# Asymptotics of Dynamical Spacetimes and Radiation

Lydia Bieri

University of Michigan
Department of Mathematics
Ann Arbor

Quantum and classical fields interacting with geometry
Curved spacetimes, field theory and beyond

IHP Paris

April 8-12, 2024

# Overview

- The Cauchy Problem for the Einstein Equations
- Gravitational Radiation and Memory
  $\Rightarrow$ A Footprint in Spacetime
- Null Asymptotics
- Dynamics of General Asymptotically-Flat Systems
- Examples
- Outlook

# GW: Measurements - Beginning of a New Era

- LIGO detected gravitational waves from binary black hole mergers for the first time in September 2015.
- Several times since then.
- LIGO and VIRGO together observed gravitational waves from a binary neutron star merger in 2017. At the same time, several telescopes registered data.
- New structures: mathematics $\Leftrightarrow$ physical observations.

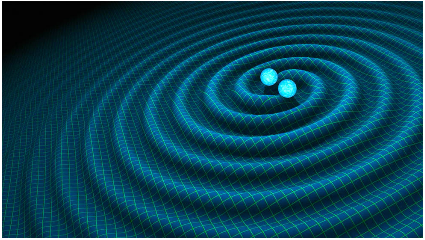


Photo: Courtesy of R. Hurt/Caltech-JPL.

## Einstein Equations

$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\, \mathbf{g}_{\mu\nu}\, \mathbf{R} = \mathbf{8}\pi\, \mathbf{T}_{\mu\nu}\ , \qquad (1)$$

Investigate dynamics of spacetimes $(M, g)$, where $M$ a 4-dimensional manifold with Lorentzian metric $g$ solving Einstein's equations (1).

Study the Cauchy problem.

For the main parts of the discussion we concentrate on solutions of the Einstein-Vacuum equations.

$$R_{\mu\nu} \quad = \quad 0\ . \qquad (2)$$

# Evolution Equations, Constraints and Lapse

Give initial data on a spacelike surface $H_0$: $(\bar{g}_{ij}, k_{ij})$

**Evolution equations of a maximal foliation:**

$$
\begin{aligned}
\frac{\partial \bar{g}_{ij}}{\partial t} &= -2\Phi k_{ij} \\
\frac{\partial k_{ij}}{\partial t} &= -\nabla_i \nabla_j \Phi + (\bar{R}_{ij} - 2k_{im}k^{m}_{\ j})\Phi
\end{aligned}
$$

**Constraint equations of a maximal foliation:**

$$
\begin{aligned}
trk &= 0 \\
\nabla^i \, k_{ij} &= 0 \\
\bar{R} &= |\, k \,|^2
\end{aligned}
$$

**Lapse equation of a maximal foliation:**

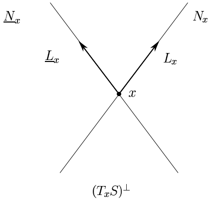
$$\triangle \Phi - |\, k \,|^2 \, \Phi = 0$$

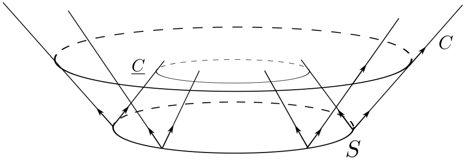
# Vectorfields

Start with an outgoing null vectorfield $L$, define a conjugate (incoming) null vectorfield $\underline{L}$ by requiring that

$$g(L, \underline{L}) = -2 \ .$$

$L$ and $\underline{L}$ are orthogonal to $S_{t,u}$.



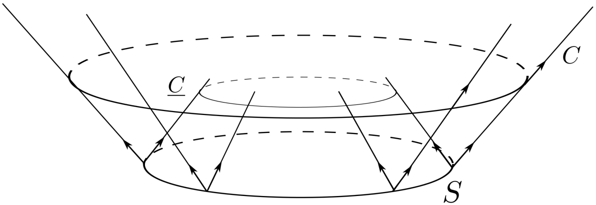

Notation: Denote $L$ by $e_4$ and $\underline{L}$ by $e_3$.
Complement $e_4$ and $e_3$ with an orthonormal frame $e_1$, $e_2$ on $S_{t,u}$
$\Rightarrow$ We obtain a **null frame**.

The **null decomposition** of a tensor relative to a null frame $e_4, e_3, e_2, e_1$ is obtained by taking **contractions** with the vectorfields $e_4, e_3$.

# Shears and Expansion Scalars

Viewing $S$ as a hypersurface in $C$, respectively $\underline{C}$:

- Denote the second fundamental form of $S$ in $C$ by $\chi$, and the second fundamental form of $S$ in $\underline{C}$ by $\underline{\chi}$.
- Their traceless parts are the shears and denoted by $\hat{\chi}$, $\hat{\underline{\chi}}$ respectively.
- The traces $tr\chi$ and $tr\underline{\chi}$ are the expansion scalars.
- Null Limits of the Shears:
  $\lim_{C_u, t\to\infty} r^2\hat{\chi} = \Sigma(u)$ (in (H) spacetimes) and
  $\lim_{C_u, t\to\infty} r\hat{\underline{\chi}} = \Xi(u)$.



