Scattering for wave equations with slowly decaying sources and data.

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Abstract

We construct solutions with prescribed radiation fields for wave equations with polynomially decaying sources close to the lightcone. In this setting, which is motivated by semi-linear wave equations satisfying the weak null condition, solutions to the forward problem have a logarithmic leading order term on the lightcone and non-trivial homogeneous asymptotics in the interior of the lightcone. The backward scattering solutions we construct from knowledge of the source and the radiation field at null infinity alone are given to second order by explicit asymptotic solutions which satisfy novel matching conditions close to the light cone. This requires a delicate analysis close to the light cone of the forward solution with sources on the light cone. We also relate the asymptotics of the radiation field towards space-like infinity to explicit homogeneous solutions in the exterior of the light cone for slowly polynomially decaying data corresponding to mass, charge and angular momentum in the applications. The somewhat surprising discovery is that these data can cause the same logarithmic radiation field as the source term. This requires a delicate analysis of the forward homogeneous solution close to the light cone using the invertibility of the Funk transform.

We consider scattering for the wave equation in three space dimensions

$$-\Box\,\phi = F,$$

We would like to give data at infinity and solve the problem backwards. However, first we must understand asymptotics for the forward problem. For fast decaying initial data and F, ϕ has a Friedlander radiation field

$$\phi(t,x) \sim \frac{\mathcal{F}_0(r-t,\omega)}{r}, \qquad \text{where } |\partial_q^k \mathcal{F}_0(q,\omega)| \lesssim \langle q\rangle^{-k-\varepsilon}, \quad x = r\omega, \quad \omega \in \mathbb{S}^2$$

The same is true if data $|\phi(0,x)| \lesssim \langle r\rangle^{-1-\varepsilon}$ (and $|\partial\phi(0,x)| \lesssim \langle r\rangle^{-2-\varepsilon}$) and

$$|\Box\,\phi| + r^{-2}|\triangle_\omega \phi| \lesssim r^{-1}\langle t+r\rangle^{-1-\varepsilon}\langle t-r\rangle^{-1-\varepsilon}, \qquad \varepsilon > 0.$$

This is seen by expressing the wave operator in spherical coordinates:

$$-\Box\,\phi = \partial_t^2 - \triangle_x = r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)(r\phi) - r^{-2}\triangle_\omega \phi,$$

and integrating, in the $t+r$ direction and in the $t-r$ direction.

In the spherically symmetric case $\triangle_\omega \phi = 0$ but in general this has to be combined with an L^2 estimate for tangential vector fields applied to ϕ.

However, general quadratic terms do not decay enough for this to hold:

Blow up
$$\Box\,\phi = (\partial_t \phi)^2.$$
Global existence
$$\Box\,\psi = (\partial_t \psi)^2 - |\nabla_x \psi|^2.$$
Null condition
$$\Box\,\phi = (\partial_t \psi)^2 - |\nabla_x \psi|^2, \qquad \Box\,\psi = 0.$$
$$\psi_t(t,x)^2 - |\nabla_x \psi|^2 \sim \frac{\mathcal{F}_0'(r-t,\omega)^2}{r^2} - \frac{\mathcal{F}_0'(r-t,\omega)^2}{r^2} \sim \frac{m(r-t,\omega)}{r^3}.$$
Here $\mathcal{F}_0'(q,\omega) = \partial_q \mathcal{F}_0(q,\omega)$.

Weak null condition
$$\Box\,\phi = (\partial_t \psi)^2, \qquad \Box\,\psi = 0.$$
$$\psi_t(t,x)^2 \sim \frac{\mathcal{F}_0'(r-t,\omega)^2}{r^2} = \frac{n(r-t,\omega)}{r^2}.$$
This is model for **Einstein's equations** in wave coordinates, for which also initial data only decays like M/r, where M is the mass.
$$\Box\,\phi = \varphi\,\partial_t \psi, \qquad \Box\,\psi = 0, \qquad \Box\,\varphi = 0.$$
$$\varphi(t,x)\,\psi_t(t,x) \sim \frac{\mathcal{G}_0(r-t,\omega)\,\mathcal{F}_0'(r-t,\omega)}{r^2} = \frac{n(r-t,\omega)}{r^2}.$$
This is model for **Maxwell-Klein-Gordon system** in Lorentz gauge, for which also initial data only decays like q/r, where q is the charge.

