Inverse scattering problems on Lorentzian manifolds

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April 11, 2024

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Consider

$$\sum_{i=1}^{N}(\sigma(x)\cdot\partial_i u)=0$$

over $\Omega\subset\mathbb{R}^N$, $N=2,3$, subject to $u|_{\partial\Omega}=f$. Define

$$\Lambda_\sigma : H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega).$$

$\Lambda_\sigma=\partial_\nu u|_{\partial\Omega}$.

Knowledge of $\sigma\in\mathcal{C}^1(\Omega)$ determines Λ_σ. (Forward problem).

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Calderón Inverse Problem: knowing Λ_σ determine σ?

Wave inverse problems.

Consider a static Lorenzian metric: $g = -a(x)dt^2 + \overline{g}$; Consider associated operators
$\mathcal{L} = \Box_g$, or $\mathcal{L} = \Box_g + a^i(x)\partial_i + V(x)$.
Theme A: Assume knowledge of "finite" scattering map. Can one reconstruct the operator? $(g,\ a^i, V??)$.

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$$\mathcal{N}[u] = \Box_g u + u^3.$$

Assume knowledge of scattering map (for small data). Can one reconstruct the operator? $(g??)$.

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First consider $g = -a(x)dt^2 + \overline{g}(x)dx^2$ static. **Finite** scattering:
Say $(\overline{M}, \overline{g})$ is a compact manifold with boundary.
Assume knowledge of the *Lorenzian* Dirichlet-to-Neumann map:
Solve:

$$\mathcal{L}[u] = 0, \mathrm{on} \overline{M} \times [0, T], u|_{t=0}, \partial_t u|_{t=0} = 0, u(x, t) = f(x, t) x \in \partial \overline{M}$$

Measure $\partial_\nu u(x, t)$ on $(x, t) \in \partial \overline{M} \times [0, T]$.

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Infinite scattering: Consider asymptotically flat space-times (M, g) with complete null infinities $\mathcal{I}^-, \mathcal{I}^+$. Consider the map S_g: $C^\infty_0(\mathcal{I}^-) \to H^1(\mathcal{I}^+)$, for suitably small initial data.

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Start with *non-linear* setting.

Brief history of *nonlinear* wave inverse problems

- Kurylev, Lassas, Uhlmann (2014-2018): Using nonlinearity and higher order linearization to solve inverse problems
- Since then, techniques using nonlinearity as a tool have been extremely popular: T Balehowsky, C Cârstea, X Chen, M de Hoop, A Feizmohammadi, C Guillarmou, P Hintz, Y Kian, H Koch, K Krupchyk, M Lassas, T Liimatainen, Y-H Lin, G Nakamura, L Oksanen, G Paternain, A Rüland, M Salo, P Stefanov, G Uhlmann, Y Wang, J Zhai, and many more

Motivation for the scattering problem

Can one recover the spacetime structure from scattering data?

A couple of examples

- Sá Barreto (2005): if space part of spacetime is asymptotically hyperbolic and setting is time-independent (static), then Carleman estimates and boundary control yield unique determination of metric: $\Phi^* g_1 = g_2$
- Sá Barreto, Wang, Uhlmann (2021): nonlinear scattering. Potential recovery for $(\partial_t^2 - \Delta)u + f(u)$, $f(u) \sim u^5$, via Melrose-type compactification (stereographic projection to compactify space)

Metric recovery from nonlinear scattering.

Theorem (A-Isozaki-Lassas-Tyni, 2024 Rough version)

Consider (M^{3+1}, g) *complete with* $g_{ab} = \eta_{ab} + d_{ab}$, $d_{ab}(t,x)$ *Schwarz. (So null infinities* $\mathcal{I}^-, \mathcal{I}^+$ *exist). Consider* $\mathcal{N}[u] = \Box_g u + A \cdot u^k$, $k \in \mathbb{N}, k \geq 4$.

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Consider incoming scattering data on u *at* $\mathcal{I}^-$. *(Incoming radiation field* $R^-[u]$*). Solve* $\mathcal{N}[u] = 0$ *in* M^{3+1} *and measure radiation* $\mathcal{R}^+[u]$ *at* $\mathcal{I}^+$.

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Idea: Well-posedness of forward problem(i. e. well-definedness of scattering map $R^-[u]|_{\mathcal{I}^-} \to R^+[u]|_{\mathcal{I}^+}$). Use nonlinearity to "generate" point sources (of tiny amplitude).