# Questions for Large Data

In order to investigate gravitational waves, we study the Cauchy problem.

• Stability theorems give precise description of null infinity.

• They are proven for small data. However, important results also hold for large data.

Large data

Main behavior along null hypersurfaces towards future null infinity

$\Rightarrow$ Largely independent from the smallness.

# Global Solutions - Stability of Minkowski Space

The celebrated result by Demetrios Christodoulou and Sergiu Klainerman, 1991, proving the global nonlinear stability of Minkowski spacetime.

**Theorem [D. Christodoulou and S. Klainerman for EV (1991)] (simplified version)**

Every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.

Generalizations of this Result:

[N. Zipser for EM (2000)] Generalization for Einstein-Maxwell case.

[L. Bieri for EV (2007)] Generalization in the Einstein-vacuum case.

All the above: geometric-analytic proofs.

Long list of other results and partial results. Works by many authors: Including but not complete: Y. Choquet-Bruhat, H. Friedrich, R. Geroch, P. Hintz, S. Hawking, H. Lindblad, F. Nicolò, R. Penrose, I. Rodnianski, A. Vasy, D. Shen, J. Szeftel, and more.

# General Spacetimes

A simplified version of the main theorem reads as follows. The original version of the theorem takes into account the detailed structures of the geometric components which requires several pages to state.

**Theorem [L. Bieri (2007, 2009)]**

Every asymptotically flat initial data obeying appropriate smallness assumptions (controlled via weighted Sobolev norms) gives rise to a globally asymptotically flat solution of the Einstein vacuum equations that is causally geodesically complete.

Small data ensures existence.

Large data

Main behavior along null hypersurfaces towards future null infinity

$\Rightarrow$ Largely independent from the smallness.

# Asymptotic Flatness

(B) (General asymptotically-flat spacetimes with finite energy.)
Asymptotically flat initial data set in the sense of (B): an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and $k$ are sufficiently smooth and for which there exists a coordinate system $(x^1, x^2, x^3)$ in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3\,(r^{-\frac{1}{2}}) \quad (3)$$

$$k_{ij} = o_2\,(r^{-\frac{3}{2}})\,. \quad (4)$$

(D Christodoulou-Klainerman) Strongly asymptotically flat initial data set in the sense of (D): an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and $k$ are sufficiently smooth and there exists a coordinate system $(x^1, x^2, x^3)$ defined in a neighbourhood of infinity such that, as $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, $\bar{g}_{ij}$ and $k_{ij}$ are:

$$\bar{g}_{ij} = (1 + \frac{2M}{r})\,\delta_{ij} + o_4\,(r^{-\frac{3}{2}}) \quad (5)$$

$$k_{ij} = o_3\,(r^{-\frac{5}{2}})\,, \quad (6)$$

where $M$ denotes the mass.

## Asymptotic Flatness

Situation (H). Consider initial data of the asymptotic type

$$\begin{aligned} \bar{g}_{ij} - \delta_{ij} &= l_{ij} + O\left(r^{-1-\varepsilon}\right) && (7) \\ k_{ij} &= O\left(r^{-2-\varepsilon}\right), && (8) \end{aligned}$$

with $l_{ij}$ being homogeneous of degree $-1$.

Situation (C). Consider initial data of the asymptotic type

$$\begin{aligned} \bar{g}_{ij} - \delta_{ij} &= O\left(r^{-\frac{1}{2}-\varepsilon}\right) && (9) \\ k_{ij} &= O\left(r^{-\frac{3}{2}-\varepsilon}\right), && (10) \end{aligned}$$

with $0 < \varepsilon < \frac{1}{2}$.

Situation (B*). As in (B) but with big $O$ instead of $o$.

# Theorems for Large Data

Stability proofs that established the relevant properties of the spacetimes:

(D) D. Christodoulou and S. Klainerman: 1993

(B) L. Bieri: 2007

*Stability Theorems*: For data as in definition (B) under a smallness condition $\Rightarrow$ established global existence and decay theorem for the Einstein vacuum equations.

*Large data*: It follows easily by a corollary that there exists a *complete domain of dependence of the complement of a sufficiently large compact subset of the initial hypersurface*. Thus, we have a solution spacetime with a portion of future null infinity corresponding to all values of the retarded time $u$ not greater than a fixed constant.

$\Rightarrow$ This provides the solid foundation to investigate the asymptotic behavior at future null infinity for large data for (B) spacetimes, and to prove theorems on the nature of gravitational radiation.

Naturally, our investigations will extend to these spacetimes coupled to neutrinos via a null fluid.

*Data of type (B)*: total energy finite, total angular momentum diverges.
*Data of type (B*)*: total energy no longer finite.

