Asymptotic Symmetries and Logarithmic Soft Theorems in Gauge Theories and Gravity

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Soft theorem

- Soft theorem is a statement in QFT that derives from studying Feynman diagrams.

$$\lim_{\omega\to 0} A_{n+1}(k,\pm) \;=\; \lim_{\omega\to 0}\left(\frac{1}{\omega}S^{\pm}_{-1} + S^{\pm}_{0} + O(\omega)\right) A_n$$

- $\frac{1}{\omega}S^{\pm}_{-1}$ is of order $1/\omega$ and diverges as $\omega \to 0$.
  $S^{\pm}_{0}$ is independent of ω and is convergent.

Soft theorem

$$\lim_{\omega\to 0} A_{n+1}(k,\pm) = \lim_{\omega\to 0}\left(\frac{1}{\omega}S^{\pm}_{-1} + S^{\pm}_{0} + O(\omega)\right) A_n$$

- Leading soft photon theorem

$$\frac{1}{\omega}S^{\pm}_{-1} = \sum_i Q_i \frac{p_i \cdot \epsilon^{\pm}(k)}{p_i \cdot k}$$

- Subleading soft photon theorem (tree-level)

$$S^{\pm}_{0} = -i \sum_i Q_i \frac{k \cdot J_i \cdot \epsilon^{\pm}(k)}{p_i \cdot k}$$

- Also leading/subleading soft graviton theorems, soft gluon theorem, soft photino theorem...

Asymptotic symmetry

- What is an asymptotic symmetry?

$$\text{asymptotic symmetry} = \frac{\text{allowed gauge symmetry}}{\text{trivial gauge symmetry}}$$

  There are exceptions, but it is a useful way to think about this.

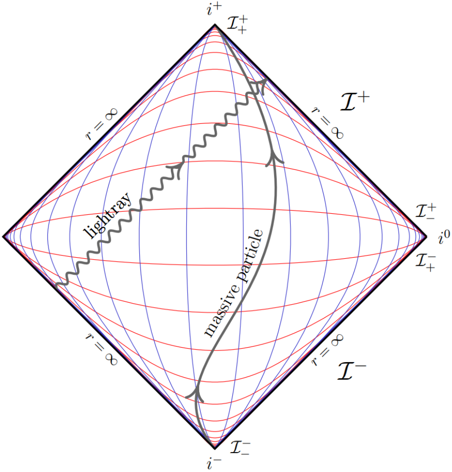
- Aren't all gauge transformations trivial?
  No. If you follow the Noether procedure, you notice that there is a falloff condition on the parameter for the symmetry to be trivial.

- This asymptotic symmetry group is often referred to as large gauge transformations (LGT). This definition allows for topologically trivial LGTs.

  This is also referred to as the improper gauge transformation.

Asymptotic symmetry

Penrose diagram of Minkowski spacetime



(drawing taken from Strominger's lecture notes)

Asymptotic symmetry

- The asymptotic symmetry of QED is the set of $U(1)$ gauge transformations whose gauge parameter does not vanish at infinity.

$$A_\mu \to A_\mu + \partial_\mu \alpha, \qquad \phi \to e^{ie\alpha}\phi, \qquad \lim_{r\to\infty} \alpha \neq 0$$

- The LGT generator does not vanish on shell; these are physical transformations.

$$Q^+[\alpha] = \int_{\mathcal{I}^+_-} \alpha * F \qquad Q^-[\alpha] = \int_{\mathcal{I}^-_+} \alpha * F$$

- Asymptotic symmetries are symmetries of the S-matrix

$$\langle \text{out}|(Q^+ S - S Q^-)|\text{in}\rangle = 0$$

This is the Ward identity of the asymptotic symmetry generators $Q^\pm$.