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Consider a Lorenzian manifold (M^{3+1}, g) with boundary ∂M^{3+1} containing space-like "bottom" and "top" $\partial M^{3+1}|_{\mathrm{bottom}}, \partial M^{3+1}|_{\mathrm{top}}$ and time-like "side" $\partial M^{3+1}|_{\mathrm{side}}$.

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Theorem (A-Feizmohammadi-Oksanen 2021, Rough version)

Assume g “tall enough”, “spatial exponential map smooth”, “no trapping” and “non-positive null sectional curvature”. Then V can be reconstructed from $\Lambda_{g,V}$ in a "thick time slab" in M^{3+1}.

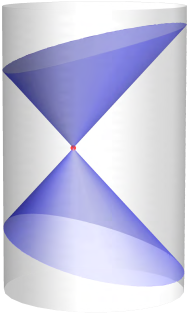
Linear scattering: The Lorenzian Calderon problem.

Key to reconstruction of V is an "optimal" unique continuation result for the metric g:

Proposition

Assume g satisfies the geometric assumptions in theorem: No trapping, no null conjugate points, $R(N, v, N, v) \geq 0, \forall v \perp N, N$ null. Choose any point $P \in M^{3+1}$ and let $\mathcal{E}_P$ be the *exterior* of null cone at P. Assume $u \in H^{-s}(M)$ solves $\mathcal{L}[u] = 0$; assume $u, \partial_\nu u$ vanish on $\mathcal{E}_P \bigcap \partial M^{3+1}$. Then u vanishes on $\mathcal{E}_P$.

Optimal from point of view of characterizing the region where one obtains vanishing.



Convexity $\to$ unique continuation $\to$ recovery of V.

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Require $u \in H^s$, $s < 0$ suitably. Can "identify" solutions to $\mathcal{L}u = 0$ with support of $\partial_t u$ *at the point* P *only*.

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Key feature: We obtain reconstruction of lower-order terms for open space of metrics. But with current ideas: We need to know the metric g, and we find the lower-order terms. (Analogous picture in the "classical" Calderón elliptic inverse problem–in all settings where it has been solved the metric is *known* and we find lower-order terms).

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In non-linear wave problems, we *find the metric.*:

Non-linear scattering: Penrose conformal compactification

Let $(\mathbb{R}^{1+3}, \eta)$ be the Minkowski space with its standard metric η in polar coordinates (t, r, θ, ϕ):

$$\eta = -dt^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2\right)$$

We make a conformal change to $\tilde{\eta} := \Omega^2 \eta$ with

$$\Omega = 4 \frac{1}{1 + (t + r)^2} \frac{1}{1 + (t - r)^2}$$

Penrose conformal compactification

- Let $\tilde{\eta} = \Omega^2 \eta$, where

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- $\Phi : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$, defined by

$$\Phi(t, r, \theta, \varphi) = (T, R, \theta, \varphi),$$

where

$$T = \arctan(t+r) + \arctan(t-r),$$
$$R = \arctan(t+r) - \arctan(t-r),$$
$$-\pi < T + R < \pi, \quad -\pi < T - R < \pi, \quad R \geq 0.$$

On $\mathbb{S}^3$ we have the standard spherical coordinates (R, θ, φ), and the metric on the cylinder $\mathbb{R} \times \mathbb{S}^3$ is of the form

$$\Phi_*(\Omega^2 \eta) = g_{\mathbb{R} \times \mathbb{S}^3} = -dT^2 + dR^2 + \sin^2(R) \left(d\theta^2 + \sin^2(\theta) d\varphi^2 \right).$$

Penrose conformal compactification

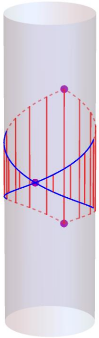
Thus

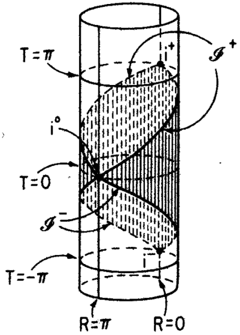
$$\Phi : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$$

is conformal diffeomorphism. We call

$$\widehat{N} = \Phi(\mathbb{R}^{1+3}) \subset \mathbb{R} \times \mathbb{S}^3$$

the **Penrose diagram** of $\mathbb{R}^{1+3}$ and Φ the Penrose map.