Null components of the Weyl curvature $W$ with the capital indices taking the values $1, 2$:

$$W_{A3B3} = \underline{\alpha}_{AB} \quad (11)$$
$$W_{A334} = 2\ \underline{\beta}_A \quad (12)$$
$$W_{3434} = 4\ \rho \quad (13)$$
$$^*W_{3434} = 4\ \sigma \quad (14)$$
$$W_{A434} = 2\ \beta_A \quad (15)$$
$$W_{A4B4} = \alpha_{AB} \quad (16)$$

Notation: Hodge duals $^*W$ and $W^*$ defined as

$$^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta}$$
$$W^*_{\alpha\beta\gamma\delta} = \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}$$

Let $\tau_-^2 = 1 + u^2$ and $r(t,u)$ is the area radius of the surface $S_{t,u}$.

Weyl curvature components

(D)

$$\begin{aligned}
\underline{\alpha}(W) &= O\left(r^{-1}\,\tau_-^{-\frac{5}{2}}\right) \\
\underline{\beta}(W) &= O\left(r^{-2}\,\tau_-^{-\frac{3}{2}}\right) \\
\rho(W) &= O\left(r^{-3}\right) \\
\sigma(W) &= O\left(r^{-3}\,\tau_-^{-\frac{1}{2}}\right) \\
\alpha(W),\ \beta(W) &= o\left(r^{-\frac{7}{2}}\right)
\end{aligned}$$

(B)

$$\begin{aligned}
\underline{\alpha} &= O\left(r^{-1}\,\tau_-^{-\frac{3}{2}}\right) \\
\underline{\beta} &= O\left(r^{-2}\,\tau_-^{-\frac{1}{2}}\right) \\
\rho,\ \sigma,\ \alpha,\ \beta &= o\left(r^{-\frac{5}{2}}\right)
\end{aligned}$$

Correspondingly, obtain decay rates for cases (B*) and (C).

# Structures in (B) Spacetimes

$$\hat{\chi} = o\,(r^{-\frac{3}{2}}) \quad (17)$$

$$\hat{\underline{\chi}} = O\,(r^{-1}\tau_{-}^{-\frac{1}{2}}) \quad (18)$$

$$\zeta = o\,(r^{-\frac{3}{2}}) \quad (19)$$

$$tr\chi = \frac{2}{r} + l.o.t. \quad (20)$$

$$tr\underline{\chi} = -\frac{2}{r} + l.o.t. \quad (21)$$

Further, we have

$$k_{AB} = \eta_{AB} \qquad \hat{\eta} = O(r^{-1}\tau_{-}^{-\frac{1}{2}})$$

$$k_{AN} = \varepsilon_A \qquad \varepsilon = o(r^{-\frac{3}{2}})$$

$$k_{NN} = \delta \qquad \delta = o(r^{-\frac{3}{2}})$$

Here, $\zeta$ is the torsion-one-form. Ricci rotation coefficients of the null frame are:

$$\chi_{AB} = g(D_A e_4, e_B)\ ,\ \underline{\chi}_{AB} = g(D_A e_3, e_B)\ ,\ \underline{\xi}_A = \frac{1}{2}g(D_3 e_3, e_A)\ ,\ \zeta_A = \frac{1}{2}g(D_3 e_4, e_A)$$

$$\underline{\zeta}_A = \frac{1}{2}g(D_4 e_3, e_A)\ ,\ \nu = \frac{1}{2}g(D_4 e_4, e_3)\ ,\ \underline{\nu} = \frac{1}{2}g(D_3 e_3, e_4)\ ,\ \varepsilon_A = \frac{1}{2}g(D_A e_4, e_3)$$

$$\frac{\partial}{\partial u}\hat{\chi} = \frac{1}{4}tr\chi \cdot \underline{\hat{\chi}} + l.o.t. \quad (22)$$

$$\frac{\partial}{\partial u}\underline{\hat{\chi}} = \frac{1}{2}\underline{\alpha} + l.o.t. \quad (23)$$

Let $K$ be the Gauss curvature of $S_{t,u}$. The Gauss equation reads

$$K + \frac{1}{4}tr\chi tr\underline{\chi} - \frac{1}{2}\hat{\chi} \cdot \underline{\hat{\chi}} = -\rho \quad (24)$$

The shears $\hat{\chi}$ and $\underline{\hat{\chi}}$ obey the equations

$$d\not{i}v\,\underline{\hat{\chi}} = \underline{\beta} + \underline{\hat{\chi}} \cdot \zeta + \frac{1}{2}(\not{\nabla}\, tr\underline{\chi} - tr\underline{\chi}\zeta) = \underline{\beta} + l.o.t. \quad (25)$$

$$d\not{i}v\,\hat{\chi} = -\beta - \hat{\chi} \cdot \zeta + \frac{1}{2}(\not{\nabla}\, tr\chi + tr\chi\zeta) \quad (26)$$

Recall that $\zeta$ is the torsion-one-form.