Null asymptotics for the wave equation with sources along light cones

$$-\Box\,\phi = \frac{n(r-t,\omega)}{r^2}, \qquad |n(q,\omega)| \lesssim \langle q\rangle^{-k-\epsilon}, \quad \epsilon>0, \quad k=1,2.$$

The solution to the forward problem has a log correction in the asymptotics

$$\phi_{rad,1}(t,r\omega) = \ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{F}_{01}(r-t,\omega)}{r} + \frac{\mathcal{F}_0(r-t,\omega)}{r}, \quad \text{as} \quad t\to\infty, \quad r\sim t,$$

In fact, using the expression for the wave operator in spherical coordinates

$$-\Box\phi_{rad,1} = r^{-1}(\partial_t+\partial_r)(\partial_t-\partial_r)(r\phi_{rad,1}) - r^{-2}\triangle_\omega\phi_{rad,1}$$

$$= -2\frac{\mathcal{F}_{01}'(r-t,\omega)}{r^2} - \frac{\mathcal{F}_{01}(r-t,\omega)}{r^3} - \ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\triangle_\omega\mathcal{F}_{01}(r-t,\omega)}{r^3} - \frac{\triangle_\omega\mathcal{F}_0(r-t,\omega)}{r^3}$$

which is $\sim n(r-t,\omega)/r^2$, if

$$-2\mathcal{F}_{01}'(q,\omega) = n(q,\omega).$$

For compactly supported data $\lim_{q\to+\infty}\mathcal{F}_{01}(q,\omega)=0$ so

$$2\mathcal{F}_{01}(q,\omega) = \int_q^{+\infty} n(q,\omega)\,dq,$$

Hence

$$2\lim_{q\to-\infty}\mathcal{F}_{01}(q,\omega) = N(\omega) := \int_{-\infty}^{+\infty} n(q,\omega)\,dq,$$

Moreover $\lim_{q\to+\infty}\mathcal{F}_0(q,\omega)=0$ and $\lim_{q\to-\infty}\mathcal{F}_0(q,\omega)$ has to match interior asymptotics determined from $N(\omega)$ as we shall see next:

Interior asymptotics for the wave equation with sources on light cones

$$-\Box\,\phi = n(r-t,\omega)/r^2, \qquad |n(q,\omega)| \lesssim \langle q\rangle^{-k-\epsilon}, \quad \epsilon>0, \quad k=1,2.$$

The forward problem with vanishing data has **homogeneous asymptotics**

$$\phi(t,x) \sim \phi_{int,1}(t,x) = \Psi_1(x/t)/t, \quad \text{as } t\to\infty, \text{ while } r/t<1.$$

In fact, $\phi_a(t,x) = a\,\phi(at,ax)$ satisfies

$$-\Box\,\phi_a = n_a(r-t,\omega)/r^2, \qquad n_a(q,\omega) = a\,n(aq,\omega).$$

As $a\to\infty$, in the sense of distribution theory

$$n_a(q,\omega) = a\,n(aq,\omega) \to \delta(q)N(\omega), \qquad \text{where} \quad N(\omega) = \int_{-\infty}^{+\infty} n(q,\omega)\,dq,$$

and $\delta(q)$ is the delta function. Hence $\phi_a \to \phi_{int,1}$, where

$$-\Box\,\phi_{int,1} = N(\omega)\,\delta(r-t)/r^2.$$

Since this is homogeneous of degree -3, $\phi_{int,1}$ is homogeneous of degree -1.

