Classical superselection sectors, memory and soft symmetries from Hamiltonian reduction

QFG2024 - IHP Paris

Michele Schiavina

Department of Mathematics, University of Pavia
& Istituto Nazionale di Fisica Nucleare, Pavia

28/03/24

UNIVERSITÀ DI PAVIA

INFN

Overview

Joint work with A. Riello general: 2207.00568 [ATMP 24] & null YM: 2303.03531 [AHP 24]

Problem: reduced phase space of gauge theories, with corners **P**

- Hamiltonian reduction paradigm becomes reduction **by stages**: **P**
  1. 'bulk' gauge ⇝ "constraint reduction"
  2. residual/large gauge ⇝ "flux superselection" **P**
- Adjusted expectation:
  1. reduced phase space is (singular/stratified) **Poisson manifold** **P**
  2. foliated by **symplectic leaves** called *flux superselection sectors* **P**
  3. **residual momentum maps** given by *Noether charges* **P**
  4. sectors labeled by **Poisson casimirs**, or gauge classes of *fluxes* **P**
  5. quantisation decomposes Hilbert space into sectors. **P**

Application to null YM: allows to recover

1. soft/asymptotic/large gauge transf. - as residual symmetries,
2. "extended" phase space / memory - as residual Hamiltonian data. **P**

General relativity is still work in progress. Technical complications.

Hamiltonian reduction primer

Hamiltonian reduction primer - Hamiltonian actions

Let $M = R^{2n}$ with symplectic form $\omega = dq^i \wedge dp_i$, i.e. $\{q^i, p_j\} = \delta^i_j$. **P**

Lie algebra action: $\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$, with *fundamental vector fields*

$$\hat{\xi}_a \equiv \rho(e_a), \quad \rho([e_a, e_b]_{\mathfrak{g}}) = [\rho(e_a), \rho(e_b)]_{\mathfrak{X}(M)}; \quad [\hat{\xi}_a, \hat{\xi}_b]_{\mathfrak{X}(M)} = f^c_{ab}\hat{\xi}_c.\mathbf{P}$$

$$\text{Hamiltonian action} \iff \iota_{\hat{\xi}_a}\omega = dH_a \iff \hat{\xi}_a = \{H_a, \cdot\}.$$

We talk of a *Hamiltonian G-space* (M, G, ρ). **P**

A Hamiltonian G space carries a *momentum map* function:

$$H\colon M \to \mathfrak{g}^* \qquad H(x)\colon \xi \to \langle H(x), \xi\rangle = H_a(x)\xi^a.\mathbf{P}$$

Equivariance: $\langle L_{\hat{\eta}} H, \xi\rangle = \langle H, [\xi, \eta]\rangle + k(\xi, \eta)$ iff cocycle k vanishes.

Hamiltonian reduction primer - familiar examples

Consider the $\mathbb{R}^n$ and $\mathfrak{so}(n)$ (algebra) actions on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ by

$$(q^i, p_i) \to (q^i + v^i, p_i) \qquad \rho(v) = v^i \frac{\partial}{\partial q^i} \qquad v \in \mathbb{R}^n,$$

$$(q^i, p_i) \to (O^i_j q^j, -O^j_i p_j) \quad \rho(O) = O^i_j q^j \frac{\partial}{\partial q^i} - O^i_j p_i \frac{\partial}{\partial p_j} \quad O \in \mathfrak{so}(n).$$

P

Momentum maps: $\langle H, \bullet \rangle \doteq \iota_{\rho(\bullet)}(p_i dq^i) \rightsquigarrow \iota_{\rho(\bullet)}(dq^i dp_i) = d\iota_{\rho(\bullet)}(p_i dq^i)$.

$$\langle H(q,p), v\rangle = v^i p_i, \qquad \text{Linear Momentum:} \quad H(q,p)(\bullet) = \langle p, \bullet\rangle \in (\mathbb{R}^n)^*$$

$$\langle H(q,p), O\rangle = p_i O^i_j q^j, \quad \text{Angular Momentum:} \quad H(q,p)(\bullet) = \langle p, \bullet q\rangle \in \mathfrak{so}(n)^*$$

P In $n = 3$ we have $O \in \mathfrak{so}(3) \simeq \mathbb{R}^3 \ni o$, given by $O^j_i \mapsto \delta^{j\ell}\epsilon_{\ell ik} o^k$ and

$$\langle p, Oq\rangle = p_j O^j_i q^i = p_j \delta^{j\ell}\epsilon_{\ell ik} o^k q^i = (q \times p) \cdot o$$

Momentum map identified with the vector $q \times p$.

Hamiltonian reduction primer - co-adjoint orbits

The dual of a Lie algebra $\mathfrak{g}^*$ is a Poisson manifold with

$$\Pi = x_c f^c_{ab} \frac{\partial}{\partial x_a} \wedge \frac{\partial}{\partial x_b}, \qquad \{x_a, x_b\} = f^c_{ab} x_c, \textbf{P}$$

foliated by co-adjoint orbits: for any $\mu \in \mathfrak{g}^*$

$$\mathcal{O}_\mu = \{\mu' \in \mathfrak{g}^* \mid \exists g \in G,\ \mathrm{Ad}^*_g \mu = \mu'\} \simeq G/G_\mu, \quad T_{\mu'}\mathcal{O}_\mu \simeq \mathfrak{g}/\mathfrak{g}_\mu.\textbf{P}$$

The foliation is symplectic with Kostant–Kirillov–Souriau form on $\mathcal{O}_\mu$,

$$\omega_{\mu'}(\mathrm{ad}_X(\mu'), \mathrm{ad}_Y(\mu')) = \langle \mu', [X, Y] \rangle, \quad \forall X, Y \in \mathfrak{g}.\textbf{P}$$

Any Poisson manifold foliated by symplectic "leaves".
Casimir functions $\{c, f\} = 0$ for all $f \in C^\infty(M)$: constant on leaves, labeled by choice of values of a complete set of Casimirs.[1]

[1] Basis of 0th Poisson cohomology.