The Bianchi equations for $\not{D}_3\rho$ as well as $\not{D}_3\sigma$ are

$$\not{D}_3\rho + \frac{3}{2}tr\underline{\chi}\rho = -\not{d}iv\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} + (\varepsilon - \zeta)\underline{\beta} + 2\underline{\xi}\beta + \frac{1}{4}(D_3R_{34} - D_4R_{33}) \quad (27)$$

$$\not{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma = -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}^*\underline{\alpha} + \varepsilon^*\underline{\beta} - 2\zeta^*\underline{\beta} - 2\underline{\xi}^*\beta + \frac{1}{4}(D_\mu R_{3\nu} - D_\nu R_{3\mu})\varepsilon^{\mu\nu}{}_{34} \quad (28)$$

The *signature* $s$ is defined to be:

# contractions with $e_4$ - # contractions with $e_3$
Let: $\xi$ any of the null components of an arbitrary Weyl tensor; $\not{D}_3\xi$, $\not{D}_4\xi$ projections to $S_{t,u}$ of $D_3\xi$ and $D_4\xi$, resp. Define $S_{t,u}$-tangent tensors:

$$\xi_3 = \not{D}_3\xi + \frac{3-s}{2}tr\underline{\chi}\xi$$

$$\xi_4 = \not{D}_4\xi + \frac{3+s}{2}tr\chi\xi \ .$$

# Limits at null infinity $\mathcal{I}^+$

Limits at null infinity $\mathcal{I}^+$ Follows from a Theorem by [B, 2020].
More general phenomenon. Several quantities, which are defined locally on the surface $S_{t,u}$, do not attain corresponding limits on a given null hypersurface $C_u$ as $t \to \infty$. However, the difference of their values at corresponding points on $S_u$ and $S_{u_0}$ does tend to a limit.
For instance, consider $\hat{\chi}$ defined locally on $S_{t,u}$. Recall (17). Even though $r^2\hat{\chi}$ does not have a limit as $r \to \infty$ on a given $C_u$, the difference at corresponding points on $S_u$ in $C_u$ and on $S_{u_0}$ in $C_{u_0}$ does have a limit. In particular, these points being joined by an integral curve of $e_3$, the said difference attains the limit

$$\int_{u_0}^{u} \not{D}_3 \hat{\chi} \, du'$$

The part of $\hat{\chi}$ with slow decay of order $o(r^{-\frac{3}{2}})$ is non-dynamical, that is, it does not evolve with $u$. We see that this part does not tend to any limit at null infinity $\mathcal{I}^+$. Similarly, the components of the curvature that are not peeling have leading order terms that are non-dynamical (and do not attain corresponding limits at $\mathcal{I}^+$). Taking off these pieces gives us the dynamical parts of these (non-peeling) curvature components.

## Theorem [L. Bieri (2007)]

For the spacetimes of types (B), the normalized curvature components $r\underline{\alpha}(W)$, $r^2\underline{\beta}(W)$ have limits on $C_u$ as $t \to \infty$:

$$\lim_{C_u, t\to\infty} r\underline{\alpha}(W) = A_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^2\underline{\beta}(W) = \underline{B}_W(u,\cdot),$$

where the limits are on $S^2$ and depend on $u$. These limits satisfy

$$|A_W(u,\cdot)| \leqslant C(1+|u|)^{-3/2} \qquad |\underline{B}_W(u,\cdot)| \leqslant C(1+|u|)^{-1/2}.$$

Moreover, the following limit exists

$$-\frac{1}{2}\lim_{C_u, t\to\infty} r\widehat{\underline{\chi}} = \lim_{C_u, t\to\infty} r\hat{\eta} = \Xi(u,\cdot)$$

Further, it follows that

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4}A_W \tag{29}$$

$$\underline{B} = -2 d\!\!/iv\, \Xi \tag{30}$$

# Gravitational Radiation

**Fluctuation of curvature of the spacetime**

propagating as a wave.

Gravitational waves:

Localized disturbances in the geometry propagate at the speed of light, along outgoing null hypersurfaces.

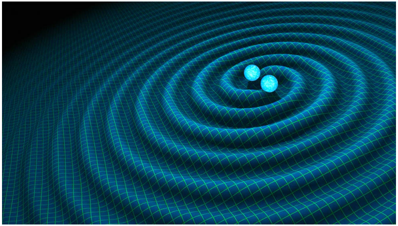
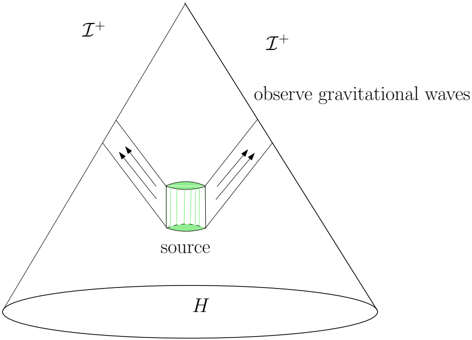
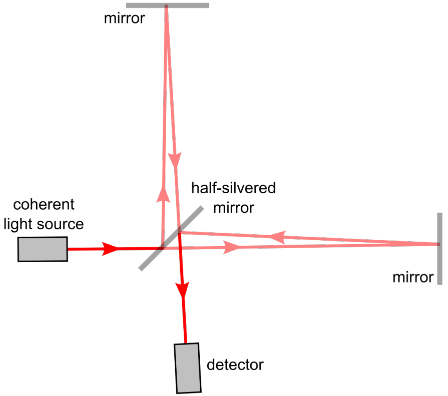
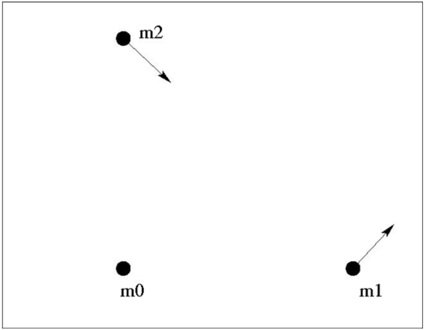


Photo: Courtesy of R. Hurt/Caltech-JPL.