We claim that $\phi_{int,1}$ has the **asymptotics towards the light cone**:

$$\phi_{int,1}(t,r\omega) \sim \ln\Big|\frac{2r}{t-r}\Big|\frac{N_{01}(\omega)}{r} + \frac{N_0(\omega)}{r}, \qquad r\to t, \quad r<t.$$

In fact convolving with the fundamental solution of $\Box$ gives a formula:

$$\phi_{int,1}(t,r\omega) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{N(\sigma)dS(\sigma)}{t-\langle\sigma,r\omega\rangle} \sim \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{N(\omega)dS(\sigma)}{t-\langle\sigma,r\omega\rangle} + \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{N(\sigma)-N(\omega)}{t-\langle\sigma,r\omega\rangle}dS(\sigma)$$

Matching of null asymptotics to interior asymptotics to first order

Asymptotics in the wave zone $r \sim t$:

$$\phi_{rad,1}(t, r\omega) = \ln\Big|\frac{2r}{\langle t-r\rangle}\Big| \frac{\mathcal{F}_{01}(r-t,\omega)}{r} + \frac{\mathcal{F}_0(r-t,\omega)}{r}, \quad \text{as} \quad t \to \infty, \quad r \sim t,$$

Interior asymptotics towards the light cone:

$$\phi_{int,1}(t, r\omega) \sim \ln\Big|\frac{2r}{t-r}\Big| \frac{N_{01}(\omega)}{r} + \frac{N_0(\omega)}{r}, \qquad r \to t, \quad r < t.$$

It follows that

$$\lim_{q \to -\infty} \mathcal{F}_{01}(q,\omega) = N(\omega)/2 = N_{01}(\omega)$$

We must have that

$$\lim_{q \to -\infty} \mathcal{F}_0(q,\omega) = N_0(\omega) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{N(\sigma)-N(\omega)}{1-\langle\sigma,\omega\rangle} dS(\sigma)$$

Second order Null asymptotics with sources along light cones

$$\Box\,\phi_{rad,1} + \frac{n(r-t,\omega)}{r^2} = \ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{R}_{01}(r-t,\omega)}{r^3} + \frac{\mathcal{R}_0(r-t,\omega)}{r^3} = \mathcal{R}_{rad,1}.$$

We seek to pick up the error with

$$\phi_{rad,2}(t,r\omega) = \ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{F}_{11}(r-t,\omega)}{r^2} + \frac{\mathcal{F}_1(r-t,\omega)}{r^2},$$

In fact, using the expression for the wave operator in spherical coordinates

$$-\Box\phi_{rad,2} = r^{-1}(\partial_t+\partial_r)(\partial_t-\partial_r)(r\phi_{rad,2}) - r^{-2}\triangle_\omega\phi_{rad,2}$$
$$= 4\ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{F}_{11}'(r-t,\omega)}{r^3} + 4\frac{\mathcal{F}_1'(r-t,\omega)}{r^3} + \cdots = \mathcal{R}_{rad,1} + O(r^{-4})$$

if

$$\mathcal{F}_{11}'(q,\omega) = ..., \qquad \mathcal{F}_1'(q,\omega) = ...$$

where the right hand side are known quantities. These can be integrated with the conditions that $\lim_{q\to+\infty}\mathcal{F}_{11}(q,\omega)=0$ and $\lim_{q\to+\infty}\mathcal{F}_1(q,\omega)=0$.

As $r \to t$ we have an expansion

$$\phi_{int,1}(t, r\omega) \sim N_{01}(\omega)\frac{1}{r}\ln\left|\frac{2\,r}{t-r}\right| + N_0(\omega)\frac{1}{r} + N_{11}(\omega)\frac{r-t}{r^2}\ln\left|\frac{2\,r}{t-r}\right| + N_1(\omega)\frac{r-t}{r^2}.$$

Plug into $\Box\phi_{int,1} = 0$ and equating powers of $(t-r)/r$ gives that the higher order coefficients can be calculated from N_{01} and N_0:

$$2N_{11}(\omega) = \triangle_\omega N_{01}(\omega), \qquad 2N_1(\omega) = 2\Delta_\omega N_{01}(\omega) + \Delta_\omega N_0 - N_{01}(\omega).$$