Hamiltonian reduction primer - orbit reduction

Theorem (Marsden, Weinstein; Meyer; Arms ... $\sim$ '70s – '80s)

Let $G \circlearrowright M$ be a free and proper Hamiltonian action with equivariant m. map $H : M \to \mathfrak{g}^*$. **P** *For every coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$ we have a* **symplectic manifold***:*

$$\underline{C}_{[\mu]} \doteq H^{-1}(\mathcal{O}_\mu)/G \simeq H^{-1}(\mu)/G_\mu, \quad e.g. \quad \underline{C}_0 = H^{-1}(0)/G.\mathbf{P}$$

Moreover M/G is **Poisson**, *and $\underline{C}_{[\mu]}$ are its symplectic leaves:*

$$M/G = \bigsqcup_{\mathcal{O}_\mu \in \mathfrak{g}^*} \underline{C}_{[\mu]} = \bigsqcup_{\mathcal{O}_\mu \in \mathfrak{g}^*} H^{-1}(\mathcal{O}_\mu)/G \qquad \textit{symplectic foliation}.\mathbf{P}$$

Reduction of T^*G yields $T^*G/G \simeq \mathfrak{g}^*$, **model for** M/G.

Orbit $\mathcal{O}_\mu \to \mathfrak{g}^*$, **model for symplectic "sector"** $H^{-1}(\mathcal{O}_\mu)/G$.

Hamiltonian reduction primer - reduction by stages

Consider a normal subgroup $G_\circ \subset G$, momentum map $H_\circ \colon M \to \mathfrak{g}_\circ^*$. **P**

Consider reduction **at zero** $\underline{C}_\circ \doteq H_\circ^{-1}(0)/G_\circ$ for subgroup. **P**

Theorem (Guillemin, Sternberg; Marsden, Ratiu, Weinstein ∼ '80*s*)

If $G_\circ \subset G$ is a normal subgroup, there is a Hamiltonian action

$$\underline{G} \circlearrowright \underline{C}_\circ \qquad \underline{G} \doteq G/G_\circ$$

with momentum map $\underline{h} \colon \underline{C}_\circ \to \underline{\mathfrak{g}}^*$ such that[2] $\pi_\circ^* \underline{h} = H|_{H_\circ^{-1}(0)}$. **P**

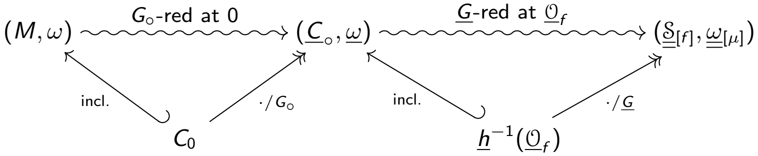
The first stage reduction $\underline{C}_\circ$ is a symplectic manifold. **P**
The second stage reduction yields the Poisson manifold:

$$\underline{\underline{M}} \doteq \underline{C}_\circ/\underline{G} = \bigsqcup_{[f]} \underline{\underline{\mathcal{S}}}_{[f]} \doteq \bigsqcup_{\underline{\mathcal{O}}_f \subset \underline{\mathfrak{g}}^*} \underline{h}^{-1}(\underline{\mathcal{O}}_f)/\underline{G}$$

Second stage sees coadjoint orbits $\underline{\mathcal{O}}_f \subset \underline{\mathfrak{g}}^*$ of $\underline{G}$.

[2] Some details are hidden.

Hamiltonian reduction primer - important remarks



P Traditionally used to reduce by semidirect product actions.
Our application is to field theory on manifolds with corners. P

Hamiltonian reduction $(T^*G)/G \simeq \mathfrak{g}^*$.
Prototype corner gauge reduction, realised exactly in 2d BF theory.

Gauge Theory

Local gauge theory with corners I

Lagrangian field theory on $\Sigma \times \mathbb{R}$, $\dim(\Sigma) \doteq n$. *Corner* if $\partial\Sigma \neq \emptyset$. **P**
Space of fields $\mathcal{F}$. Local Lagrangian $\mathbf{L} \in \Omega^{0,\mathrm{top}}_{\mathrm{loc}}(\mathcal{F} \times (\Sigma \times \mathbb{R}))$.

$$\mathbb{d}\mathbf{L} = \mathbf{EL} + d\boldsymbol{\theta}.\mathbf{P}$$

Will not be working on "covariant phase space" ~~$\mathbf{EL} = 0$~~.
On Σ: **Geometric phase space** $\mathcal{P}$ w. (local) symplectic form[3] $\boldsymbol{\omega} = \mathbb{d}\boldsymbol{\theta}$
Shell defines constraint submanifold, or "Cauchy data" $\mathcal{C} \subset \mathcal{P}$. **P**

For *gauge* field theory assume we have

1. A (local) Lie group action on $\mathcal{F}$
2. An induced (local) Lie group action $\mathcal{G} \circlearrowright (\mathcal{P}, \boldsymbol{\omega})$ **P**

Note: induced action is not always a group action.
OK for YM, but not for GR: point 2 fails, algebroid/groupoid **on shell**.

[3] Terms and conditions apply. [Kijowski–Tulczyjew]

Local gauge theory with corners II

Hamiltonian formulation yields $(\mathcal{P}, \boldsymbol{\omega}, \boldsymbol{H}, \mathcal{G})$ locally Hamiltonian $\mathcal{G}$-space:

1. $\mathcal{P} = \Gamma(\Sigma, F)$ sections of a vector bundle (for simplicity), **P**
2. $\mathcal{G}$ a local Lie group with a local action on $\mathcal{P}$ with $\mathrm{Lie}(\mathcal{G}) \doteq \mathfrak{G}$, **P**
3. $\boldsymbol{\omega} \in \Omega^{2,\mathrm{top}}_{\mathsf{loc}}(\mathcal{P} \times \Sigma)$ a local symplectic density on $\mathcal{P}$, **P**
4. $\boldsymbol{H} \in \Omega^{0,\mathrm{top}}_{\mathsf{loc}}(\mathcal{P} \times \Sigma, \mathfrak{G}^*)$ a $\mathfrak{G}^*$-valued local form on $\mathcal{P}$. **P**

