Estimates for Low Regularity Wave Maps on $\mathbb{R} \times \mathbb{S}^3$

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Quantum and Classical Fields Interacting with Geometry
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Introduction

- Preliminaries
- Classical Null Form Estimates of Klainerman–Machedon
- Main Results
- Fourier Analysis via Peter–Weyl Theory
- Method of Proof
- Concluding Remarks
- Based on arXiv:2307.13052 + .

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Definition (Wave Maps on Minkowski Space)

Given a Riemannian manifold (M, g), a function on Minkowski space $\mathbb{R}^{1+n}$

$$\phi : \mathbb{R}_t \times \mathbb{R}^n_x \to M$$

satisfies the wave map equation if

$$\Box\phi^i = \Gamma^i_{jk}(\phi)\partial^\alpha\phi^j\partial_\alpha\phi^k = \Gamma^i_{jk}(\phi)Q_0(\phi^j, \phi^k), \tag{1}$$

where Γ^i_{jk} are the Christoffel symbols of g and α's are contracted using the Minkowski metric. Equation (1) is the Euler–Lagrange equation for

$$L_M[\phi] = \int_{\mathbb{R}\times\mathbb{R}^n} |\partial_t\phi|^2_g - |\nabla_x\phi|^2_g \, \mathrm{d}t \, \mathrm{d}x.$$

Lorentzian analogue of harmonic maps.

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- nonlinear σ-models in theoretical physics, magnetism, materials...
- extremely well-studied when the background is Minkowski (Christodoulou, Kenig, Klainerman, Krieger, Lindblad, Machedon, Metcalfe, Nirenberg, Ponce, Rodnianski, Selberg, Shatah, Sideris, Sterbenz, Struwe, Tao, Tataru, Vega...)
- for general nonlinearities in $n = 3$ cannot expect well-posedness unless data is in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for $s > 2$ (Ponce–Sideris '93, Lindblad '93, '96)
- in general even if local solutions exist ($s > 2$), they may blow up in finite time (John '81)
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- a natural problem is to consider initial data

$$(\phi_0, \phi_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$$

  and seek a solution $\phi \in C^0([-T,T]; H^s(\mathbb{R}^n)) \cap C^1([-T,T]; H^{s-1}(\mathbb{R}^n))$, possibly with $T = \infty$.

- is invariant with respect to the scaling

$$\phi(t,x) \longrightarrow \phi_\lambda(t,x) = \phi(\lambda t, \lambda x), \qquad \lambda \in \mathbb{R}$$

  $\rightsquigarrow$ self-similar blow-up solutions possible for large data (if no symmetry, cf. Christodoulou–Tahvildar-Zadeh '93)
  $\rightsquigarrow$ for global existence must focus on small data

- has a conserved energy:

$$E[\phi] = \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t\phi|_g^2 + |\nabla_x\phi|_g^2 \, \mathrm{d}x, \qquad \frac{\mathrm{d}}{\mathrm{d}t}E[\phi] = 0.$$

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Criticality

For $s > \frac{n+2}{2}$ local well-posedness is "easy" and follows from standard energy estimates + Sobolev embedding argument.

Question

How much can one reduce s? The $\dot{H}^s(\mathbb{R}^n)$ norm of ϕ scales as

$$\|\phi_\lambda\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{s-\frac{n}{2}}\|\phi\|_{\dot{H}^s(\mathbb{R}^n)},$$

i.e. exponent $s = \frac{n}{2}$ is critical.

- For $s > \frac{n}{2}$ can trade time of local existence against size of initial data;
- For $s \leqslant \frac{n}{2}$ the problem is non-local in time;
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The wave map equation on Minkowski space is ill-posed with data in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ for $s < \frac{n}{2}$.

Theorem

The wave map equation on Minkowski space is locally well-posed for initial data in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ for initial data in $s > \frac{n}{2}$.