Right picture: R. Wald *General relativity*, 1984

Penrose conformal compactification

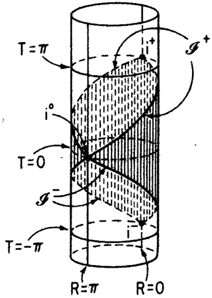
Thus

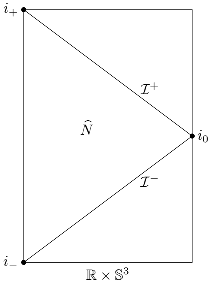
$$\Phi : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$$

is conformal (isometric) diffeomorphism. We call

$$\widehat{N} = \Phi(\mathbb{R}^{1+3}) \subset \mathbb{R} \times \mathbb{S}^3$$

the Penrose diagram of $\mathbb{R}^{1+3}$.





Notation for the wave equation

Let $(\mathbb{R}^{1+3}, g)$ be a globally hyperbolic Lorentzian manifold and consider the nonlinear wave equation

$$\square_g u(t,y) + a(t,y)u(t,y)^\kappa = 0, \quad (t,y) \in \mathbb{R}^{1+3}.$$

where $\kappa \geq 4$ is an integer and $a(t,y)$ a smooth rapidly decaying function.

Here $\square_g$ is the D'Alembertian wave operator

$$\square_g u = -\sum_{a,b=0}^{n} \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^a} \left(\sqrt{|\det(g)|} g^{ab} \frac{\partial u}{\partial x^b} \right)$$

Towards a scattering problem

- Let η be the standard Minkowski metric on $\mathbb{R}^{n+1}$.
- Let g be a globally hyperbolic Lorentzian metric on $\mathbb{R}^{n+1}$, such that $g_{ij}(x) - \eta_{ij}$ is a Schwartz rapidly decaying function and
- Let $\tilde{g} = \Omega^2 g$ be a conformal metric to g and let $\widehat{g} = \Phi_* \tilde{g}$ be the pushforward metric on the Penrose diagram.

Then u satisfies the nonlinear wave equation

$$\Box_g u + a u^\kappa = 0$$

iff $\tilde{u} = (\Omega^{-1} u) \circ \Phi^{-1}$ satisfies

$$(\Box_{\widehat{g}} + B)\tilde{u} + A\tilde{u}^\kappa = 0$$

in $\widehat{N}$, where

$$A := (\Phi^{-1})^*(a\Omega^{\kappa-3}), \quad B := \frac{1}{6}(\Phi^{-1})^*(R_{\Omega^2 g} - \Omega^{-2} R_g).$$

A geometric scattering problem

We say that a function $u \in H^m_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3)$ is a solution of the scattering problem on $(\mathbb{R} \times \mathbb{R}^3, g)$, with the past radiation field h_-,

$$\begin{cases} \Box_g u(x) + a(x) \cdot u(x)^\kappa = 0, & \text{in } \mathbb{R} \times \mathbb{R}^3, \\ u(x) \sim h_-(x) & \text{as } x \text{ goes to } \mathcal{I}^- \end{cases}$$

if the function $\tilde{u} = (\Omega^{-1} u) \circ \Phi^{-1}$ satisfies $\tilde{u} \in H^m(\widehat{N})$ and it is a solution of the Goursat-Cauchy boundary value problem

$$\begin{cases} \Box_{\widehat{g}} \tilde{u}(x) + B(x)\tilde{u}(x) + A(x) \cdot \tilde{u}(x)^\kappa = 0, & \text{in } \widehat{N}, \\ \tilde{u}|_{\mathcal{I}^-} = \tilde{h}_-, \end{cases}$$

Scattering problem has a solution

Lemma

Let g be a globally hyperbolic Lorentzian metric on $\mathbb{R}^{3+1}$, where $g-\eta$ is in the Schwartz class. Let $(\widehat{N},\Omega^2 g)$ be the Penrose diagram. Let $-\pi < T_- < t_- < 0 < t_+ < T_+ < \pi$. There is $0<\varepsilon<<1$, m large such that the following holds: Let $h\in H^m(\mathcal{I}^-)$ be such that $\mathrm{supp}(h)\subset\{x\in\mathcal{I}^- \mid T_- < \mathrm{t}(x) < t_-\}$ and $\|h\|_{H^m(\mathcal{I}^-)}<\varepsilon$ (m large enough).
Then the non-linear scattering problem