Flow and equivariance now hold pointwise: for $\xi \in \mathfrak{G}$

$$\iota_{\rho(\xi)}\boldsymbol{\omega} = \langle \mathbb{d}\boldsymbol{H}, \xi \rangle \qquad \text{local Hamiltonian form}$$

$$\mathbb{L}_{\rho(\xi)}\boldsymbol{H} = \mathrm{ad}^*_\xi \boldsymbol{H} + d\boldsymbol{k}(\xi) \qquad \text{Equivariance up to corners}$$

P Note 1: *Local* pairing $\langle \mathbb{d}\boldsymbol{H}, \xi \rangle$: may depend on derivatives $\partial\xi$.
$\rightsquigarrow$ Generally not $C^\infty(\Sigma)$-linear! **P**

Note 2: Integrate $\omega \doteq \int_\Sigma \boldsymbol{\omega}$ and $H \doteq \int_\Sigma \boldsymbol{H}$.
$\rightsquigarrow$ Momentum map. Weakly equivariant .

Running Example I: Spacelike Yang–Mills Theory

Consider Lie group G, with inner product $\mathrm{tr}\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$.
G-connections $A\in\mathcal{A}\doteq\mathrm{Conn}(P\to\Sigma)$ with Σ spacelike.
Generalised electric fields E are $\mathfrak{g}$-valued (top-1)-forms on Σ. **P**

We have the *geometric phase space*:[4]

$$\mathcal{E}\doteq\Omega^{n-1}(\Sigma,\mathfrak{g}),\qquad \mathcal{P}\equiv T^*\mathcal{A}\doteq\mathcal{A}\times\mathcal{E}\ni(A,E),\qquad \boldsymbol{\omega}=\mathrm{tr}(\mathbb{d}A\mathbb{d}E).\mathbf{P}$$

The gauge action of $\mathcal{G}\doteq G_0^\Sigma\equiv C_0^\infty(\Sigma,G)$ reads

$$(A,E,\xi)\longmapsto\rho(\xi)(A,E)=(d_A\xi,\mathrm{ad}(\xi)\cdot E),\qquad \xi\in\mathfrak{G}=\mathfrak{g}^\Sigma.$$

Locally Hamiltonian with (equivariant) momentum map

$$\iota_{\rho(\xi)}\boldsymbol{\omega}=\langle\mathbb{d}\boldsymbol{H},\xi\rangle,\qquad \langle\boldsymbol{H},\xi\rangle=\mathrm{tr}(Ed_A\xi).\mathbf{P}$$

[4] $A=\tilde{A}|_\Sigma$, $E=(\star F_A)|_\Sigma$ from YM theory on $\Sigma\times\mathbb{R}$ and $\mathbf{L}=F_{\tilde{A}}\wedge\star F_{\tilde{A}}$.

Reduced phase space

Assume for a moment that $\partial\Sigma = \emptyset$. Physical configurations on $\mathcal{P}$ are recovered as the vanishing locus of **Noether's current $\boldsymbol{H}$**. **P**

If Noether's current is a locally Hamiltonian equivariant, momentum form, physical configurations on $\mathcal{P}$ are characterised by:

$$\text{Noether Thm} \implies \boldsymbol{H} \quad \text{d-exact on shell} \implies H \doteq \int_\Sigma \boldsymbol{H} \approx 0.\mathbf{P}$$

H is an equivariant momentum map, so Hamiltonian reduction yields the space of physical configurations modulo gauge:

$$\partial\Sigma = \emptyset, \quad \mathcal{C} \doteq H^{-1}(0) \quad \text{constraint set}, \quad \underline{\mathcal{C}} = H^{-1}(0)/\mathcal{G}.\mathbf{P}$$

This is the **reduced phase space** of the theory.
In this case, this is a **symplectic manifold**. **P**

Complications arise when $\partial\Sigma \neq \emptyset$.

Reduction by Stages

Reduction with corners via reduction by stages

Problem: $H^{-1}(0)$ *is no longer the* <u>*correct constraint locus*</u>*!*
"Zero-flux conditions" imposed by $H = 0$! **P**

Proposition (Constraint / Flux splitting [Riello, MS])

There is a natural bulk/boundary splitting:

$$\boldsymbol{H} = \boldsymbol{H}_\circ + d\boldsymbol{h}$$

such that $\mathcal{C} \doteq \boldsymbol{H}_\circ^{-1}(0)$ *coincides with the constraint set of the theory. We call* $\boldsymbol{H}_\circ$ *the constraint form and* $d\boldsymbol{h}$ *the flux form.* **P**

Problem: $\boldsymbol{H}_\circ$ is NOT a momentum form for $\mathcal{G}$ anymore!

$$\text{Noether Thm} \implies \mathcal{C} = \boldsymbol{H}_\circ^{-1}(0) \quad \text{first-class constraint set.}\textbf{P}$$

Question: Is there a subgroup $\mathcal{G}_\circ \subset \mathcal{G}$, for which $\mathcal{C}$ is zero level set of induced momentum map $J_\circ \colon \mathcal{P} \to \mathfrak{G}_\circ^*$, so that $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{G}_\circ$ symplectic?