Second order Interior asymptotics Further homogeneous asymptotics

$\phi_2(t,x) = \phi(t,x) - \phi_{int,1}(t,x) \sim \phi_{int,2}(t,x) = \Psi_2(x/t)/t^2; \quad t \to \infty, \; r/t < 1.$

In fact, $\phi_{2,a}(t,x) = a^2\phi_2(at,ax)$ satisfies

$-\Box\,\phi_{2,a} = m_a(r-t,\omega)/r^2, \quad \text{where} \quad m_a(q,\omega) = a\big(a\,n(aq,\omega) - \delta(q)N(\omega)\big).$

As $a \to \infty$, $\int m_a(q,\omega)\psi(q)dq \to \psi_q(0)$, i.e. in the sense of distribution theory

$$m_a(q,\omega) \to -\delta'(q)\,M(\omega), \qquad \text{where} \quad M(\omega) = \int_{-\infty}^{+\infty} q\,n(q,\omega)\,dq.$$

Hence $\phi_{2,a} \to \phi_{int,2}$, where

$$-\Box\,\phi_{int,2} = -M(\omega)\delta'(r-t)/r^2, \qquad (t,r) \neq (0,0).$$

Since this is homogeneous of degree -4, $\phi_{int,2}$ is homogeneous of degree -2.

We claim that $\phi_{int,2}$ has the **asymptotics towards the light cone**:

$$\phi_{int,2}(t,r\omega) \sim \frac{M_0(\omega)}{r^2}\frac{r}{r-t} + \frac{M_{11}(\omega)}{r^2}\ln\Big|\frac{2\,r}{t-r}\Big| + \frac{M_1(\omega)}{r^2}, \qquad r \to t, \quad r < t.$$

In fact taking the time derivative of the expression for $\phi_{int,1}$ gives

$$\phi_{int,2}(t,r\omega) = -\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{M(\sigma)\,dS(\sigma)}{(t-\langle\sigma,r\omega\rangle)^2}, \qquad r < t.$$

Higher order asymptotics in the wave zone $r \sim t$:

$$\phi_{rad}(t, r\omega) = \ln\Big|\frac{2\,r}{\langle t-r\rangle}\Big|\frac{\mathcal{F}_{01}(r-t,\omega)}{r} + \frac{\mathcal{F}_0(r-t,\omega)}{r} + \ln\Big|\frac{2\,r}{\langle t-r\rangle}\Big|\frac{\mathcal{F}_{11}(r-t,\omega)}{r^2} + \frac{\mathcal{F}_1(r-t,\omega)}{r^2}.$$

Here $\mathcal{F}_{11}, \mathcal{F}_1$ are determined from $\mathcal{F}_{01}, \mathcal{F}_0$ up to integration constants that are determined by matching to interior and exterior homogeneous solutions.

Higher order asymptotics in the interior $r < t$:

$$\phi_{int}(t, r\omega) \sim \ln\Big|\frac{2\,r}{t-r}\Big|\frac{N_{01}(\omega)}{r} + \frac{N_0(\omega)}{r} + \ln\Big|\frac{2\,r}{t-r}\Big|\frac{r-t}{r^2}N_{11}(\omega) + \frac{r-t}{r^2}N_1(\omega)$$
$$+ \frac{M_0(\omega)}{r}\frac{1}{r-t} + \frac{M_{11}(\omega)}{r^2}\ln\Big|\frac{2\,r}{t-r}\Big| + \frac{M_1(\omega)}{r^2}.$$

Here N_{11}, N_1 are determined from N_{01}, N_0, and M_{11} is determined from M_0. Moreover, for the interior, N_0 is determined from N_{01}, and M_1 from M_0.