$$\begin{cases}(\Box_g+B)u+Au^\kappa=0, \quad \text{in } \{x\in\widehat{N}:\ \mathrm{t}(x)<T_+\},\\ u\big|_{\mathcal{I}^-}=h,\\ u=0 \text{ in } \{x\in\widehat{N}:\ \mathrm{t}(x)<T_-\}\end{cases} \tag{1}$$

has a unique solution depending continuously on h

Defining the scattering operator

We define the (geometric) scattering operator on $\widehat{N}$ by

$$S : C_c^\infty(\mathcal{I}^-) \supset U \to C^\infty(\mathcal{I}^+),$$
$$S(u\big|_{\mathcal{I}^-}) = u\big|_{\mathcal{I}^+}, \quad u = h \in U,$$

for a neighbourhood U of the zero function in $C_c^\infty(\mathcal{I})$. Here u solves the (non-linear) scattering problem

$$\begin{cases} (\Box_g + B)u + Au^\kappa = 0, \quad \text{in } \{x \in \widehat{N} : \ \mathrm{t}(x) < T_+\}, \\ u\big|_{\mathcal{I}^-} = h, \\ u = 0 \text{ in } \{x \in \widehat{N} : \ \mathrm{t}(x) < T_-\} \end{cases} \tag{2}$$

Metric reconstruction result

Theorem (A, Isozaki, Lassas, Tyni–2024)

If $\mathrm{supp}(a) = \mathbb{R}^{1+3}$*, the non-linear scattering operator* S*, defined in a neighborhood of the zero function in* $C_0^\infty(\mathcal{I}^-)$*, determines the conformal class of* g*.*

Idea of proof: Compactification and a non-physical extension

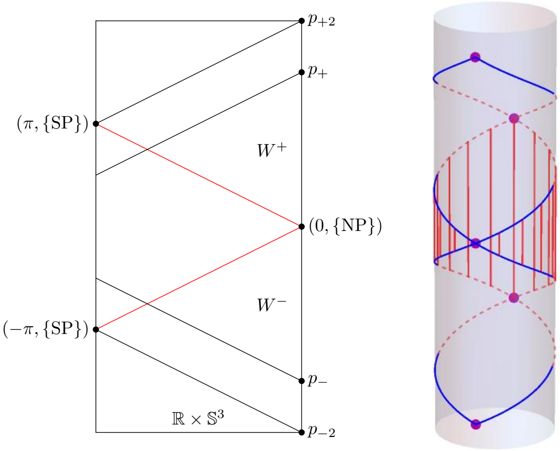
Assume that

- the metric g is globally hyperbolic Lorentzian metric and $g_{ij} - \eta_{ij}$ belong to the Schwartz class
- $\widehat{g} := \Phi_* \widetilde{g}$ is the push-forward metric of $\widetilde{g} := \Omega^2 g$ on the Penrose diagram

Then $\widehat{g}$ can be smoothly extended to $\mathbb{R} \times \mathbb{S}^3$ by defining $g_e = \widehat{g}$ in $\widehat{N}$ and $g_e = \eta$, for $x \in \mathbb{R} \times \mathbb{S}^3 \setminus \widehat{N}$

Measurements beyond infinity

We can do an artificial extension of the Penrose diagram $\widehat{N}$ by gluing it into the cylinder $\mathbb{R} \times \mathbb{S}^3$:



Scattering operator determines a source-to-solution map

Given a source f supported in the non-physical past, we solve a linear wave equation

$$\begin{cases} (\Box_{g_e} + B)u = f, & \text{in } N_{\mathrm{ext}}, \\ \mathrm{supp}(u) \subset J^+(\mathrm{supp}(f)). \end{cases}$$

up to $\mathcal{I}^-$. Restricting u to $\mathcal{I}^-$, this is equivalent to

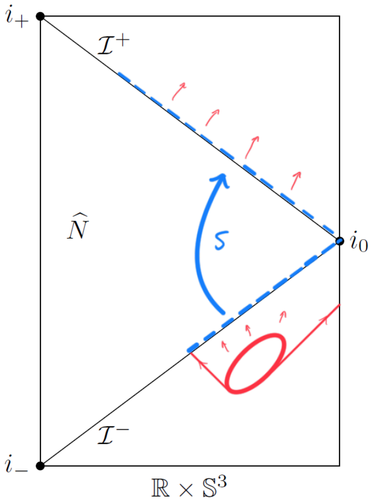
$$\begin{cases} (\Box_{g_e} + B)u = f, & \text{in } N_{\mathrm{ext}}, \\ u\big|_{\mathcal{I}^-} = h^-, \\ u(T_-, x) = \partial_t u(T_-, x) = 0, \end{cases}$$

which is a scattering problem in the Penrose diagram. (Here $T_- \leq \inf \mathrm{t}(\mathrm{supp}(\mathrm{f}))$).