First Stage: Constraint Reduction

Answer: Yes! **P**

Theorem (Constraint reduction [Riello MS])

Let $h_{\mathcal{C}} \doteq \iota^*_{\mathcal{C}} \int_\Sigma d\boldsymbol{h}$. Under certain regularity assumptions:

1. $\mathfrak{G}_\circ \doteq \mathrm{AnnIm}(h_{\mathcal{C}}) \subset \mathfrak{G}$ *is the maximal Lie ideal whose associated momentum map $J_\circ$ is* **constraining:** $J_\circ^{-1}(0) = \mathcal{C}$. **P**

   Normal subgroup $\mathcal{G}_\circ \subset \mathcal{G}$: **constraint gauge group**.
   Quotient group $\underline{\mathcal{G}} \doteq \mathcal{G}/\mathcal{G}_\circ$: **flux gauge group P**

2. *There is a* **residual Hamiltonian action** $\underline{\mathcal{G}} \circlearrowright \underline{\mathcal{C}} = \mathcal{C}/\mathcal{G}_\circ$, *with momentum map $\underline{h}\colon \underline{\mathcal{C}} \to \underline{\mathfrak{G}}^*$, such that $h_{\mathcal{C}} = \pi^*_\circ \underline{h}$.*

   We call $\underline{h}$ the **flux map** *and $\mathfrak{F} \doteq \mathrm{Im}(\underline{h})$ the* **flux space**. **P**

3. *Equivariance controlled by the cocycle $k \doteq \int d\boldsymbol{k}$.*
   [Recall: $\boldsymbol{H}$ is equivariant up to corner] **P**

We will call $\underline{\mathcal{C}}$ the **constraint-reduced** phase space.

Yang–Mills II: Constraint/flux split

The Hamiltonian momentum form splits as:

$$\boldsymbol{H} = \boldsymbol{H}_\circ + d\boldsymbol{h}, \qquad \langle \boldsymbol{H}_\circ, \xi\rangle = \mathrm{tr}(d_A E \xi), \qquad \langle d\boldsymbol{h}, \xi\rangle = -d\mathrm{tr}(E\xi),$$

$$\mathcal{C} = \boldsymbol{H}_\circ^{-1}(0) = \{(A,E) \in \mathcal{P} \mid d_A E = 0\} : \text{Gauss' Constraint}\mathbf{P}$$

Note: Imposing $\boldsymbol{H} = 0$ forces $E|_{\partial\Sigma} = 0$: *zero flux*.
Indeed $\langle h, \xi\rangle \doteq \int_{\partial\Sigma} \iota^*_{\partial\Sigma}\mathrm{tr}(E\xi)$ is the (smeared) "electric" flux. **P**

Denote $\xi \in \mathfrak{g} \hookrightarrow \mathfrak{G} \iff d\xi = 0$. The constraint gauge ideal $\mathfrak{G}_\circ$ reads:

$$\mathfrak{G}_\circ = \mathrm{Ann}(\mathfrak{F}) = \begin{cases} \{\xi \in \mathfrak{G} \mid \xi|_{\partial\Sigma} = 0\} & \text{G semisimple} \\ \{\xi \in \mathfrak{G} \mid \exists \chi \in \mathfrak{g} : \xi|_{\partial\Sigma} = \chi|_{\partial\Sigma}\} & \text{G Abelian}\mathbf{P} \end{cases}$$

and thus the flux gauge algebra $\underline{\mathfrak{G}}$ reads (true also for null case!)

$$\underline{\mathfrak{G}} = \mathfrak{G}/\mathfrak{G}_\circ = \begin{cases} C^\infty(\partial\Sigma, \mathfrak{g}) & G \text{ semisimple} \\ C^\infty(\partial\Sigma, \mathfrak{g})/\mathfrak{g} & G \text{ Abelian} \end{cases}$$

Yang–Mills III: Constraint reduction

[Singer; Narasimhan, Ramadas; Gomes, Hopfmüller, Riello; Riello-MS]

Given A, radiative electric fields $\mathcal{H}_A = \{d_A E = 0 = E|_{\partial\Sigma}\}$.
Radiative/Coulombic (Helmholz–Hodge) orthogonal decomposition. **P**
$E = E_{\rm rad} + \star d_A\varphi$, with $\varphi \in C^\infty(\Sigma, \mathfrak{g})$ the Coulombic potential

$$\begin{cases} \Delta_A\varphi = \star d_A E \approx 0 & \text{in } \Sigma, \\ \mathrm{n}\cdot d_A\varphi = E_\partial & \text{at } \partial\Sigma \end{cases}$$

parametrised by $E_\partial \in \mathcal{E}_\partial = \Omega^{\rm top}(\partial\Sigma, \mathfrak{g})$. **P** Then

$$\mathcal{C} \simeq_{\rm loc} \underbrace{\mathcal{H}_A \times \mathcal{A}}_{\mathcal{P}_{\rm rad}} \times \mathcal{E}_\partial \quad \Longrightarrow \quad \underline{\mathcal{C}} \simeq_{\rm loc} \underbrace{\mathcal{P}_{\rm rad}/\mathcal{G}_\circ}_{\underline{\mathcal{P}}_{\rm rad}} \times \mathcal{E}_\partial \textbf{P}$$

For G **Abelian**, $A = A_{\rm rad} + d\varsigma$, with $\varsigma \in C^\infty(\Sigma, \mathfrak{g})$ solution of Neumann–Laplace, **P** one obtains **globally!**

$$\underline{\mathcal{C}} \simeq \underline{\mathcal{P}}_{\rm rad} \times T^*\underline{\mathfrak{G}} \qquad \text{"Edge modes" (?)}$$

$$\underline{\omega} \overset{\rm ab}{=} \int_\Sigma \mathbb{d}E_{\rm rad} \wedge \mathbb{d}A_{\rm rad} + \int_{\partial\Sigma} \mathbb{d}E_\partial \wedge \mathbb{d}\varsigma_\partial,$$

with $\underline{\mathcal{P}}_{\rm rad} \doteq \mathcal{P}_{\rm rad}/\mathcal{G} \ni (A_{\rm rad}, E_{\rm rad})$.