Matching conditions For $j = 0, 1$ (here $M_{01}(\omega) = 0$)

$$\lim_{q\to-\infty} \mathcal{F}_j(q,\omega)q^{-j} = N_j(\omega), \qquad \lim_{q\to-\infty} \mathcal{F}_{j1}(q,\omega)q^{-j} = N_{j1}(\omega),$$

$$\lim_{q\to-\infty} \big(\mathcal{F}_j(q,\omega)q^{-j} - N_j(\omega)\big)q = M_j(\omega),$$

$$\lim_{q\to-\infty} \big(\mathcal{F}_{j1}(q,\omega)q^{-j} - N_{j1}(\omega)\big)q = M_{j1}(\omega).$$

Theorem (L-Schlue)

Suppose that

$$|\,(\langle q\rangle\partial_q)^k\partial_\omega^\alpha n(q,\omega)|\lesssim\langle q\rangle^{-3}.$$

$$\mathcal{F}_0(q,\omega)=\big(N_0(\omega)+M_0(\omega)q^{-1}\big)\chi_{q<0}+\mathcal{H}_0(q,\omega)\,,$$

$$|(\langle q\rangle\partial_q)^k\partial_\omega^\alpha\mathcal{H}_0(q,\omega)|\lesssim\langle q\rangle^{-2},\qquad\int_{-\infty}^{\infty}\mathcal{H}_0(q,\omega)\,dq=\mathcal{P}(\omega)\,.$$

Here N_0, M_0, and $\mathcal{P}$ are determined from the source function n alone.
Let $\mathcal{F}_{01}(q,\omega)=\int_q^\infty n(\tilde q,\omega)d\tilde q$, $N(\omega)=\int_{-\infty}^\infty n(\tilde q,\omega)d\tilde q$, $M(\omega)=\int_{-\infty}^\infty \tilde q n(\tilde q,\omega)d\tilde q$.

Then the equation

$$-\Box\,\phi=\frac{n(r-t,\omega)}{r^2}$$

has a solution with asymptotics in the wave zone

$$\phi(t,r\omega)\sim\ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{F}_{01}(r-t,\omega)}{r}+\frac{\mathcal{F}_0(r-t,\omega)}{r},\quad \textit{as } t\to\infty,\quad \textit{while } r\sim t,$$

and interior asymptotics:

$$\phi(t,r\omega)\sim\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{N(\sigma)\,dS(\sigma)}{t-\langle\sigma,r\omega\rangle},\quad \textit{as } t\to\infty,\quad \textit{while } r/t<1,$$

and in the exterior $\phi(t,r\omega)\sim 0$, as $t\to\infty$, while $r/t>1$.

Proof Construct an **approximate solution**:

$$\phi_{app} = \chi_a \phi_{rad} + (1-\chi_a)\phi_{int}, \qquad \text{where} \qquad \chi_a(t,x) = \chi\big((r-t)/r^a\big),$$

and $\chi(s) = 1$, when $s \geq -1/2$ and $\chi(s) = 0$ when $s \leq -1$, $a = 1/2$.

$$\Box\,\phi_{rad} + \frac{n(r-t,\omega)}{r^2} \sim \ln\Big|\frac{2\,r}{\langle t-r\rangle}\Big|\frac{\langle t-r\rangle}{r^4}, \qquad \text{and} \qquad \Box\,\phi_{int} = 0.$$

With $\phi_{diff} = \phi_{rad} - \phi_{int}$ we have with $Q(\partial f, \partial g) = \partial_t f\, \partial_t g - \nabla_x f \cdot \nabla_x g$

$$\Box\phi_{app} = \chi_a \Box\phi_{rad} + (1-\chi_a)\Box\phi_{int} + \Box\chi_a\, \phi_{diff} + 2Q(\partial\chi_a, \partial\phi_{diff}).$$

With $\mathcal{H}_{j1} = \mathcal{F}_{j1} - N_{j1}q^j - M_{j1}q^{j-1}$ and $\mathcal{H}_j = \mathcal{F}_j - N_j q^j - M_j q^{j-1}$:

$$\phi_{diff}(t,r\omega) = \ln\Big|\frac{2\,r}{t-r}\Big|\frac{\mathcal{H}_{01}(r-t,\omega)}{r} + \frac{\mathcal{H}_0(r-t,\omega)}{r} + \ln\Big|\frac{2\,r}{t-r}\Big|\frac{\mathcal{H}_{11}(r-t,\omega)}{r^2} + \frac{\mathcal{H}_1(r-t,\omega)}{r^2},$$

which decays more in the matching region due to more decay in $q = r-t$:

$$|\mathcal{H}_0| + |\mathcal{H}_{01}| + \langle q\rangle^{-1}\big(|\mathcal{H}_1| + |\mathcal{H}_{11}|\big) + \langle q\rangle\big(|\mathcal{H}_0'| + |\mathcal{H}_{01}'|\big) + |\mathcal{H}_1'| + |\mathcal{H}_{11}'| \lesssim \langle q\rangle^{-2}.$$