# From Mathematical Theory to Physics and Observation

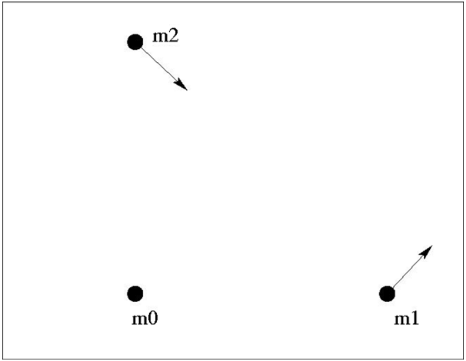


# Memory Effect of Gravitational Waves

Gravitational waves traveling from their source to our experiment. Three test masses in a plane as follows. The test masses will experience

1. Instantaneous displacements (while the wave packet is traveling through)
2. Permanent displacements (cumulative, stays after wave packet passed). This is called the memory effect of gravitational waves. Two types of this memory.




# Gravitational Wave Memory

- **Ordinary (formerly called "linear")** effect
  => was known for a long time in the slow motion limit [Ya.B. Zel'dovich, A.G. Polnarev 1974]

- **Null (formerly called "nonlinear")** effect
  => was found by [D. Christodoulou 1991].

- **Early Works on Memory**:

  L. Blanchet, T. Damour,

  V. B. Braginsky, L. P. Grishchuk, C. M. Will , A. G. Wiseman, K. S. Thorne, J. Frauendiener.

# Memory - Continued

- Other Related Early Works: [A. Ashtekar and various co-authors (1970s and 1980s)] Studies of asymptotic symmetries in GR and infrared problems in quantum field theory.

- Detection: 2016: A paper by P. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen suggests a method for detecting gravitational wave memory with aLIGO by stacking events.

# Memory - Continued

Recent results and new memory effects:

- Contribution from electro-magnetic field to null effect
  => was found by [L. Bieri, P. Chen, S.-T. Yau 2010 and 2011].

- Contribution from neutrino radiation to null effect
  => was found by [L. Bieri, D. Garfinkle 2012 and 2013].

- For the first time outside of GR, for pure Maxwell equations:
  We find an electromagnetic analog of gravitational wave memory.
  [L. Bieri, D. Garfinkle 2013]
  $\Rightarrow$ charged test masses observe a residual kick.

- Fast massive particles mimic null memory: [A. Tolish, L. Bieri, D. Garfinkle, R.M. Wald 2014]

- Other theories: In recent years, A. Strominger relates memory effect, soft theorem and asymptotic symmetry to each other. Many papers by many authors.

- Recent works on memory include many authors. See speakers at this conference.... and many more. A growing field of research....

# Memory - Permanent Displacement

Asymptotically Flat Spacetimes

The permanent displacement $\triangle x$ of test masses is related to the difference $(Chi^- - Chi^+)$ at $\mathcal{I}^+$:

$$\triangle x = -\frac{d_0}{r}\left(Chi^- - Chi^+\right) , \tag{31}$$

where $d_0$ denotes the initial distance between the test masses.

Contributions to the permanent displacement $\triangle x$:

AF systems with $O(r^{-1})$ decay: The ordinary memory is sourced by the change in the radial component of the electric part of the Weyl tensor. The null memory is sourced by $F$, the energy per unit solid angle radiated to infinity (including shear and component of energy-momentum tensor).

NEW, (B) spacetimes: In addition, there is magnetic memory. All memories (electric and magnetic) diverge at rate $\sqrt{|u|}$. Additional structures.

# Parity of Gravitational Waves and Memory

Let $(M, g)$ denote our solution spacetimes.

The Weyl tensor $W_{\alpha\beta\gamma\delta}$ is decomposed into its electric and magnetic parts, which are defined by

$$E_{ab} := W_{atbt} \tag{32}$$

$$H_{ab} := \frac{1}{2}\varepsilon^{ef}{}_{a}W_{efbt} \tag{33}$$

Here $\varepsilon_{abc}$ is the spatial volume element and is related to the spacetime volume element by $\varepsilon_{abc} = \varepsilon_{tabc}$. The electric part of the Weyl tensor is the crucial ingredient in the equation governing the distance between two objects in free fall. In particular, their spatial separation denoted by $\Delta x^a$:

$$\frac{d^2 \Delta x^a}{dt^2} = -E^{a}{}_{b}\Delta x^b \tag{34}$$

In this decomposition, it is

$$E_{NN} = \rho \ \ , \ \ H_{NN} = \sigma \ .$$

# Electric and Magnetic Memory

Memory effect caused by the electric part of the curvature tensor
$\Rightarrow$ called *electric parity memory (i.e. electric memory)*.