Second Stage: Flux Superselection

First stage reduction output: $(\underline{\mathcal{C}}, \underline{\omega}, \underline{h})$ Hamiltonian $\underline{\mathcal{G}}$-space. **P**

Consider the coadjoint orbit $\mathcal{O}_f \in \underline{\mathfrak{G}}^*$ of a flux $f \in \mathfrak{F} \subset \underline{\mathfrak{G}}^*$.
All on-shell configurations whose flux is in $\mathcal{O}_f$ are acted upon by $\underline{\mathcal{G}}$:[5]

$$\underline{\mathcal{S}}_{[f]} = \underline{h}^{-1}(\mathcal{O}_f) \quad \rightsquigarrow \quad \underline{\underline{\mathcal{S}}}_{[f]} = \underline{\mathcal{S}}_{[f]}/\underline{\mathcal{G}}$$

Superselection sector (SSS)**P**

Theorem (Flux Superselection [Riello, MS])

The fully-reduced phase space $\underline{\underline{\mathcal{C}}} = \underline{\mathcal{C}}/\underline{\mathcal{G}} = \mathcal{C}/\mathcal{G}$ *is a Poisson manifold whose symplectic leaves are the superselection sectors:* $\underline{\underline{\mathcal{C}}} = \bigsqcup_{\mathcal{O}_f \subset \underline{\mathfrak{G}}^*} \underline{\underline{\mathcal{S}}}_{[f]}$.

P The second-stage, **fully-reduced**, phase space is only Poisson!
Fully gauge-invariant symplectic leaves. **P**
Labels are **Casimirs of the Poisson structure**, i.e. central elements of the Poisson algebra $C^\infty(\underline{\underline{\mathcal{C}}})$. Hilbert space decomposition into "blocks".
(Think Casimirs of the Noether charge algebra.)

[5] Ignoring multiple connected components.

Yang–Mills IV: A closer look to the first stage

Abelian case

Radiative/Coulombic split leads to constraint reduction:

$$\underline{\mathcal{C}} \simeq \underline{\mathcal{P}}_{\mathsf{rad}} \times \mathcal{E}_\partial, \qquad \underline{\mathcal{P}}_{\mathsf{rad}} = \mathcal{P}_{\mathsf{rad}}/\mathcal{G}_\circ = (\mathcal{H}_A \times \mathcal{A})/\mathcal{G}_\circ.\mathbf{P}$$

$\underline{\mathcal{G}}$ acts freely on $\underline{\mathcal{P}}_{\mathsf{rad}}$. Then $\underline{\mathcal{C}} \to \underline{\underline{\mathcal{P}}}_{\mathsf{rad}} \doteq \underline{\mathcal{P}}_{\mathsf{rad}}/\underline{\mathcal{G}}$ is a fibre bundle

$$\underline{\mathcal{C}} \simeq_{\mathsf{loc}} T^*\underline{\mathcal{G}} \times \underline{\underline{\mathcal{P}}}_{\mathsf{rad}} \simeq \mathcal{E}_\partial \times \underline{\mathcal{G}} \times \underline{\underline{\mathcal{P}}}_{\mathsf{rad}} \overset{ab}{\ni} (E_\partial, e^{\varsigma_\partial}, E_{\mathsf{rad}}, A_{\mathsf{rad}}),$$

$$\underline{\omega} \overset{\mathsf{ab}}{=} \int_\Sigma \mathbb{d}E_{\mathsf{rad}} \wedge \mathbb{d}A_{\mathsf{rad}} + \int_{\partial\Sigma} \mathbb{d}E_\partial \wedge \mathbb{d}\varsigma_\partial, \mathbf{P}$$

Constraint-reduced phase space $\simeq_{\mathsf{loc}}$ “Extended phase space”

Yang–Mills V: A closer look to the second stage

Abelian Case

Residual momentum map ($\mathcal{E}_\partial \simeq \underline{\mathfrak{G}}^*$):

$$\underline{h} : \underbrace{\underline{\mathfrak{G}}^* \times \underline{\mathcal{G}} \times \underline{\mathcal{P}}_{\mathrm{rad}}}_{\underline{\mathcal{C}}} \to \underbrace{C^\infty(\partial\Sigma, \mathfrak{g})^*}_{\underline{\mathfrak{G}}^*}, \quad (E_\partial, e^{\varsigma_\partial}, E_{\mathrm{rad}}, A_{\mathrm{rad}}) \mapsto \int_{\partial\Sigma} \mathrm{tr}(E_\partial \cdot)\mathbf{P}$$

The prototypical reduction yields

$$T^*\underline{\mathcal{G}}/\underline{\mathcal{G}} \simeq_{\mathrm{loc}} (\underline{\mathfrak{G}}^* \times \underline{\mathcal{G}})/\underline{\mathcal{G}} \simeq \underline{\mathfrak{G}}^*\mathbf{P}$$

The second stage, fully reduced phase space locally reads:

$$\underline{\underline{\mathcal{C}}} = \underline{\mathcal{C}}/\underline{\mathcal{G}} \simeq_{\mathrm{loc}} \underline{\mathfrak{G}}^* \times \underline{\mathcal{P}}_{\mathrm{rad}}$$

With the foliation: $\underline{\mathcal{S}}_{[f]} \simeq_{\mathrm{loc}} \mathcal{O}_f \times \underline{\mathcal{P}}_{\mathrm{rad}} \hookrightarrow \underline{\mathfrak{G}}^* \times \underline{\mathcal{P}}_{\mathrm{rad}} \simeq_{\mathrm{loc}} \underline{\underline{\mathcal{C}}}$. $\mathbf{P}$

Fully reduced phase space $\simeq_{\mathrm{loc}}$ Radiative$\times$(Charge algebra)*

A note on quantization

Assume a quantization of $\underline{\mathcal{C}}$ is given $Q : C^\infty(\underline{\mathcal{C}}) \to \mathcal{B}(\mathcal{H})$.
Symplectic manifolds have trivial Poisson center (only constants). **P**

$$
\begin{array}{ccccc}
\underline{\mathcal{C}} & (C^\infty(\underline{\mathcal{C}}), \{\cdot,\cdot\}) & \xrightarrow{Q} & (\mathcal{B}(\mathcal{H}), [\cdot,\cdot]) & \text{irrep} \\
\Big\downarrow \underline{\pi} & \Big\uparrow \underline{\pi}^* & & \Big\uparrow & \\
\underline{\underline{\mathcal{C}}} & (C^\infty(\underline{\underline{\mathcal{C}}}), \{\cdot,\cdot\}) & \xrightarrow{Q\circ\underline{\pi}^*} & (\underline{\mathcal{B}}(\mathcal{H}), [\cdot,\cdot]) & \text{induced rep} \\
 & \Big\uparrow & & \Big\uparrow & \\
 & Z(C^\infty(\underline{\underline{\mathcal{C}}})) & \longrightarrow & \mathcal{Z}(\underline{\mathcal{B}}(\underline{\mathcal{H}})) & \text{center}
\end{array}
$$