We convolve with the backwards fundamental solution to solve from infinity

$$\Box(\phi-\phi_{app}) = O(\ln t\,\langle t-r\rangle t^{-4}\chi_{|t-r|<r^a}) + O(\langle t+r\rangle^{-2}\langle t-r\rangle^{-3}\chi_{|t-r|>r^a}),$$

which gives $|\phi-\phi_{app}| \lesssim \langle t+r\rangle^{-2}\ln\langle t+r\rangle$.

Radiation field from initial data for the homogeneous equation

$$\Box\,\phi = 0, \qquad \phi\big|_{t=0} = f, \quad \partial_t\phi\big|_{t=0} = g\,,$$

is given by

$$\mathcal{F}_0(q,\omega) = \mathcal{R}[g](q,\omega) - \partial_q\mathcal{R}[f](q,\omega),$$

where $\mathcal{R}[g]$ denotes the Radon transform of the data g:

$$\mathcal{R}[g](q,\omega) = \int \delta\big(q - \langle \omega, y\rangle\big)g(y)\,dy = \int_{\langle\omega,y\rangle=q} g(y)\,dS(y)\,.$$

For this to be well defined we need g and ∇f to decay like $\langle y\rangle^{-2-\epsilon}$, $\epsilon > 0$
We also see that $\mathcal{F}_0(q,\omega)$ cannot be arbitrary because its integral is independent of ω:

$$\int \mathcal{F}_0(q,\omega)\,dq = \int g(y)\,dy.$$

Exterior asymptotics homogeneous of degree -1

$$\Box\phi_{ext,1} = 0, \quad r > t, \quad \phi_{ext,1}\Big|_{t=0} = \frac{M(\omega)}{r}, \quad \partial_t\phi_{ext,1}\Big|_{t=0} = \frac{N(\omega)}{r^2}.$$

It follows from using the fundamental solution that with $z_0 = \sqrt{1-(t/r)^2}$,

$$\phi_{ext,1}(t, r\omega) = r^{-1}\mathcal{I}_0[N](\omega, z_0) - \omega^i t^{-1}\mathcal{I}_0[\not\nabla_i M](\omega, z_0) + t^{-1}\mathcal{I}_1[M](\omega, z_0),$$

where $\not\nabla_i M(\omega) = r^{-1}\not\partial_i M(\omega)$, $\not\partial_i = \partial_i - \omega\partial_r$, and

$$\mathcal{I}_k[N](\omega, z_0) = \frac{1}{2\pi}\int_{\langle\omega,\sigma\rangle > z_0} \frac{\langle\omega,\sigma\rangle^k N(\sigma)\, dS(\sigma)}{\sqrt{\langle\omega,\sigma\rangle^2 - {z_0}^2}} = \int_{z_0}^1 \frac{z^k N(\omega, z)\, dz}{\sqrt{z^2 - {z_0}^2}},$$

where

$$N(\omega, z) = \int_{\langle\sigma,\omega\rangle = z} N(\sigma)\, ds(\sigma) \Big/ \int_{\langle\sigma,\omega\rangle = z} ds(\sigma).$$

We have

$$\frac{1}{r}\mathcal{I}_0[N](\omega, z_0) \sim \frac{1}{2r}\ln\Big|\frac{2r}{r-t}\Big| N(\omega, 0) + \frac{1}{r}\widetilde{N}(\omega), \qquad \widetilde{N}(\omega) = \int_0^1 \frac{N(\omega, z) - N(\omega, 0)}{z}dz$$

$$\frac{1}{t}\mathcal{I}_1[M](\omega, z_0) \sim \frac{1}{r}\int_0^1 M(\omega, z)dz$$