- Bourgain '93 ($n = 3$), Klainerman–Machedon '95 ($n \geqslant 4$), $s > \frac{n}{2}$,
- Zhou '97: $n = 2$, $s \geqslant \frac{9}{8}$,
- Klainerman–Selberg '97: all $n \geqslant 2$, $s > \frac{n}{2}$,
- Keel–Tao '98: $n = 1$, $s > \frac{n}{2}$.

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For $n \geqslant 2$ and "reasonable" target manifolds M the wave map equation on Minkowski space is globally ($\iff$ locally) well-posed for small initial data in $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$.

- Tao '00: $n \geqslant 5$, $M = \mathbb{S}^{m-1}$,
- Klainerman–Rodnianski '00: $n \geqslant 5$, more general M,
- Tao '01: $n \geqslant 2$, $M = \mathbb{S}^{m-1}$,
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- Tao '00: $n = 1$, $s = \frac{n}{2}$ is ill-posed.

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Open questions: scattering

- Related to global existence is the question of scattering: do there exist scattering states $\phi^{\pm}|_{\mathscr{I}} \in \mathcal{H}$ in some space $\mathcal{H}$ such that

$$\lim_{t\pm\infty} \|\phi(t) - \phi^{\pm}|_{\mathscr{I}}\|_{\mathcal{H}} = 0 \ ?$$

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- AFAIK scattering in $\dot{H}^{3/2} \times \dot{H}^{1/2}$ open
- For GR, would like a geometric interpretation of scattering as a characteristic initial value problem
- Would also like well-posedness results on more general backgrounds, i.e. $\phi : (N, h) \to (M, g)$ critical points of

$$\int_N \nabla_\mu \phi^\alpha \nabla_\nu \phi^\beta g_{\alpha\beta} h^{\mu\nu} \,\mathrm{dvol}_N \iff \Box_h \phi = \Gamma_g(\phi) Q_0^h(\phi, \phi)$$

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Some previous work on curved spacetimes:

- Shatah–Struwe '02: $N = \mathbb{R} \times \mathbb{R}^n$ flat, $n \geqslant 4$, $s = \frac{n}{2}$, *moving frame approach*,
- Geba '09: $3 \leqslant n \leqslant 5$, $s > \frac{n}{2}$, $N = \mathbb{R} \times \mathbb{R}^n$, h a perturbation of η,
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The wave map equation on $\mathbb{R} \times \mathbb{S}^3$

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is locally well-posed for small initial data in $H^s(\mathbb{S}^3) \times H^{s-1}(\mathbb{S}^3)$ for $s > \frac{3}{2}$, or $\dot{H}^{3/2}(\mathbb{S}^3) \times \dot{H}^{1/2}(\mathbb{S}^3)$.

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Null Form Estimates

The key nonlinearity to understand is

$$Q_0(\phi,\psi) = \partial_\alpha \phi \, \partial^\alpha \psi = \partial_t \phi \, \partial_t \psi - \nabla_x \phi \cdot \nabla_x \psi.$$

Breakthrough result of Klainerman–Machedon ('95) relied on:

Theorem (Klainerman–Machedon, '93)

For ϕ, ψ satisfying $\Box\phi = 0 = \Box\psi$ with data $(\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1)$, $(\psi, \partial_t\psi)|_{t=0} = (\psi_0, \psi_1)$ the null form $Q_0(\phi,\psi)$ satisfies the estimate

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- gain of ≈ 1 derivative
- similar estimates also hold for the "Yang–Mills/MKG" null forms

$$Q_{\alpha\beta}(\phi,\psi) = \nabla_\alpha \phi \nabla_\beta \psi - \nabla_\beta \phi \nabla_\alpha \psi$$

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- forthcoming work on $\mathbb{R} \times \mathbb{S}^3$ with J.-P. Nicolas

Null Form Estimates

The key nonlinearity to understand is

$$Q_0(\phi, \psi) = \partial_\alpha \phi \, \partial^\alpha \psi = \partial_t \phi \, \partial_t \psi - \nabla_x \phi \cdot \nabla_x \psi.$$