P Reducibility of $\underline{\mathcal{B}}(\mathcal{H})$ induces a decomposition

$$\mathcal{H} = \bigoplus_\alpha \mathcal{H}^\alpha, \qquad \mathcal{H}^\alpha \quad C^\infty(\underline{\underline{\mathcal{C}}}) - \text{irrep}\mathbf{P}$$

E.g. 2d BF theory for G compact $\rightsquigarrow$ Peter–Weyl theorem

$$\underline{\mathcal{C}} \simeq T^*G,\ \underline{\underline{\mathcal{C}}} \simeq \mathfrak{g}^*, \quad \mathcal{H} = L^2(G) \simeq \bigoplus_\lambda (\mathcal{H}^\lambda)^* \otimes \mathcal{H}^\lambda, \quad \mathcal{H}^\lambda \quad \text{G-unirrep.}$$

Null Yang–Mills theory

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0, 1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F_u^i \wedge \mathbb{d}\mathrm{a}_i) \Big) \boldsymbol{vol}_\Sigma.\mathbf{P}$$

The *Ashtekar–Streubel* phase space $(\mathcal{P}_{\mathrm{AS}}, \varpi_{\mathrm{AS}})$ is

$$\mathcal{P}_{\mathrm{AS}} \doteq \widehat{\mathcal{A}}, \quad \varpi_{\mathrm{AS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i) \boldsymbol{vol}_\Sigma.$$

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0,1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F_u^i \wedge \mathbb{d}\mathrm{a}_i)\Big)\boldsymbol{vol}_\Sigma.\mathbf{P}$$

The *Ashtekar–Streubel* phase space $(\mathcal{P}_{\mathrm{AS}}, \varpi_{\mathrm{AS}})$ is

$$\mathcal{P}_{\mathrm{AS}} \doteq \widehat{\mathcal{A}}, \quad \varpi_{\mathrm{AS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i)\boldsymbol{vol}_\Sigma.$$

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0,1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F_u^i \wedge \mathbb{d}\mathrm{a}_i)\Big) \boldsymbol{vol}_\Sigma.\mathbf{P}$$

The *Ashtekar–Streubel* phase space $(\mathcal{P}_{\mathrm{AS}}, \varpi_{\mathrm{AS}})$ is

$$\mathcal{P}_{\mathrm{AS}} \doteq \widehat{\mathcal{A}}, \quad \varpi_{\mathrm{AS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i) \boldsymbol{vol}_\Sigma.$$

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0, 1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F_u^i \wedge \mathbb{d}\mathrm{a}_i) \Big) \boldsymbol{vol}_\Sigma.\mathbf{P}$$

The *Ashtekar–Streubel* phase space $(\mathcal{P}_{\mathrm{AS}}, \varpi_{\mathrm{AS}})$ is

$$\mathcal{P}_{\mathrm{AS}} \doteq \widehat{\mathcal{A}}, \quad \varpi_{\mathrm{AS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i) \boldsymbol{vol}_\Sigma.$$

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0,1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F^i_u \wedge \mathbb{d}\mathrm{a}_i)\Big)\boldsymbol{vol}_\Sigma.\mathbf{P}$$

The extended-*Ashtekar–Streubel* phase space $(\mathcal{P}_{\mathrm{eAS}}, \varpi_{\mathrm{eAS}})$ is

$$\mathcal{P}_{\mathrm{eAS}} \doteq \widehat{\mathcal{A}} \times T^* G_0^S, \quad \varpi_{\mathrm{eAS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i)\boldsymbol{vol}_\Sigma + \Omega_S.$$

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0, 1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F_u^i \wedge \mathbb{d}\mathrm{a}_i)\Big) \boldsymbol{vol}_\Sigma.\mathbf{P}$$

The linearly extended-*Ashtekar–Streubel* phase space $(\mathcal{P}^{\mathrm{lin}}_{\mathrm{eAS}}, \varpi^{\mathrm{lin}}_{\mathrm{eAS}})$ is

$$\mathcal{P}^{\mathrm{lin}}_{\mathrm{eAS}} \doteq \widehat{\mathcal{A}} \times T^*\mathfrak{g}^S, \quad \varpi^{\mathrm{lin}}_{\mathrm{eAS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i) \boldsymbol{vol}_\Sigma + \omega_S.$$

Null Yang–Mills theory I - relevant symplectic spaces

Let $\dim(\Sigma) = 3$ and $\Sigma = S \times I$ null, $I = [0,1]$.
$\{x^i, u\}$ coord on Σ, morally $\mathcal{I}$. **P**

Write gauge fields and "electric fields" as

$$A = A_u du + \mathrm{a} \in \mathcal{A}_\ell \times \widehat{\mathcal{A}}, \qquad \mathrm{a} \in \widehat{\mathcal{A}} \doteq C^\infty(I, \Omega^1(S, \mathfrak{g}))$$
$$E \in \mathcal{E} \doteq C^\infty(I, \Omega^{top}(S, \mathfrak{g})).\mathbf{P}$$

Then, the geometric phase space of null YM theory $(\mathcal{P}_{\mathrm{nYM}}, \omega_{\mathrm{nYM}})$ is

$$\mathcal{P}_{\mathrm{nYM}} \doteq \mathcal{A}_\ell \times \widehat{\mathcal{A}} \times \mathcal{E}, \quad \omega_{\mathrm{nYM}} \doteq \int_\Sigma \Big(\mathrm{tr}(\mathbb{d}E \wedge \mathbb{d}A_u) + \mathrm{tr}(\mathbb{d}F_u^i \wedge \mathbb{d}\mathrm{a}_i)\Big) \boldsymbol{vol}_\Sigma.\mathbf{P}$$