Hence

$$\phi_{ext,1}(t, r\omega) \sim \frac{1}{2r}\ln\Big|\frac{2r}{r-t}\Big|\big(\mathcal{F}[N](\omega) - \mathcal{G}[M](\omega)\big) + \frac{1}{r}\big(\dots\big)$$

where the **Funk transform** (Funk 1911 thesis with Hilbert)

$$\mathcal{F}[N](\omega) = \frac{1}{2\pi}\int_{\langle\sigma,\omega\rangle=0} N(\sigma)\, ds(\sigma),$$

is **invertible on even functions** and the related (s.c.**Gunk**) transform

$$\mathcal{G}[M](\omega) = \omega^i \mathcal{F}[\nabla\!\!\!/_i M](\omega) = \frac{1}{2\pi}\int_{\langle\sigma,\omega\rangle=0} \langle\omega, \nabla\!\!\!/ M(\sigma)\rangle\, ds(\sigma).$$

is **invertible on odd functions**. (Bailey, Eastwood, Grover, Mason 2003)
In the lower order term two more transforms show up:

$$\mathcal{S}[N](\omega) = \frac{1}{2}\int_{-1}^{1} \frac{N(\omega, z) - N(\omega, 0)}{z}\, dz\,.$$

which is the inverse of $\mathcal{G}$ on odd functions and

$$\mathcal{T}[M](\omega) = -\omega^i \mathcal{S}[M_i](\omega)\,, \quad \text{where} \quad M_i(\sigma) = -M(\sigma)\sigma^i + \nabla\!\!\!/_i M(\sigma)$$

which is the inverse of $\mathcal{F}$ on even functions. On odd functions $\widetilde{N} = \mathcal{S}(N)$.

Exterior asymptotics homogeneous of degree -2

$$\Box\phi_{ext,2} = 0, \quad r > t, \quad \phi_{ext,2}\big|_{t=0} = \frac{K(\omega)}{r^2}, \quad \partial_t\phi_{ext,2}\big|_{t=0} = \frac{L(\omega)}{r^3}.$$

This is obtained by taking the time derivative $\psi = \partial_t\phi_{ext,1}$ of $\phi_{ext,1}$

$$\Box\psi = 0, \quad r > t, \quad \psi\big|_{t=0} = \frac{N(\omega)}{r^2}, \quad \partial_t\psi\big|_{t=0} = \frac{\triangle_\omega M(\omega)}{r^3}.$$

We want to solve

$$\triangle_\omega M(\omega) = L(\omega)$$

which would require that

$$\int_{S^2} L(\omega)dS(\omega) = 0$$

One can reduce to this case by subtracting a multiple of the exact solution

$$\varphi = \frac{1}{r(r+t)}$$

Theorem (L-Schlue)

Let $N_0^{ext}(\omega)$, $M_0^{ext}(\omega)$, $N_{01}(\omega)$, C_0 *and* $\mathcal{H}_0(q,\omega)$ *be given, such that* $|(\langle q\rangle\partial_q)^k\partial_\omega^\alpha\mathcal{H}_0(q,\omega)|\lesssim\langle q\rangle^{-2}$. *Set* $\mathcal{F}_{01}(q,\omega)=N_{01}(\omega)$ *and*

$$\mathcal{F}_0(q,\omega)=\big(N_0^{int}(\omega)+M_0^{int}(\omega)q^{-1}\big)\chi_{q<0}+\big(N_0^{ext}(\omega)+M_0^{ext}(\omega)q^{-1}\big)\chi_{q>0}+\mathcal{H}_0(q,\omega),$$

where

$$N_0^{int}(\omega)=\frac{1}{2\pi}\int_{\mathbb{S}^2}\frac{N_{01}(\sigma)-N_{01}(\omega)}{1-\langle\sigma,\omega\rangle}dS(\sigma),\quad \text{and}\quad M_0^{int}(\omega)=M_0^{ext}(\omega)+C_0.$$

Then the equation

$$-\Box\phi=0$$

has a solution with asymptotics in the wave zone $r\sim t$*:*

$$\phi(t,r\omega)\sim\ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{F}_{01}(q,\omega)}{r}+\frac{\mathcal{F}_0(r-t,\omega)}{r},\quad \text{as}\quad t\to\infty,$$

and corresponding interior and exterior homogeneous asymptotics.