Then the scattering operator determines $h^+ := u|_{\mathcal{I}^+} = S(u|_{\mathcal{I}^+})$. Finally, solving the linear Cauchy-Goursat problem

$$\begin{cases} (\Box_{g_e} + B)u = 0, & \text{in } N_{\mathrm{ext}}, \\ u|_{\mathcal{I}^+} = h^+, \end{cases}$$

shows that we determine u in the non-physical future.



Rough sketch of the inverse scattering problem

- Scattering operator determines the source-to-solution map
- The source-to-solution map determines the scattering relation: **using the nonlinearity**
  - A. Feizmohammadi, M. Lassas, L. Oksanen: *Inverse problems for non-linear hyperbolic equations with disjoint sources and receivers*. Forum of Mathematics, Pi 9 (2021), Paper No. e10, 52

Higher order linearization

A k-fold linearization of the nonlinear equation

$$(\Box_g + B)u + Au^k = \sum_{j=1}^{k} \varepsilon_j f_j \tag{3}$$

with respect to ε_j yields

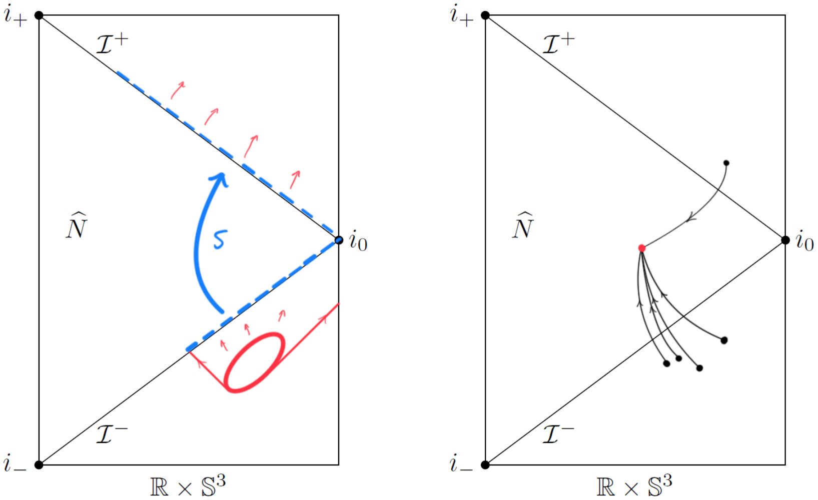
$$(\Box_g + B)w + Av_1 v_2 \cdots v_k = 0 \tag{4}$$

where

$$(\Box_g + B)v_j = f_j, \quad \text{in } \widehat{N}$$

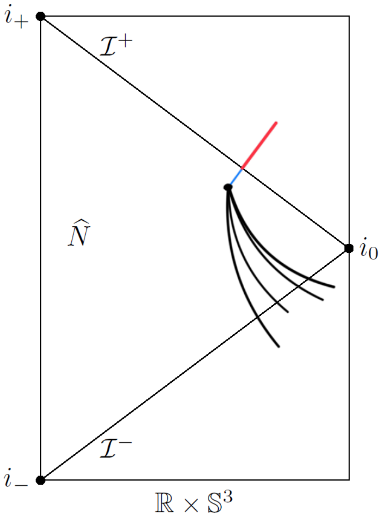
The products $v_1 v_2 \cdots v_k$ can be used to produce point sources.

Scattering relation from the $\mathcal{I}^-$-to-$\mathcal{I}^+$ map



Rough sketch of the inverse scattering problem

- The scattering relation determines the arrival time functions
- Arrival time functions determine light observation sets (and the differentiable structure of the manifold)



Rough sketch of the inverse scattering problem

- The light observation sets determine parts of lightcones, which themselves determine the full lightcones
- Knowledge of the lightcones determines the metric up to a conformal factor
- If the nonlinear term $A \equiv 1$, then one could also recover the conformal factor of the metric