The *Ashtekar–Streubel* phase space $(\mathcal{P}_{\mathrm{AS}}, \varpi_{\mathrm{AS}})$ is

$$\mathcal{P}_{\mathrm{AS}} \doteq \widehat{\mathcal{A}}, \quad \varpi_{\mathrm{AS}} \doteq \int_\Sigma \mathrm{tr}((\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}^i) \boldsymbol{vol}_\Sigma.$$

$$\mathcal{P}_{\mathrm{nYM}} \overset{A_u=0}{\rightsquigarrow} \mathcal{P}_{\mathrm{AS}}, \qquad \mathbb{d}E\mathbb{d}A_u + \mathbb{d}F_u^i \mathbb{d}\mathrm{a}_i \overset{A_u=0}{\rightsquigarrow} (\partial_u \mathbb{d}\mathrm{a}_i)\mathbb{d}\mathrm{a}^i$$ *Meaning?*

Null Yang–Mills theory II - Hamiltonian setup

Geometric phase space $\mathcal{P}_{\mathrm{nYM}}$ with $\mathcal{G} = C_0^\infty(\Sigma, G) \equiv G_0^\Sigma$ action.[6]

Momentum forms: $$\langle \boldsymbol{H}(A,E),\xi\rangle = \underbrace{\mathrm{tr}(\boldsymbol{G}\,\xi)\boldsymbol{vol}_\Sigma}_{\langle \boldsymbol{H}_\circ,\xi\rangle} - \underbrace{\big(\partial_u \mathrm{tr}(E\,\xi) + \partial^i \mathrm{tr}(F_{\ell i}\,\xi)\big)\boldsymbol{vol}_\Sigma}_{d\langle \boldsymbol{h},\xi\rangle},$$

Gauss constraint: $$\mathcal{C} = \{\boldsymbol{G} \equiv \partial_u E + [A_u, E] + \mathcal{D}^i F_{ui} = 0\} = \boldsymbol{H}_\circ^{-1}(0).\mathbf{P}$$

Corner: $\partial\Sigma = S \times S$. "Initial and final" values of (vector valued) fields:

$$\varphi^{\mathrm{in}} \doteq \varphi|_{u=-1}, \quad \varphi^{\mathrm{fin}} \doteq \varphi|_{u=1}, \quad \varphi^{\mathrm{diff}} = \varphi^{\mathrm{fin}} - \varphi^{\mathrm{in}}.\mathbf{P}$$

Parametrizing $\mathfrak{g}^{\partial\Sigma} = \mathfrak{g}^S \times \mathfrak{g}^S$: $\xi^\partial = (\xi^{\mathrm{in}}, \xi^{\mathrm{fin}}) \mapsto (\xi^{\mathrm{fin}}, \xi^{\mathrm{diff}})$:

$$\langle h(A,E),\xi\rangle = -\int_S \mathrm{tr}(E^{\mathrm{fin}}\xi^{\mathrm{fin}} - E^{\mathrm{in}}\xi^{\mathrm{in}}) = -\int_S \mathrm{tr}(E^{\mathrm{diff}}\xi^{\mathrm{fin}} + E^{\mathrm{in}}\xi^{\mathrm{diff}}).$$

[6] $\mathcal{D} \doteq d + [\mathrm{a}, \cdot]$.

Theorem (Abelian case - Riello, MS)

The constraint-reduced phase space $(\underline{\mathcal{C}}, \underline{\omega})$ of null **abelian** *YM theory is symplectomorphic to the linearly extended Ashtekar-Streubel phase space:*

$$\underline{\mathcal{C}} \simeq \mathcal{P}^{\mathrm{lin}}_{\mathrm{eAS}} \doteq \widehat{\mathcal{A}} \times T^*\mathfrak{g}^S \ni (\mathrm{a}, \lambda, \mathrm{e}), \qquad ((\mathfrak{g}^S)^* \simeq \mathfrak{g}^S)$$

$$\varpi^{\mathrm{lin}}_{\mathrm{eAS}}(\mathrm{a}, \lambda, \mathrm{e}) = \int_\Sigma \sqrt{\gamma}\, \gamma^{ij}(\partial_u \mathbb{d}\mathrm{a}_i) \wedge \mathbb{d}\mathrm{a}_j + \int_S \sqrt{\gamma}\, \mathbb{d}\mathrm{e} \wedge \mathbb{d}\lambda.$$

P *It carries the Hamiltonian action of $\underline{\mathfrak{G}} \simeq \mathfrak{g}^S \times \mathfrak{g}^S \ni (\xi_{\mathrm{in}}, \xi_{\mathrm{fin}})$,*

$$\begin{cases} \mathrm{a} \mapsto \mathrm{a} + d\xi_{\mathrm{fin}} \\ \lambda \mapsto \lambda + \xi_{\mathrm{fin}} - \xi_{\mathrm{in}} \\ \mathrm{e} \mapsto \mathrm{e} \end{cases} \qquad \langle \underline{h}_{\mathrm{eAS}}, (\xi_{\mathrm{in}}, \xi_{\mathrm{fin}}) \rangle = \int_S \sqrt{\gamma}\, \big((\partial^i \mathrm{a}_i^{\mathrm{diff}})\xi_{\mathrm{fin}} - \mathrm{e}(\xi_{\mathrm{fin}} - \xi_{\mathrm{in}}) \big).\mathbf{P}$$

The on-shell electromagnetic field (E, F) at $(u, x) \in \Sigma \subset \mathcal{I}$ is given by

$$(E, F) = (\mathrm{e} + \partial^i \mathrm{a}_i^{\mathrm{in}} - \partial^i \mathrm{a}_i(u), d\mathrm{a}) \implies E^{\mathrm{diff}} \equiv E^{\mathrm{fin}}(\mathrm{a}, \mathrm{e}) - E^{\mathrm{in}}(\mathrm{e}) = -\partial^i \mathrm{a}_i^{\mathrm{diff}}.\mathbf{P}$$

Electromagnetic memory [Bieri,Garfinkle;Pasterski] is (a component of) the momentum map $\underline{h}_{\mathrm{eAS}}$ for the action of $\mathfrak{g}^S \times \mathfrak{g}^S$ on the Linearly Extended Ashtekar Streubel Phase space.