Breakthrough result of Klainerman–Machedon ('95) relied on:

Theorem (Klainerman–Machedon, '93)

For ϕ, ψ satisfying $\Box \phi = 0 = \Box \psi$ with data $(\phi, \partial_t \phi)|_{t=0} = (\phi_0, \phi_1)$, $(\psi, \partial_t \psi)|_{t=0} = (\psi_0, \psi_1)$ the null form $Q_0(\phi, \psi)$ satisfies the estimate

$$\|Q_0(\phi, \psi)\|_{L^2(\mathbb{R}^4)} \lesssim \|(\phi_0, \phi_1)\|_{H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)} \|(\psi_0, \psi_1)\|_{H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)}$$

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Null Form Estimates

"Deeper" estimates to get close to criticality:

Theorem (Foschi–Klainerman, '00)

For ϕ, ψ satisfying $\Box\phi = 0 = \Box\psi$ with data $(\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1)$ and $(\psi, \partial_t\psi)|_{t=0} = (\psi_0, \psi_1)$ there holds

$$\|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} Q_0(\phi,\psi)\|_{L^2(\mathbb{R}^{1+n})} \lesssim \|(\phi_0,\phi_1)\|_{H^{\alpha_1}(\mathbb{R}^n)\oplus H^{\alpha_1-1}(\mathbb{R}^n)} \times \|(\psi_0,\psi_1)\|_{H^{\alpha_2}(\mathbb{R}^n)\oplus H^{\alpha_2-1}(\mathbb{R}^n)}$$

for any α_i, β_0, $\beta_\pm$ satisfying

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n+3}{2}, \qquad \beta_- \geqslant -\frac{n+1}{4}, \qquad \beta_0 > -\frac{n-1}{2},$$

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Curved spacetimes

For curved spacetimes less is known. Basic estimate was obtained by Sogge, Georgiev–Schirmer, Sogge–Smith, Tataru.

Theorem (Sogge '93, Georgiev–Schirmer '93)

For $\Box\phi = 0 = \Box\psi$ *on* $\mathbb{R} \times \mathbb{S}^3$,

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Proof uses FIOs to localize and flatten the metric and then Klainerman & Machedon's original techniques. (Sogge treats more general compact manifolds K in place of $\mathbb{S}^3$ of any dimension, but no estimates with multipliers.)

Question

Do Foschi–Klainerman estimates hold on curved backgrounds?

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Do Foschi–Klainerman estimates hold on curved backgrounds?

Main theorem

Theorem (T. '23/'24)

For free waves ϕ, ψ satisfying

$$\Box_{\mathbb{R}\times\mathbb{S}^3}\phi + \phi = 0 = \Box_{\mathbb{R}\times\mathbb{S}^3}\psi + \psi$$

with data $(\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1)$ and $(\psi, \partial_t\psi)|_{t=0} = (\psi_0, \psi_1)$ on $\mathbb{R}\times\mathbb{S}^3$ the estimate

$$\begin{aligned}\|J^{\beta_0} W^{\beta_w} Q_0(\phi,\psi)\|_{L^2([-\pi,\pi]\times\mathbb{S}^3)} &\lesssim \|(\phi_0,\phi_1)\|_{H^{\alpha_1}(\mathbb{S}^3)\oplus H^{\alpha_1-1}(\mathbb{S}^3)} \\ &\quad \times \|(\psi_0,\psi_1)\|_{H^{\alpha_2}(\mathbb{S}^3)\oplus H^{\alpha_2-1}(\mathbb{S}^3)},\end{aligned}$$

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Cancellations in Fourier space

Key observation in $\mathbb{R}^3$: for free waves ϕ, ψ the spacetime Fourier symbol of $Q_0(\phi, \psi) = \partial_t\phi\,\partial_t\psi - \nabla_x\phi \cdot \nabla_x\psi$ is

$$q_0^{\pm}(\eta, \zeta) = \pm|\eta||\zeta| - \eta \cdot \zeta,$$

which vanishes when η and ζ are parallel. Captures cancellations in Q_0 between parallel waves. Classical proof of null form estimates goes in 3 steps:

Step 1: positive/negative frequency splitting

For $\Box\phi = 0$ with data $(\phi, \partial_t\phi)|_{t=0} = (0, \phi_1)$ the solution is

$$\hat{\phi}(t, \xi) = \frac{\sin(|\xi|t)}{|\xi|}\hat{\phi}_1(\xi) = \frac{1}{2i}(\hat{\phi}^+(t, \xi) - \hat{\phi}^-(t, \xi)),$$

where

$$\phi^{\pm}(t, x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{\pm it|\xi| + ix\cdot\xi}}{|\xi|}\hat{\phi}_1(\xi)\,d\xi$$

By bilinearity, enough to understand $Q_0(\phi^{\pm}, \psi^{\pm})$.

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Cancellations in Fourier space

Step 2: spacetime FT of $Q_0(\phi^\pm, \psi^\pm)$

Using $2Q_0(\phi^\pm, \psi^\pm) = \Box(\phi^\pm \psi^\pm)$, the inverse convolution formula gives

$$\begin{aligned}\mathcal{F}_{t,x}\left(Q_0(\phi^\pm, \psi^\pm)\right)(\tau, \xi) &= \frac{1}{2}(\tau^2 - |\xi|^2)\mathcal{F}_{t,x}(\phi^\pm) * \mathcal{F}_{t,x}(\psi^\pm) \\ &= \pi^2 \int_{\mathbb{S}^2} \alpha^2 \hat{\phi}_1\left(\frac{\alpha}{2}\omega\right) \hat{\psi}_1\left(\xi - \frac{\alpha}{2}\omega\right) \mathrm{d}^2\omega,\end{aligned}$$

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Step 3: Plancherel & Cauchy–Schwarz

$$\begin{aligned}\|Q_0(\phi^\pm, \psi^\pm)\|^2_{L^2(\mathbb{R}^4_{t,x})} &\simeq \|\mathcal{F}_{t,x}(Q_0(\phi^\pm, \psi^\pm))\|^2_{L^2(\mathbb{R}^4_{\tau,\xi})} \\ &\lesssim \int_{\mathbb{R}^3} \mathrm{d}\xi \int_0^\infty \mathrm{d}\alpha \int_{\mathbb{S}^2} \mathrm{d}^2\omega\, \alpha^4 \left|\hat{\phi}_1\left(\frac{\alpha}{2}\omega\right)\right|^2 \left|\hat{\psi}_1\left(\xi - \frac{\alpha}{2}\omega\right)\right|^2 \\ &\lesssim \int_{\mathbb{R}^3} \mathrm{d}\xi \int_{\mathbb{R}^3} \mathrm{d}\xi' |\xi'|^2 |\hat{\phi}_1(\xi')|^2 |\hat{\psi}_1(\xi - \xi')|^2 \\ &\lesssim \|\phi_1\|^2_{H^1(\mathbb{R}^3)} \|\psi_1\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

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$$\begin{aligned}\|Q_0(\phi^\pm, \psi^\pm)\|^2_{L^2(\mathbb{R}^4_{t,x})} &\simeq \|\mathcal{F}_{t,x}(Q_0(\phi^\pm, \psi^\pm))\|^2_{L^2(\mathbb{R}^4_{\tau,\xi})} \\ &\lesssim \int_{\mathbb{R}^3} \mathrm{d}\xi \int_0^\infty \mathrm{d}\alpha \int_{\mathbb{S}^2} \mathrm{d}^2\omega\, \alpha^4 \left|\hat{\phi}_1\left(\frac{\alpha}{2}\omega\right)\right|^2 \left|\hat{\psi}_1\left(\xi - \frac{\alpha}{2}\omega\right)\right|^2 \\ &\lesssim \int_{\mathbb{R}^3} \mathrm{d}\xi \int_{\mathbb{R}^3} \mathrm{d}\xi' |\xi'|^2 |\hat{\phi}_1(\xi')|^2 |\hat{\psi}_1(\xi - \xi')|^2 \\ &\lesssim \|\phi_1\|^2_{H^1(\mathbb{R}^3)} \|\psi_1\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Cancellations in Fourier space