Asymptotic symmetries and soft theorems

- There is a remarkable link that have been established between many asymptotic symmetries and soft theorems.

$$\langle \text{out}|(Q^+ S - S Q^-)|\text{in}\rangle = 0 \qquad \Longleftrightarrow \qquad \text{soft theorem}$$

- Convergent LGT ("superphaserotation") $\Longleftrightarrow$ Leading soft photon theorem
  [He, Mitra, Porfyriadis, Strominger]
- BMS supertranslations $\Longleftrightarrow$ Leading soft graviton theorem
  [He, Mitra, Lysov, Strominger]
- Divergent LGT $\Longleftrightarrow$ Tree-level subleading soft photon theorem
  [Campiglia, Laddha]
- BMS superrotations $\Longleftrightarrow$ Tree-level subleading soft graviton theorem
  [Kapec, Lysov, Pasterski, Strominger]
- This even extends to higher dimensions [He, Mitra]
- $\cdots$

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Divergent LGT $\iff$ Tree-level subleading soft theorem

- Consider a gauge transformation $\delta A_\mu = \partial_\mu \alpha$ where $\alpha = O(r)$ for large r. The Lorenz gauge condition $\nabla^2 \alpha = 0$ dictates its form,

$$\alpha(u, r, \hat{x}) = r\lambda(\hat{x}) + u(1 + \tfrac{D^2}{2})\lambda(\hat{x}) + \cdots$$

- If we compute the charge on a constant time slice Σ_t, the divergent term organizes into t times the LGT ("superphaserotation") charge. Regulating this by introducing a cutoff $t \to \Lambda^{-1}$ and taking $t \to \infty$ with $u = t - r =$ fixed, we find [Campiglia, Laddha]

$$Q[\alpha] = \Lambda^{-1} Q[\lambda] + \frac{1}{2} \int_{\mathcal{I}^+} du\, d^2\hat{x}\, \lambda(\hat{x}) \left[D^A \overset{2}{j}_A - u D^2 \overset{2}{j}_u + u \partial_u D^2 D^A \overset{0}{A}_A \right].$$

The divergent Λ^{-1} term can be removed by phase space renormalization [Peraza] .

- The Ward identity of this charge is equivalent to the tree-level subleading soft photon theorem. [Lysov, Pasterski, Strominger]

Loop corrections to subleading soft theorem

- It turns out that the subleading soft theorem receives one-loop corrections that are infrared-divergent that have to be regularized [Bern, Davies, Nohle]

$$\lim_{\omega\to 0} A_{n+1}(k,\pm) \overset{\text{1-loop}}{=} \lim_{\omega\to 0}\left(\frac{1}{\omega}S_{-1}^{\pm} + S_0^{\pm} + \frac{1}{\epsilon}S_{0,\text{div}}^{\pm} + O(\omega)\right)A_n$$

The soft factor is divergent at loop-order! Is the soft theorem ill-defined?

- This seems to be a consequence of assuming power series in ω. Allowing non-analytic terms of order $\ln\omega$ in the soft photon energy yields logarithmic soft factors [Sahoo, Sen]

$$\lim_{\omega\to 0} A_{n+1}(k,\pm) \overset{\text{1-loop}}{=} \lim_{\omega\to 0}\left(\frac{1}{\omega}S_{-1}^{\pm} + \ln\omega S_{\ln}^{\pm} + S_0^{\pm} + O(\omega)\right)A_n$$

There is a logarithmic soft theorem that is more leading compared to the subleading soft theorem.

Loop corrections to subleading soft theorem

- Here is what the logarithmic soft photon factor looks like:

$$S_{\ln}^{\pm} = \frac{i}{4\pi} \sum_{\substack{i\neq j \\ \eta_i\eta_j=1}} \frac{Q_i^2 Q_j p_i^2 p_j^2 \left[(p_i\cdot\epsilon^{\pm})(p_j\cdot k) - (i\leftrightarrow j)\right]}{(p_i\cdot k)((p_i\cdot p_j)^2 - p_i^2 p_j^2)^{\frac{3}{2}}}$$

$$- \frac{i}{8\pi^2} \sum_{i\neq j} \frac{Q_i^2 Q_j}{p_i\cdot k} (\epsilon^{\pm}\cdot J_i\cdot k) \frac{(p_i\cdot p_j)}{\sqrt{(p_i\cdot p_j)^2 - p_i^2 p_j^2}} \ln\left(\frac{p_i\cdot p_j + \sqrt{(p_i\cdot p_j)^2 - p_i^2 p_j^2}}{p_i\cdot p_j - \sqrt{(p_i\cdot p_j)^2 - p_i^2 p_j^2}}\right)$$

- What is the associated asymptotic symmetry?

