Simplifying harmonic gauge perturbations around black holes

[arXiv:1711.00585, 1801.09800, 2004.09651] + WIP

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Harmonic Gauge and its Advantages

- On a Lorentzian (M, g), $R_{\mu\nu} = 0$ vacuum, consider scalar z ($s = 0$), Maxwell v_μ ($s = 1$) and Einstein $p_{\mu\nu}$ ($s = 2$) perturbations:

$$(SW) \quad \Box z = 0,$$

$$(Max)\ (VW) \quad \Box v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0$$

$$(v_\mu = \nabla_\mu \varepsilon \rightsquigarrow \Box\varepsilon = 0 \text{ residual gauge dynamics}),$$

$$(Ein)\ (LW) \quad \Box p_{\mu\nu} - 2\,{}^4R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\,\nabla_{(\mu}\nabla^\lambda \overline{p}_{\nu)\lambda} = 0$$

$$(p_{\mu\nu} = \nabla_{(\mu} v_{\nu)} \rightsquigarrow \Box v_\mu = 0 \text{ residual gauge dynamics}).$$

- Under harmonic gauges ($\nabla^\mu v_\mu = 0$ and $\nabla^\nu \overline{p}_{\mu\nu} = \nabla^\nu(p_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \operatorname{tr} p) = 0$) we get the vector wave and Lichnerowicz wave equations.
- **Advantages:** well known regularity properties for solutions in harmonic gauge
- **Disadvantages:** reduction to master equations and separation of variables is usually done in Regge-Wheeler (Schwarzschild) or radiation (Kerr) gauges; not obvious in harmonic gauge.

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Schwarzschild background

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Full separation of variables for any $s = 0, 1, 2$:

$$\Phi(t, r, \theta, \varphi) = \{\phi_{\omega lm}(r) Y^{lm}(\theta, \varphi)\} e^{-i\omega t}$$

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- But gauge invariant modes decouple and satisfy spin-s Regge-Wheeler equations $\mathcal{D}_s \phi^s(r) = 0$.

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Radial Mode Equation: $VW_\omega[v] = 0$

Explicitly, $v_\mu \to v(r) = (v_t, v_r, u \mid w)$:

(odd) $$\partial_r \mathcal{B}_l r^2 f \partial_r w + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \mathcal{B}_l w + \mathcal{B}_l \frac{2M}{r} w = 0,$$

(even) $$\begin{bmatrix} -\partial_r \frac{1}{f}\, r^2 f \partial_r\, v_t \\ \partial_r f\, r^2 f \partial_r\, v_r \\ \partial_r \mathcal{B}_l\, r^2 f \partial_r\, u \end{bmatrix} + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \begin{bmatrix} -\frac{1}{f}\, v_t \\ f\, v_r \\ \mathcal{B}_l\, u \end{bmatrix}$$
$$+ i\omega \frac{2M}{f} \begin{bmatrix} v_r \\ -v_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2f^2 & 2\mathcal{B}_l f \\ 0 & 2\mathcal{B}_l f & \mathcal{B}_l \frac{2M}{r} \end{bmatrix} \begin{bmatrix} v_t \\ v_r \\ u \end{bmatrix} = 0,$$

where $f(r) = 1 - \frac{2M}{r}$ and $\mathcal{B}_l = l(l+1)$.

Radial Mode Equation: $LW_\omega[p] = 0$ (odd sector)

Explicitly, $p_{\mu\nu} \to p(r) = (h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G \mid h_t, h_r, h_2)$:

$$\begin{bmatrix} \partial_r(-2\frac{\mathcal{B}_l}{f}\, r^2 f\partial_r)\, h_t \\ \partial_r(2\mathcal{B}_l f\, r^2 f\partial_r)\, h_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\, r^2 f\partial_r)\, h_2 \end{bmatrix} - \mathcal{B}_l \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -4\frac{\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 \\ 0 & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix}$$

$$-i\omega\frac{4M}{f} \begin{bmatrix} 0 & -\mathcal{B}_l & 0 \\ \mathcal{B}_l & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix} + \omega^2 \frac{r^2}{f} \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix} = 0$$

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Radial Mode Equation: $LW_\omega[p] = 0$ (even sector)

$$
\begin{bmatrix} \partial_r(-2\,r^2 f\partial_r)\,h_{tr} \\ \partial_r(-2\frac{\mathcal{B}_l}{f}\,r^2 f\partial_r)\,j_t \\ \partial_r(\frac{1}{f^2}\,r^2 f\partial_r)\,h_{tt} \\ \partial_r(f^2\,r^2 f\partial_r)\,h_{rr} \\ \partial_r(2\,r^2 f\partial_r)\,K \\ \partial_r(2\mathcal{B}_l f\,r^2 f\partial_r)\,j_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\,r^2 f\partial_r)\,G \end{bmatrix} - \mathcal{B}_l \begin{bmatrix} -2\,h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\,j_t \\ \frac{1}{f^2}\,h_{tt} \\ f^2\,h_{rr} \\ 2\,K \\ 2\mathcal{B}_l f\,j_r \\ \frac{\mathcal{A}_l}{2}\,G \end{bmatrix}
$$

$$
\begin{bmatrix} \frac{2(f^2+1)}{f} & -4\mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\ -4\mathcal{B}_l & -\frac{4\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4M^2}{2f^3r^2} & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{r} & \frac{2}{f}\frac{2M}{r} & 0 & 0 \\ 0 & 0 & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{e} & \frac{f(\frac{4M^2}{r^2}-8f^2)}{2} & 4f(1-\frac{3M}{r}) & 4\mathcal{B}_l f^2 & 0 \\ 0 & 0 & \frac{2}{f}\frac{2M}{r} & 4f(1-\frac{3M}{r}) & -4(1-\frac{4M}{r}) & -4\mathcal{B}_l f & 0 \\ 0 & 0 & 0 & 4\mathcal{B}_l f^2 & -4\mathcal{B}_l f & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 0 & 0 & 0 & 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{bmatrix} \begin{bmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{bmatrix}
$$