Note: $\underline{\mathcal{G}} = G^S \times G^S$ so can (carefully) proceed by stages again! **P**

Theorem (Decomposing second stage reduction, Riello MS)

The Hamiltonian reduction of $(\underline{\mathcal{C}}, \underline{\omega}, \underline{\mathcal{G}}) \simeq (\mathcal{P}^{\text{lin}}_{\text{eAS}}, \varpi^{\text{lin}}_{\text{eAS}}, G_0^{\partial\Sigma}/G)$*, with respect to the Hamiltonian action of the initial copy of* $\mathfrak{g}^S$ *at* $\mathrm{e} = 0$*, yields the Ashtekar–Streubel symplectic space* $(\widehat{\mathcal{A}}, \varpi_{\text{AS}})$*,*

$$(\mathcal{P}^{\text{lin}}_{\text{eAS}}, \varpi^{\text{lin}}_{\text{eAS}})//_0 G^{S_{\text{in}}} \simeq (\widehat{\mathcal{A}}, \varpi_{\text{AS}}).\mathbf{P}$$

It carries the residual Lie algebra action of (the final copy) $\mathfrak{g}^S$*:*

$$\underline{\varrho}_{\text{AS}}(\xi_{\text{fin}})\mathrm{a} = d\xi_{\text{fin}}, \mathbf{P}$$

with momentum map given by the electromagnetic memory

$$\langle \underline{h}_{\text{AS}}, \xi_{\text{fin}} \rangle = \int_S \sqrt{\gamma}\ ((\partial^i \mathrm{a}_i^{\text{diff}})\xi_{\text{fin}}).\mathbf{P}$$

Reproduces relationship soft symmetries $\leftrightarrow$ memory [Strominger et al] in terms of residual gauge in (partially reduced) AS phase space. **P**
Explains map $\mathcal{P}_{\text{nYM}} \rightsquigarrow \mathcal{P}_{\text{AS}}$: $A_u = 0$ gauge fixing for the group of gauge transformations trivial at the "future" celestial sphere (not both!). **P**
Question: How far can we generalise this to the non-abelian case?

Theorem (Non Abelian case - Riello, MS)

The symplectic reduction of $(\underline{\mathcal{C}}, \underline{\omega}, \underline{\mathcal{G}}) \simeq (\mathcal{P}_{\mathrm{eAS}}, \varpi_{\mathrm{eAS}}, G_0^{\partial\Sigma})$*, with respect to the Hamiltonian action of the initial copy of* $\mathfrak{g}^S$ *at* $\mathrm{e}=0$ *yields the Ashtekar–Streubel symplectic space* $(\widehat{\mathcal{A}}, \varpi_{\mathrm{AS}})$*,*

$$(\mathcal{P}_{\mathrm{eAS}}, \varpi_{\mathrm{eAS}})//G_0^{S_{\mathrm{in}}} \simeq (\widehat{\mathcal{A}}, \varpi_{\mathrm{AS}}).\mathbf{P}$$

It carries the residual Lie algebra action of (the final copy) $\mathfrak{g}_0^S$*:*

$$\underline{\varrho}_{\mathrm{AS}}(\xi_{\mathrm{fin}})\mathrm{a} = \mathcal{D}\xi_{\mathrm{fin}} \doteq d\xi_{\mathrm{fin}} + [\mathrm{a}, \xi_{\mathrm{fin}}].\mathbf{P}$$

with momentum map given by "the non-Abelian memory":

$$\langle \underline{h}_{\mathrm{AS}}, \xi_{\mathrm{fin}} \rangle = \int_S \sqrt{\gamma}\, \mathrm{tr}((\mathcal{D}^i\partial_u \mathrm{a}_i)^{\int} \xi_{\mathrm{fin}}), \quad (\mathcal{D}^i\partial_u \mathrm{a}_i)^{\int} \doteq \int_{-1}^{1} \mathcal{D}^i\partial_u \mathrm{a}_i du\mathbf{P}$$

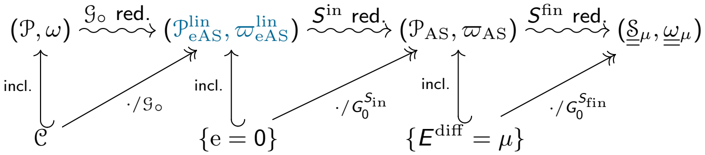
Attention! this is NOT the "color memory" [Pasterski, Raclariu, Strominger]!
In a mode-decomposition:

$$(\mathcal{D}^i L_\ell \mathrm{a}_i)^{\int} = \partial^i \mathrm{a}_i^{\mathrm{diff}} + \sum_{k\geq 0} \left[\mathfrak{Re}(2\widetilde{\mathrm{a}}(k)^i), \mathfrak{Im}(2\widetilde{\mathrm{a}}(k)_i)\right] \neq \partial^i \mathrm{a}_i^{\mathrm{diff}}$$

Conclusions

Conclusions

1. In good cases, Noether theorems set us up for Hamiltonian reduction, where the (fully) reduced phase space is $\underline{\underline{\mathcal{C}}} = \mathcal{C}/\mathcal{G}$. **P**
2. In the presence of corners, there is a mismatch between $\mathcal{C}$ and the zero-level set of the FULL Noether current $\boldsymbol{H}$. **P**
3. Reduction fails to output a symplectic manifold, but rather $\underline{\underline{\mathcal{C}}} = \mathcal{C}/\mathcal{G} = \underline{\mathcal{C}}/\underline{\mathcal{G}}$ is Poisson. **P**
4. For null, abelian YM theory this yields



with μ a fixed value for the Electromagnetic memory. **P**

5. Quantization (at $\mathrm{e} = 0$) should yield decomposition in memory eigenvalues, **simply owing to the Hamiltonian structure**.

Thanks!