Memory effect caused by the magnetic part of the curvature tensor
$\Rightarrow$ called *magnetic parity memory (i.e. magnetic memory)*.

So far

AF systems with $O(r^{-1})$ decay towards infinity
$\Rightarrow$ only electric parity memory, no magnetic memory occurs.

New (B, 2020)

AF spacetimes of slower decay like (B) spacetimes
$\Rightarrow$ magnetic memory occurs naturally.

Overall memory is growing and new structures arise.

Shown for the Einstein vacuum equations and Einstein-null-fluid equations describing neutrino radiation. The new results hold as well for the Einstein equations coupled to other fields of slow decay towards infinity and obeying other corresponding properties.

Next, we are going to derive electric and magnetic parity memory for

1) the Einstein vacuum equations and

2) the Einstein-null-fluid equations describing neutrino radiation.

# Main Theorem and Proof

Recall from above that $(Chi^- - Chi^+)$ is related to permanent displacement.

*Simplified and first version of the main result:*

$(Chi^- - Chi^+)$ *determined by equations at* $\mathcal{I}^+$
• *including terms sourced by "electric part of curvature" (always present)*
• *including terms sourced by "magnetic part of curvature" (only for slow fall-off)*

*On* $S^2$ *at* $\mathcal{I}^+$*: Let* $Z := \not{div}\,(Chi^- - Chi^+)$. *Equations for* $Z$ *involve*

$$\begin{aligned} \not{div}\, Z &= \{\textit{structures involving electric part of curvature}\} \\ \not{curl}\, Z &= \{\textit{structures involving magnetic part of curvature}\} \\ &\quad \textit{plus further new structures} \end{aligned}$$

*Next:*

• Ideas and main steps of the derivation of the main results.

• Presented as a "flow", focussing on the main structures.

• Official Version of the Main Theorem.

# Derivation of Electric Memory

Einstein vacuum equations:

Consider the Bianchi equation for $\not{D}_3\rho$.

Notation $\rho_3 := \not{D}_3\rho \ + \ \frac{3}{2}tr\underline{\chi}\rho$.

In the Bianchi equation for $\not{D}_3\rho$

$$\not{D}_3\rho \ + \ \frac{3}{2}tr\underline{\chi}\rho \quad = \quad -\not{div}\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} \ + \ (\varepsilon - \zeta)\underline{\beta} \ + \ 2\underline{\xi}\beta \qquad (35)$$

we focus on the higher order terms,

$$\rho_3 \ = \ - \underbrace{\not{div}\,\underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} - \underbrace{\frac{1}{2}\hat{\chi}\cdot\underline{\alpha}}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} + \ l.o.t.$$

A short computation shows that

$$\rho_3 \; = \; - \underbrace{\not{d}iv \, \underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} \; - \underbrace{\frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}})}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} + \underbrace{\frac{1}{4} tr\chi |\hat{\underline{\chi}}|^2}_{=O(r^{-3}\tau_-^{-1})} + l.o.t.$$

Thus it is

$$\rho_3 + \frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}}) \quad = \quad -\not{d}iv \, \underline{\beta} + \frac{1}{4} tr\chi |\hat{\underline{\chi}}|^2 \; = \; O(r^{-3}\tau_-^{-\frac{1}{2}}) \qquad (36)$$

Structures:

For small data, $\rho_3$ as well as $\frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}})$ take a well-defined limit at $\mathcal{I}^+$ when multiplied with $r^3$.

For large data, that is not the case, but many more terms of order $r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}}$ exist in $\rho_3$ as well as in $\frac{\partial}{\partial u}(\hat{\chi} \cdot \hat{\underline{\chi}})$ and potentially terms of order $r^{-\frac{5}{2}}\tau_-^{-1-\alpha}$ with $\alpha \geqslant 0$ in $\rho_3$. However, as a consequence of equation (36) all these terms on the left hand side of (36) cancel.

Limit at $\mathcal{I}^+$ of the left hand side of (36)

$\Rightarrow$ leading order term originates from $\rho_3$ and is of order $O(r^{-3}\tau_-^{-\frac{1}{2}})$.

# Future Null Infinity and Electric Memory

Notation for the corresponding limit of the LHS of (36):

$$\mathcal{P}_3 \quad := \quad \lim_{C_u, t \to \infty} r^3 \big( \rho_3 + \frac{\partial}{\partial u} (\hat{\chi} \cdot \underline{\hat{\chi}}) \big) \tag{37}$$

$$\mathcal{P} \quad := \quad \int_u \mathcal{P}_3 \, du \tag{38}$$

Note that $\mathcal{P}$ is defined on $S^2 \times \mathbb{R}$ up to an additive function $C_{\mathcal{P}}$ on $S^2$ (thus the latter is independent of $u$). Later, when taking the integral $\int_{-\infty}^{+\infty} \mathcal{P}_3 \, du$, the term $C_{\mathcal{P}}$ will cancel.