Remark Previously we had shown that if

$$|(\langle q\rangle\partial_q)^k\partial_\omega^\alpha\mathcal{F}_0(q,\omega)|\lesssim\langle q\rangle^{-\gamma},\quad 1/2<\gamma<1$$

then the equation

$$-\Box\,\phi=0$$

has a solution with asymptotics in the wave zone

$$\phi(t,r\omega)\sim\frac{\mathcal{F}_0(r-t,\omega)}{r},\quad\text{as}\quad t\to\infty,\quad r\sim t,$$

This does not cover the case of data decaying only like $1/r$, i.e. $\gamma=0$, but one should be able to prove the same result for $\gamma>0$.
Friedlander proved that if data has finite energy $\|\phi\|_{H_E}^2=\|\partial\phi(0,\cdot)\|_{L^2}$ then

$$\partial_t\phi(t,r\omega)\sim\frac{\mathcal{G}_0(r-t,\omega)}{r},\quad\text{where}\quad\int_{\mathbb{S}^2}\int_{-\infty}^{\infty}\mathcal{G}_0(q,\omega)^2dq\,dS(\omega)\lesssim\|\phi\|_{H_E}.$$

This covers the case when $\phi(0,x)\sim 1/r$ and is consistent with

$$\phi(t,r\omega)\sim\ln\Big|\frac{2r}{\langle t-r\rangle}\Big|\frac{\mathcal{N}_{01}(\omega)}{r}+\frac{\mathcal{F}_0(r-t,\omega)}{r},\quad\text{as}\quad t\to\infty,\quad r\sim t,$$

since when taking the time derivative the log disappears.

Wave-Klein-Gordon system (Chen-L)

We obtained scattering results for coupled wave Klein-Gordon systems:

$$-\Box u = (\partial_t \phi)^2 + \phi^2, \qquad -\Box \phi + \phi = u\phi,$$

in a setting where the interior asymptotics of the Klein-Gordon field affects the asymptotics for the wave equation in the interior and the asymptotics of the wave equation cause a logarithmic correction to the phase of the Klein-Gordon field. With $\rho = \sqrt{t^2 - |x|^2}$ and $y = x/t$ it was proven that

$$u(t,x) \sim \frac{U(y)}{\rho}, \quad r/t < 1, \quad \text{and} \quad u(t,x) \sim \frac{\mathcal{F}_0(t-r,\omega)}{r}, \quad t \sim r, \quad \text{as} \quad t \to \infty,$$

and

$$\phi(t,x) \sim \rho^{-\frac{3}{2}} \big(e^{i\rho - \frac{i}{2} U(y) \ln \rho} a_+(y) + e^{-i\rho + \frac{i}{2} U(y) \ln \rho} a_-(y) \big),$$

where $a_\pm(y)$ decay as $|y| \to 1$, and

$$-\Box\Big(\frac{U(y)}{\rho}\Big) = 2\rho^{-3}\big(1 + (1 - |y|^2)^{-1}\big) a_+(y) a_-(y).$$

Related problems

Asymptotics for Einstein (L)

Asymptotics for MKG (Candy-L-Kauffman)

Asymptotics for Nonlinear Klein-Gordon (Delort, L-Soffer, L-Luhrman-S)

Scattering for Nonlinear Klein-Gordon in 1D (L-Soffer)

Scattering for Einstein in 4D (Wang)

Scattering for Null condition and weak null condition (L-Schlue)

Scattering for quasilinear models (Yu)

Scattering for MKG (He)

Scattering for WKG (Chen-L)

Scattering for MKG (Wei-Fang)

Scattering for Einstein (work in progress)