Step 2: spacetime FT of $Q_0(\phi^\pm, \psi^\pm)$

Using $2Q_0(\phi^\pm, \psi^\pm) = \Box(\phi^\pm \psi^\pm)$, the inverse convolution formula gives

$$\begin{aligned}\mathcal{F}_{t,x}\left(Q_0(\phi^\pm, \psi^\pm)\right)(\tau, \xi) &= \frac{1}{2}(\tau^2 - |\xi|^2)\mathcal{F}_{t,x}(\phi^\pm) * \mathcal{F}_{t,x}(\psi^\pm) \\ &= \pi^2 \int_{\mathbb{S}^2} \alpha^2 \hat{\phi}_1\left(\frac{\alpha}{2}\omega\right) \hat{\psi}_1\left(\xi - \frac{\alpha}{2}\omega\right) \mathrm{d}^2\omega,\end{aligned}$$

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Global method on $\mathbb{R} \times \mathbb{S}^3$

Observation

Using $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, may try to replicate the method on $\mathbb{R} \times \mathrm{SU}(2)$ by exploiting global Lie group structure. $\mathrm{SU}(2)$ non-abelian, so no Pontryagin duality; need Peter–Weyl theory.

Very brief recap of Peter–Weyl theory

G a compact Lie group.

Definition

The unitary dual $\hat{\mathrm{G}}$ of G is the set of equivalence classes of unitary irreducible representations of G.

Definition

Let $f \in L^1(\mathrm{G})$. For each $\pi \in \hat{\mathrm{G}}$ the Fourier coefficient $\hat{f}(\pi)$ is the operator

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The matrix coefficients of unitary irreducible representations of G *are dense in* $L^2(\mathrm{G})$*:*

$$L^2(\mathrm{G}) = \overline{\bigoplus_{\pi \in \hat{\mathrm{G}}} \mathcal{M}_\pi}^{L^2},$$

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With $G = SU(2) = \mathbb{S}^3$:

- the characters $e^{\pm ix\cdot\xi}$ of $\mathbb{R}^3$ are replaced with irreps π on SU(2)
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With $G = SU(2) = \mathbb{S}^3$:

- an explicit choice of π_m's is given by Wigner's D-matrices

$$\mathcal{D}^{(j)}(\{\alpha\beta\gamma\})_{\mu'\mu} = \sum_x (-1)^x \frac{\sqrt{(j+\mu)!(j-\mu)!(j+\mu')!(j-\mu')!}}{(j-\mu'-x)!(j+mu-x)!x!(x+\mu'-m)!} \times e^{i\mu'\alpha} \cos^{2j+\mu-\mu'-2x} \frac{1}{2}\beta \cdot \sin^{2x+\mu'-\mu} \frac{1}{2}\beta \cdot e^{i\mu\gamma}$$

  [Wigner '59, *Group Theory and Atomic Spectra*]

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Time periodicity

Recall in step 3 (Plancherel & Cauchy–Schwarz) defined

$$2\xi' = \alpha\omega = \left(\frac{\tau^2 - |\xi|^2}{\tau - \xi\cdot\omega}\right)\omega \ :$$

- $\alpha(\tau,\xi)$ mixes time and space Fourier variables on $\mathbb{R}^{1+3}$
- On $\mathbb{R}\times SU(2)$ the space Fourier variable m is discrete, but time Fourier variable is continuous
- but for solutions of the modified wave equation on $\mathbb{R}\times\mathbb{S}^3$,

$$\Box\phi + \phi = 0,$$

are periodic in time:

$$\partial_t^2\hat{\phi}(\pi_m) + (1 + m(m+2))\hat{\phi}(\pi_m) = 0 \implies \partial_t^2\hat{\phi}(\pi_m) = -(m+1)^2\hat{\phi}(\pi_m)$$

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Modified equation

On $\mathbb{R} \times \mathbb{S}^3$ study instead $Q_0(\phi, \psi)$ for ϕ, ψ satisfying

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i.e.