Taking the limit of $\big( r^3$ (36)$\big)$ on $C_u$ as $t \to \infty$, each term on the right hand side takes a well-defined limit. This yields

$$\mathcal{P}_3 \quad = \quad -\not{d}\mathrm{iv}\, \underline{B} + 2|\Xi|^2 \tag{39}$$

Moreover, using our previous theorem on the limits at $\mathcal{I}^+$ it follows that

$$\mathcal{P}_3 \; = \; \mathcal{R}_{\frac{1}{2}}(u, \cdot) + \mathcal{R}_{\beta}(u, \cdot) + \underbrace{l.o.t.}_{\text{more structures}}$$

Next, we define

$$Chi_3 := \lim_{C_u, t \to \infty} \left( r^2 \frac{\partial}{\partial u} \hat{\chi} \right) \quad (40)$$

$$Chi := \int_u Chi_3 \, du \quad (41)$$

We have (see before)

$$\underline{B} = -2 d\not{i}v \, \Xi \quad , \quad Chi_3 = -\Xi \quad (42)$$

Using these with the above we obtain

$$\mathcal{P}_3 = -2 d\not{i}v \, d\not{i}v \, Chi_3 + 2|\Xi|^2 \quad (43)$$

Integrating (43) with respect to $u$ gives

$$(\mathcal{P}^- - \mathcal{P}^+) - \int_{-\infty}^{+\infty} |\Xi|^2 \, du = d\not{i}v \, d\not{i}v \, (Chi^- - Chi^+) \quad (44)$$

In $(\mathcal{P}^- - \mathcal{P}^+)$ an abundance of new terms, leading order $|u|^{+\frac{1}{2}}$.

# Derivation of Magnetic Memory

Consider the Bianchi equation for $\not{D}_3\sigma$.

Notation $\sigma_3 = \not{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma$. In the Bianchi equation for $\sigma_3$

$$\sigma_3 \;=\; -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\; {}^*\underline{\alpha} + \varepsilon\; {}^*\underline{\beta} - 2\zeta\; {}^*\underline{\beta} - 2\underline{\xi}\; {}^*\beta$$

we concentrate on the higher order terms

$$\sigma_3 \quad = \quad -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\; {}^*\underline{\alpha} + l.o.t. \tag{45}$$

A short computation yields

$$\sigma_3 + \frac{\partial}{\partial u}(\hat{\chi}\wedge\underline{\hat{\chi}}) \quad = \quad -c\not{u}rl\,\underline{\beta} \;=\; O(r^{-3}\tau_-^{-\frac{1}{2}}) \tag{46}$$

For $\hat{\chi}\wedge\underline{\hat{\chi}}$ the orders of the terms are at the level of $\hat{\chi}\cdot\underline{\hat{\chi}}$ above.

Multiply the left hand side of (46) by $r^3$ and and take the limit on each $C_u$ for $t \to \infty$ denoting this limit by $\mathcal{Q}_3$. Then introduce $\mathcal{Q}$ as follows:

$$\mathcal{Q}_3 := \lim_{C_u, t\to\infty} r^3\big(\sigma_3 + \frac{\partial}{\partial u}(\hat{\chi} \wedge \hat{\underline{\chi}})\big) \quad (47)$$

$$\mathcal{Q} := \int_u \mathcal{Q}_3 \, du \quad (48)$$

Note that $\mathcal{Q}$ is defined on $S^2 \times \mathbb{R}$ up to an additive function $C_{\mathcal{Q}}$ on $S^2$ (thus the latter is independent of $u$). Later, when taking the integral $\int_{-\infty}^{+\infty} \mathcal{Q}_3 \, du$, the term $C_{\mathcal{Q}}$ will cancel.

Taking the limit of $\big(r^3 \text{ (46)}\big)$ on $C_u$ as $t \to \infty$, the term on the right hand side takes a well-defined limit. This yields

$$\mathcal{Q}_3 = -c\not{u}rl\, \underline{B} \quad (49)$$

Moreover, using our previous theorem on the limits at $\mathcal{I}^+$ it follows that

$$\mathcal{Q}_3 = \mathcal{S}_{\frac{1}{2}}(u, \cdot) + \mathcal{S}_\beta(u, \cdot) + \underbrace{l.o.t.}_{\text{more structures}}$$

Continue to compute using equation (49):

Consider (49) and employ the derived relations between $\hat{\chi}$, $\underline{\hat{\chi}}$ and $\underline{\beta}$ as well as the corresponding limits (30) and (42) to compute

$$\mathcal{Q}_3 \ = \ -2 \ c\not{u}rl \ di\not{v} \, Chi_3 \tag{50}$$

Integrating (50) with respect to $u$ yields

$$(\mathcal{Q}^- - \mathcal{Q}^+) \ = \ c\not{u}rl \ di\not{v} \, (Chi^- - Chi^+) \tag{51}$$

In $(\mathcal{Q}^- - \mathcal{Q}^+)$ an abundance of new terms, leading order $|u|^{+\frac{1}{2}}$.