Massive scalar QED with long-range interactions

- We work with massive scalar QED.
- To study large-time behavior of massive particles, we employ the "blowup" of i^+ in terms of hyperbolic coordinate $(\tau, \rho, \hat{x})$,

$$\tau = \sqrt{t^2 - r^2}, \qquad \rho = \frac{r}{\sqrt{t^2 - r^2}}$$

- $\tau \to \infty$ with ρ fixed follows the future trajectory of massive particles.
- For massless particles (photons) we use the usual retarded coordinates $(u, r, \hat{x})$ where $u = t - r$. Then $r \to \infty$ with fixed u follows the trajectory of massless particles.

Massive charge with long-range interactions

- The divergent LGT in Lorenz gauge has the large-r behavior

$$\delta A_\mu = \partial_\mu \alpha, \qquad \alpha(u,r,\hat{x}) = r\lambda(\hat{x}) + u(1+\tfrac{D^2}{2})\lambda(\hat{x}) + O(\tfrac{\ln r}{r})$$

- For massive charges it has the following large-τ behavior

$$\delta\phi = ie\alpha\phi, \qquad \alpha(\tau,\rho,\hat{x}) = \tau\bar{\lambda}(\rho,\hat{x}) + O(\tfrac{\ln\tau}{\tau})$$

- We match the two asymptotics using the Euclidean AdS_3 bulk-to-boundary propagator $G^{(3)}(\rho,\hat{x};\hat{q})$

$$\bar{\lambda}(\rho,\hat{x}) = \int d^2\hat{q}\, G^{(3)}(\rho,\hat{x};\hat{q})\lambda(\hat{x})$$

such that $\tau\bar{\lambda} \to r\lambda$ as $\rho \to \infty$.

Massive charge with long-range interactions

- At large times $\tau \to \infty$, one usually approximates matter fields to be free, i.e. as a solution of the Klein-Gordon equation

$$(\nabla^2 - m^2)\phi = 0$$

But actually, interactions with the gauge field lead to the equation as $\tau \to \infty$

$$(\mathcal{D}^2 - m^2)\phi = (\nabla^2 - m^2 - 2ieA_\tau \partial_\tau + \cdots)\phi = 0$$

where the gauge field has falloff $A_\tau = \frac{1}{\tau} \overset{1}{A}_\tau + \cdots$.

- This means the phase space does not consist of free massive fields at i^+; it consists of the dressed fields [Campiglia, Laddha]

$$\phi(\tau, \rho, \hat{x}) \overset{\tau \to \infty}{=} e^{ie \ln \tau \overset{1}{A}_\tau} \underbrace{\frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \left[e^{-im\tau} b_0(\rho, \hat{x}) + e^{im\tau} d_0^\dagger(\rho, \hat{x}) \right]}_{\text{free field}}$$

Massive charge with long-range interactions

- This motivates the following ansatz for the large-τ behavior of dressed matter fields

$$\phi = \frac{\sqrt{m}}{2(2\pi)^{3/2}} \left[e^{-im\tau} \left(\frac{\overset{\ln}{b}_0 \ln\tau}{\tau^{3/2}} + \frac{b_0}{\tau^{3/2}} + \frac{\overset{\ln}{b}_1 \ln\tau}{\tau^{5/2}} + \frac{b_1}{\tau^{5/2}} + \cdots \right) + e^{im\tau} \left(b \to d^\dagger \right) \right]$$

  Plugging this into the equations of motion, we obtain a series of algebraic equations in which all other coefficients can be solved in terms of b_0 and $d_0^\dagger$.

- The symplectic form of massive scalar fields is

$$\Omega_{i^+} = \lim_{\tau\to\infty} \int d\Sigma_\tau \, \tau^3 \left(-\delta\phi^\dagger \wedge \delta\dot{\phi} + \text{h.c.} \right)$$

- One can see that both the symplectic form and the charge are formally divergent at $\tau \to \infty$. Therefore we regulate this using an infrared cutoff $\tau \to \Lambda^{-1}$.

