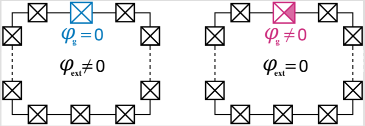
Possible applications

[Chernodub, JG, Kharzeev 2019]

Coulomb interactions btw Cooper pairs

described by the qubit Hamiltonian

$$\hat{H} = 4E_C\hat{n}^2 + E_J[1 - \cos(\varphi - \varphi_g)] + E_L\varphi^2$$



Chiral magnetic junction

- fluxonium qubits relate the phase offset to the externally applied flux Φ as $\varphi_g = 2\pi\Phi/\Phi_0$
- nonzero phase bias φ_g imposes a large anharmonicity on the energy-level

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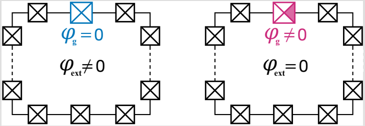

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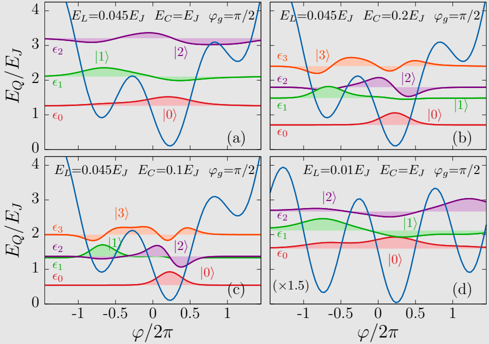

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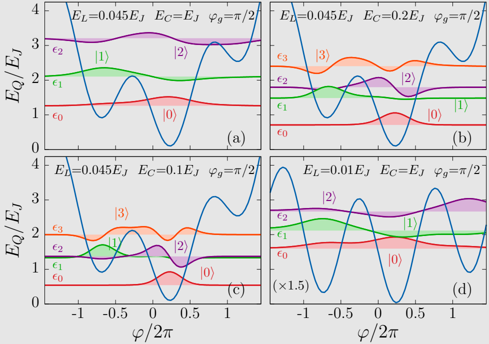

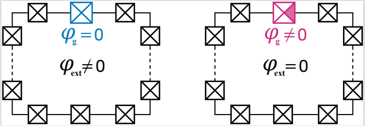

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Advantage

- CMJs eliminate the need for an external magnetic flux
- suppress noise resulting from offset flux by factor $\approx 10^{-2}$

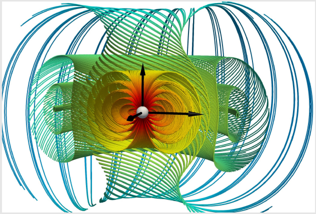
Conclusion

New excitations in non-centrosymmetric superconductors

Toroflux

- knotted magnetic configurations analogous to Chandrasekhar-Kendall
- simplest modes are induced by magnetic dipoles
- perhaps observable in muon spectroscopy

Toroflux



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- feature an helical magnetic field, with a possible field inversion
- existence of non-monotonic forces
  $\Rightarrow$ formation of bound-states, cluster,...

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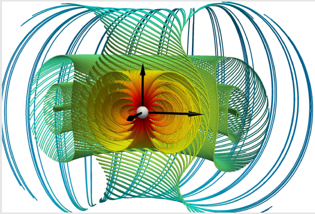

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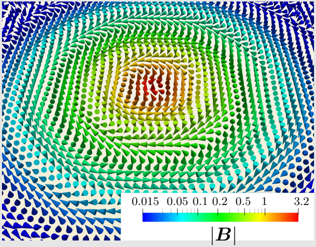

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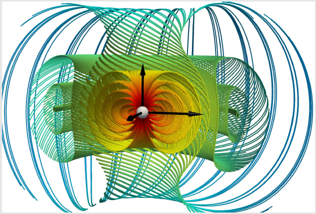
NCS have possible interesting applications

Chiral Magnetic Junction, low qubits?

Toroflux



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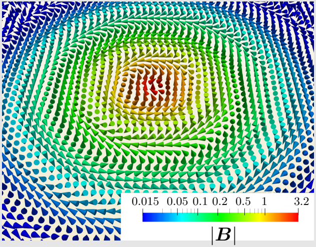
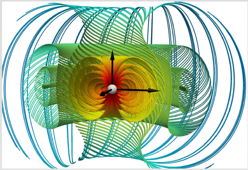
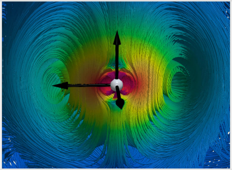
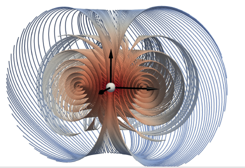
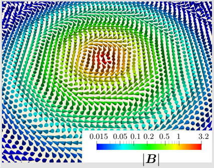


Thank you for your attention!

based on

- **JG**, A. Korneev, A. Samoilenka, A. Molochkov, E. Babaev, and M. Chernodub
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