$$
-i\omega\frac{4M}{f} \begin{bmatrix} 0 & 0 & -\frac{1}{f} & -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathcal{B}_l & 0 \\ \frac{1}{f} & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{bmatrix} + \omega^2\frac{r^2}{f} \begin{bmatrix} -2\,h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\,j_t \\ \frac{1}{f^2}\,h_{tt} \\ f^2\,h_{rr} \\ 2\,K \\ 2\mathcal{B}_l f\,j_r \\ \frac{\mathcal{A}_l}{2}\,G \end{bmatrix} = 0
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Simplified Radial Mode Equations

- Vector wave equation [arXiv:1711.00585]:
  - $VW^{\text{odd}}_{\omega} \sim \mathcal{D}_1 \quad VW^{\text{even}}_{\omega} \sim$
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Simplification of a Differential Equation

- A simplification is an isomorphism $E[\phi] = 0 \sim \tilde{E}[\tilde{\phi}] = 0$ from a more complicated PDE to a simpler PDE.
- **Q:** What is a(n iso)morphism between Differential Equations?

$$E[\phi] = 0 \quad \overset{k}{\rightsquigarrow} \quad \tilde{E}[\tilde{\phi}] = 0$$

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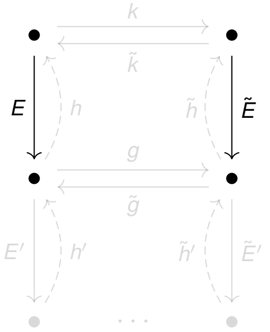

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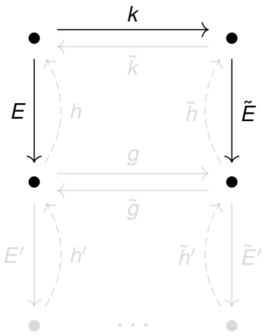

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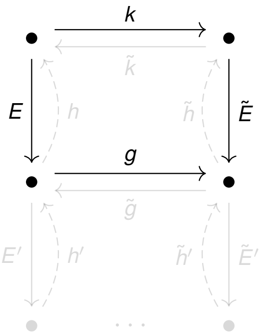

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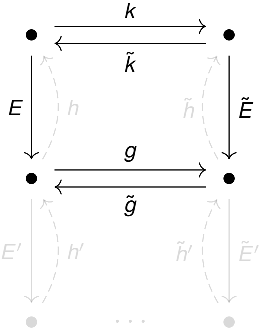

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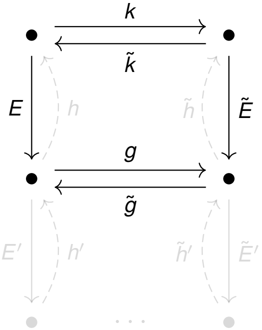

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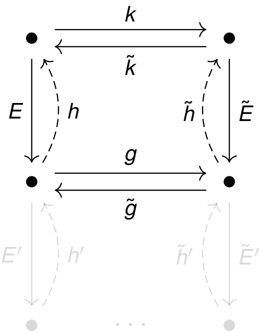

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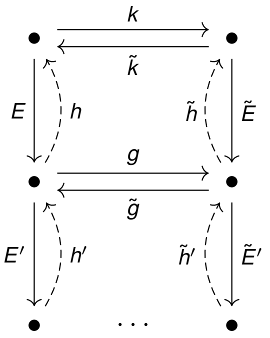

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Kerr background

- **Kerr:** axially symmetric, stationary black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -\frac{\Delta_r}{\Sigma}(\mathrm{d}\tau + y^2 \mathrm{d}\psi)^2 + \frac{\Delta_y}{\Sigma}(\mathrm{d}\tau - r^2 \mathrm{d}\psi)^2 + \Sigma\left(\frac{(\mathrm{d}r)^2}{\Delta_r} + \frac{(\mathrm{d}y)^2}{\Delta_y}\right),$$

to Boyer-Lindquist coords: $\tau = t - a\varphi$, $y = a\cos\theta$, $\psi = \varphi/a$,

$$\Sigma = r^2 + y^2, \quad \Delta_y = a^2 - y^2, \quad \Delta_r = r(r-2M) + a^2.$$

- Partial separation of variables by symmetry ($s = 0, 1, 2$):

$$\Phi = \phi_{\omega m}(r, y) e^{-i\omega t} e^{im\psi}$$

- Teukolsky scalars ($\Phi^{\pm s} = \ldots$) decouple,

$$\Phi^{\pm s}_{\omega m}(r, y) = R^{\pm s}_{\omega m\lambda}(r) Y^{\pm s}_{\omega m\lambda}(y),$$

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- Superficially, harmonic gauge perturbation equations do not fully separate on Kerr.
- What if we could isolate all the modes in harmonic gauge (gauge, gauge invariant modes, constraint violating modes) and fully separate each of the resulting equations?
  Then harmonic gauge perturbations would fully separate indirectly.
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- **Q:** On Kerr, can we achieve full separation of variables and upper triangular simplification like on Schwarzschild?
- A crucial step ($s = 1$): do $v_\mu = \nabla_\mu \varepsilon$ (gauge), $z = \nabla^\mu v_\mu$ (constraint violating) and $\Phi^{\pm 1}$ (Teukolsky invariants) exhaust all degrees of freedom in the solutions of $\Box v_\mu = 0$?
  Precise question, by analogy with Schwarzschild:

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