Massive charge with long-range interactions

- The hard charge is obtained by $\Omega_{i^+}(\delta, \delta_\alpha) = \delta Q_H$ with $\delta_\alpha \phi = ie\tau\bar{\lambda}\phi$. We find

$$Q_H[\bar{\lambda}] = \ln \Lambda^{-1} Q_H^{(\ln)}[\bar{\lambda}] + Q_H^{(0)}[\bar{\lambda}] + \cdots$$

- As in tree-level, one finds a linear term $\Lambda^{-1} Q_H^{(1)}$ associated with the leading soft photon theorem, which has been renormalized away using a corner term.

- $Q_H^{(0)}$ is the tree-level hard charge of the divergent LGTs, associated with tree-level subleading soft photon theorem.

- $Q_H^{(\ln)}$ is the hard part of the logarithmic charge proposed by Laddha and Campiglia, whose Ward identity is the Sen-Sahoo log soft photon theorem.

Soft charge with long-range interactions

- Now we turn to the photon side of the story (the soft charge).

- The presence of interactions lead to logarithmic terms in the large-r expansions of the gauge field and field strength,

$$F_{ur} = \frac{1}{r^2}\overset{2}{F}_{ur} + \frac{\ln r}{r^3}\overset{\ln}{F}_{ur} + \frac{1}{r^3}\overset{3}{F}_{ur} + \cdots$$

$$F_{rA} = \frac{\ln r}{r^2}\overset{\ln}{F}_{rA} + \frac{1}{r^2}\overset{2}{F}_{rA} + \cdots$$

$$A_A = \overset{0}{A}_A + \frac{\ln r}{r}\overset{\ln}{A}_A + \frac{1}{r}\overset{1}{A}_A + \cdots$$

and to the large-u expansions as well

$$\overset{2}{F}_{rA} \overset{u\to\pm\infty}{=} u\overset{2,-1}{F}{}^{\pm}_{rA} + \ln u\,\overset{2,\ln}{F}{}^{\pm}_{rA} + \cdots$$

$$\overset{0}{A}_A \overset{u\to\pm\infty}{=} \overset{0,0}{A}{}^{\pm}_A + \frac{1}{u}\overset{0,1}{A}{}^{\pm}_A + \cdots$$

Soft charge with long-range interactions

- The divergent LGT in Lorenz gauge

$$\delta A_\mu = \partial_\mu \alpha, \qquad \alpha(u,r,\hat{x}) = r\lambda(\hat{x}) + u(1+\tfrac{D^2}{2})\lambda(\hat{x}) + \cdots$$

  with the new falloffs has following soft charge

$$Q_S[\alpha] \ = \ \frac{1}{2}\int_{\mathcal{I}^+} du\, d^2\hat{x}\, \lambda(\hat{x})\, u\partial_u D^2 D^A \overset{0}{A}_A$$

  where again the linear term $\Lambda^{-1}Q_S^{(1)}[\lambda]$ has been removed with a corner term.

- One finds that this expression is the same as what you would get from free massive fields, so has anything changed?

Soft charge with long-range interactions

- The soft charge and the relevant new term introduced to $\overset{0}{A}_A$ are

$$Q_S = \frac{1}{2}\int_{\mathcal{I}^+} du\, d^2\hat{x}\, \lambda\, u\partial_u D^2 D^A \overset{0}{A}_A, \qquad \overset{0}{A}_A \overset{u\to\pm\infty}{=} \overset{0,0}{A}{}^{\pm}_A + \frac{1}{u}\overset{0,1}{A}{}^{\pm}_A + \cdots$$

- The new term makes this charge divergent, so we regulate it using a large-time cutoff Λ^{-1}. By choosing a suitably large but finite $u_0 < \Lambda^{-1}$, we write

$$\begin{aligned}\int_{-\infty}^{\infty} du\, u\partial_u \overset{0}{A}_A &= \left(\int_{-\Lambda^{-1}}^{-u_0} + \int_{u_0}^{\Lambda^{-1}}\right) du\, u\partial_u \overset{0}{A}_A + \cdots \\ &= \ln \Lambda^{-1}\left(\overset{0,1}{A}{}^{+}_A - \overset{0,1}{A}{}^{-}_A\right) + \cdots \\ &= \ln \Lambda^{-1} \int_{-\infty}^{\infty} du\, (-\partial_u u^2 \partial_u \overset{0}{A}_A) + \cdots\end{aligned}$$