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$$\phi^\pm(t, x) = \sum_{m \geqslant 0} e^{\pm i(m+1)t} \operatorname{Tr}\left(\hat{\phi}_1(\pi_m)\pi_m(x)\right)$$

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In $\mathbb{R}^{1+3}$ this relies on "inverse" convolution formula

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In non-abelian setting for f, $g : \mathrm{G} \to \mathbb{R}$ one can define

$$(f * g)(x) = \int_{\mathrm{G}} f(y) g(xy^{-1}) \, \mathrm{d}\mu(y).$$

Then the forward convolution formula

$$\widehat{(f * g)}(\pi) = \hat{f}(\pi) \circ \hat{g}(\pi)$$

holds; $\hat{f}(\pi)$, $\hat{g}(\pi)$ are operators.

In general there is insufficient structure on $\hat{\mathrm{G}}$ to define $\hat{f} * \hat{g} \rightsquigarrow$ no "inverse" convolution formula.

On $\mathbb{R} \times \mathrm{SU}(2)$ need to compute $\mathcal{F}_{t,x}(\phi^{\pm}\psi^{\pm})$ directly: OK using inverse convolution in $\mathbb{R}$ factor, schematically

$$\mathcal{F}_{t,x}(\phi^{\pm}\psi^{\pm})(\pi_m)_n = \sum_l \underbrace{\hat{\phi}_1(\pi_l)\hat{\psi}_1(\pi_{n-l}) \int_{\mathrm{SU}(2)} \pi_l \otimes \pi_{n-l} \otimes \pi_m^{\dagger} \, \mathrm{d}\mu}_{\stackrel{def}{=} (m+1)(\varpi_l(\pi_m)_n)_{pq}}.$$

Recall $\pi_l \otimes \pi_{n-l}$ "smears" over a range of irreps, with probability amplitude weights.

Non-abelian step 3

Step 3: Plancherel & Cauchy–Schwarz

After a calculation, must handle a Clebsch–Gordan expansion of the form

$$\sum_l (m+1)(\varpi_l(\pi_m)_n)_{pq} = \sum_l \hat{\phi}_1(\pi_l)_{ji} \hat{\psi}_1(\pi_{n-l})_{(q-j)(p-i)} \mathcal{C}^{l\,(n-l)}_{m\,i\,(p-i)} \mathcal{C}^{l\,(n-l)}_{m\,j\,(q-j)}$$

- fundamentally different from calculation in abelian case
- does not localize around a single π, even asymptotically
- requires estimating "matrix convolutions"

Using orthogonality of Clebsch–Gordan coefficients $\mathcal{C}$, can recover a discrete Young's inequality for convolutions for $\varpi_l(\pi_m)_n$:

Lemma

For the matrices $\varpi_l(\pi_m)_n$ there hold the estimates

$$\sum_m |||(m+1)\varpi_l(\pi_m)_n|||^2 \leqslant |||\hat{\phi}_1(\pi_l)|||^2 \, |||\hat{\psi}_1(\pi_{n-l})|||^2$$

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Here $\sum_m$ not $\sum_l$ $\rightsquigarrow$ loss of arbitrarily small amount of regularity.

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This allows to define wave-Sobolev spaces $H^{s,b}$ of Bourgain & Klainerman et al on $\mathbb{R} \times \mathbb{S}^3$:

$$\|u\|_{H^{s,b}(\mathbb{R}\times\mathbb{S}^3)} = \|(m+1)^{s+\frac{1}{2}} \langle (m+1) - |n| \rangle^b \widetilde{u}(\pi_m)_n\|_{\ell^2_m \ell^2_n}$$

Compare to $\mathbb{R}^{1+3}$:

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