Symbolic-Numeric Aspects in Computation of Invariant Pairs for Matrix Polynomials

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The generalized eigenvalue/eigenvector problem for matrix polynomials consists in computing eigenpairs of the form $(x, \lambda) \in \mathbb{C}^n \times \mathbb{C}$, such that

$$P(\lambda)x = (A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_\ell\lambda^\ell)x = 0, \tag{1}$$

where $A_i \in \mathbb{C}^{n\times n}$ for $i = 1, \ldots, \ell$ are given coefficient matrices.
This problem can be extended by the concept of invariant pairs. A pair $(X, S) \in \mathbb{C}^{n\times k} \times \mathbb{C}^{k\times k}$ is called invariant if it satisfies the relation

$$\mathbb{P}(X, S) := A_0X + A_1XS + A_2XS^2 + \cdots + A_{\ell-1}XS^{\ell-1} + A_\ell XS^\ell = 0 \tag{2}$$

Invariant pairs generalize the notion of invariant subspaces to matrix polynomials. Working with invariant subspaces instead of eigenvectors offers conceptual and numerical benefits. For instance, eigenvectors associated with a multiple eigenvalue are unstable under perturbations. In contrast, the corresponding invariant subspace remains stable under perturbations, provided that the algebraic eigenvalue multiplicities of the invariant subspace coincide with those of the matrix. Also, invariant pairs present an interest in the context of differential systems.
The definition (2) was used in [1] by Timo Betcke and Daniel Kressner to analyse and give a numerical solution to the problem of computing invariant pairs. Their approach is based on linearization, extraction of an approximate solution and iterative refinement.
An equivalent formulation for (2), that we will use in this work, is:

$$\mathbb{P}(X, S) := \frac{1}{2\pi\imath}\oint_\Gamma P(\lambda)X(\lambda I - S)^{-1}d\lambda \tag{3}$$

where $\Gamma \subseteq \mathbb{C}$ is a contour containing the spectrum of S in its interior.
Using definition (3) instead of (2) has a number of advantages. For instance,

we can choose the contour Γ to find specific eigenvalues. Also, this definition makes it easier to analyse the problem, to compute a condition number and a backward error, which play an important role for a symbolic-numeric approach.
To solve (3) numerically, we propose Newton's method with line search that defines $(\Delta X, \Delta S)$ by

$$\mathbb{P}(X, S) + t\, \mathbb{DP}_{X,S}(\Delta X, \Delta S) = 0$$

where t solves the minimization problem with cost function

$$p(t) = \|\mathbb{P}(X + t\Delta X, S + t\Delta S)\|_F^2,$$

and

$$\mathbb{DP}_{X,S}(\Delta X, \Delta S) = \frac{1}{2\pi\imath} \oint_\Gamma P(\lambda) \left(\Delta X + X(\lambda I - S)^{-1}\Delta S\right) (\lambda I - S)^{-1} d\lambda.$$

References

[1] T. Betcke, D. Kressner. Perturbation, extraction and refinement of invariant pairs for matrix polynomials. *Linear Algebra Appl.*, 435, 514-536, 2011.