Soft charge with long-range interactions

- Thus the divergent part of the soft part can be pulled out as

$$Q_S[\alpha] = \ln\Lambda^{-1} Q_S^{(\ln)}[\lambda] + Q_S^{(0)}[\lambda] + \cdots$$

where

$$Q_S^{(\ln)}[\lambda] = \frac{1}{2}\int_{\mathcal{I}^+} du\, d^2\hat{x}\, \lambda\, \partial_u u^2 \partial_u D^2 D^A \overset{0}{A}_A,$$
$$Q_S^{(0)}[\lambda] = \frac{1}{2}\int_{\mathcal{I}^+} du\, d^2\hat{x}\, \lambda\, u\partial_u D^2 D^A (\overset{0}{A}_A)_{\text{tree}}$$

- Here $Q_S^{(\ln)}$ is soft part of the logarithmic charge whose Ward identity is known to yield Sen-Sahoo log soft theorem. $Q_S^{(0)}$ is the tree-level soft charge of the divergent LGTs, associated with tree-level subleading soft photon theorem.

- Note that the two are the same operators with different projectors $\partial_u u^2 \partial_u$ and $u\partial_u$, which in Fourier space pick out $\ln\omega$ and ω^0 terms respectively.

Divergent LGT with long-range interactions

- The total charge becomes

$$Q[\alpha] = \ln\Lambda^{-1}\left[Q_H^{(\ln)}(\bar{\lambda}) + Q_S^{(\ln)}(\lambda)\right] + Q_H^{(0)}(\bar{\lambda}) + Q_S^{(0)}(\lambda) + \cdots$$

- The coefficient of $\ln\Lambda^{-1}$ is exactly the "logarithmic charge" proposed by Campiglia and Laddha whose Ward identity yields the logarithmic soft theorem.

- The Ward identity $\langle \text{out}|(Q^+S - SQ^-)|\text{in}\rangle = 0$ amounts to

$$\lim_{\omega\to 0} \ln\Lambda^{-1}\partial_\omega\omega^2\partial_\omega A_{n+1}(\omega,\pm) = \ln\Lambda^{-1} S_{\ln}^{\pm} A_n + O(\Lambda^0)$$

which is finite as the cutoff is removed $\Lambda \to 0$.

- The Ward identity of the divergent LGT is the log soft theorem.

Gravity

- The story in gravity is exactly analogous. The superrotation charge takes the form

$$\mathcal{Q}[Y] = \ln \Lambda^{-1} \left[\mathcal{Q}_H^{(\ln)}(\bar{Y}) + \mathcal{Q}_S^{(\ln)}(Y) \right] + \mathcal{Q}_H^{(0)}(\bar{Y}) + \mathcal{Q}_S^{(0)}(Y) + \cdots$$

  where $\bar{Y}^\alpha$ on i^+ is obtained from Y^A on $\mathcal{I}^+$ by smearing with a vector bulk-to-boundary propagator.

- The only notable difference from QED is that there is no term linear in Λ^{-1} to renormalize.

- The Ward identity of superrotation is the log soft graviton theorem.

Summary

- Therefore, we have established the following correspondence from phase space: Divergent LGT/superrotation $\Longleftrightarrow$ Logarithmic soft photon/graviton theorem
- More precisely, the charges $Q^{\pm}$ that generate the subleading asymptotic symmetry has the following Ward identities:
  - Free fields: $Q^{\pm}$ are finite, and

$$\langle \text{out}|(Q^{+}S - SQ^{-})|\text{in}\rangle = 0 \quad \rightarrow \quad \text{tree-level subleading soft theorem}$$

  - Fields with interactions: $Q^{\pm}$ have logarithmic divergence $\ln \Lambda^{-1}$, and

$$\langle \text{out}|(Q^{+}S - SQ^{-})|\text{in}\rangle = 0 \quad \rightarrow \quad \text{logarithmic soft theorem}$$

- The asymptotic symmetry that corresponds to tree-level soft theorem leads to loop corrections once long-range interactions are taken into account.

Thank you for your attention!