We obtain

$$(\mathcal{Q}^-_{\sigma_1} - \mathcal{Q}^+_{\sigma_1}) \ + \ (\mathcal{Q}^-_{\sigma_2} - \mathcal{Q}^+_{\sigma_2}) \ - \ \frac{1}{2}(G^- - G^+) \tag{52}$$
$$= \ c\not{u}rl \ di\not{v} \, (Chi^- - Chi^+)$$

- Behavior of $(\mathcal{Q}^- - \mathcal{Q}^+)$ as well as $c\not{u}rl \ di\not{v} \, (Chi^- - Chi^+)$:
  Fix a point on the sphere $S^2$ at fixed $u_0$ and consider $\mathcal{Q}(u_0)$. Next, take $\mathcal{Q}(u)$ at the corresponding point for some value of $u \neq u_0$. Keep $u_0$ fixed and let $u$ tend to $+\infty$, respectively to $-\infty$. Then the difference $\mathcal{Q}(u) - \mathcal{Q}(u_0)$ is no longer finite, but it grows with $|u|^{+\frac{1}{2}}$. A corresponding argument holds for $Chi(u) - Chi(u_0)$.
- $(G^- - G^+)$ is finite. Contributions rooted in magnetic Weyl curvature and shears (shears: sourced by $\int_u \frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}}) \ du$).
- In AF systems with fall-off $O(r^{-1})$ towards infinity, each term in the above equation is identically zero.
- $\mathcal{Q}$ part features terms of diverging order $|u|^{+\frac{1}{2}}$, $|u|^{+\beta}$ for $0 < \beta < \frac{1}{2}$. Rooted in magnetic Weyl curvature.

# Gravitational Wave Memory: Electric and Magnetic

The above gives the main ingredients in the proof of the following theorem.

**Theorem [L. Bieri]**

The following holds for (B) spacetimes.
$(Chi^- - Chi^+)$ is determined by the following equations on $S^2$ (see next slide).

Electric and Magnetic Parts

Next, we are going to COMBINE the two parts.

# Gravitational Waves: New Structures

There exist functions $\Phi$ and $\Psi$ such that

$$\not{d}iv\,(Chi^- - Chi^+) = \not{\nabla}\,\Phi + \not{\nabla}^{\perp}\Psi.$$

Let $Z := \not{d}iv\,(Chi^- - Chi^+)$. Note that then the following holds:

$$\not{d}iv\,Z \;=\; \not{\triangle}\,\Phi \quad , \qquad \not{c}url\,Z \;=\; \not{\triangle}\,\Psi \;\;.$$

We obtain the system on $S^2$, solve by Hodge theory,

$$\begin{aligned}
\not{d}iv\,(Chi^- - Chi^+) &= \not{\nabla}\,\Phi + \not{\nabla}^{\perp}\Psi && (53)\\
\not{c}url\,\not{d}iv\,(Chi^- - Chi^+) &= \not{\triangle}\,\Psi \\
&= (\mathcal{Q} - \bar{\mathcal{Q}})^- - (\mathcal{Q} - \bar{\mathcal{Q}})^+ && (54)\\
\not{d}iv\,\not{d}iv\,(Chi^- - Chi^+) &= \not{\triangle}\,\Phi \\
&= (\mathcal{P} - \bar{\mathcal{P}})^- - (\mathcal{P} - \bar{\mathcal{P}})^+ \\
&\quad -2(F - \bar{F}) && (55)
\end{aligned}$$

New quantities at diverging and finite orders in $\mathcal{Q}$ and $\mathcal{P}$ parts.

For the more general spacetimes of slow decay (like (B)) we conclude:

1. There is the new magnetic memory effect growing with $|u|^{\frac{1}{2}}$ sourced by $\mathcal{Q}$ and finite contributions from both $\mathcal{Q}$ and $G$.

2. $\mathcal{Q}$ has further diverging terms at lower order.

3. There is the electric memory, previously established. This electric part is growing with $|u|^{\frac{1}{2}}$ sourced by $\mathcal{P}$, further lower-order growing terms and finite contributions from $\mathcal{P}$ and from $F$ (the latter may be unbounded for systems of decay $O(r^{-\frac{1}{2}})$).

4. $c\not{u}rl\ di\not{v}\,(Chi^- - Chi^+)$ being non-trivial allows for the magnetic structures to appear in gravitational radiation and to enter the permanent changes of the spacetime. Thus, these more general spacetimes generate memory of magnetic type.

Points 1, 2, 4 are NEW.

Point 3, the leading order behavior as well as the null memory were established in (B, 2018). The finer structures are new.

# Adding Neutrinos

(B 2020) Einstein-null-fluid equations describing neutrino radiation:

$$R_{\mu\nu} \; = \; 8\pi \, T_{\mu\nu} \; .$$

Describe the neutrinos in this equation, represented via the energy-momentum tensor given by

$$T^{\mu\nu} \;\; = \;\; \mathcal{N} K^{\mu} K^{\nu} \tag{56}$$

with $K$ being a null vector and $\mathcal{N} = \mathcal{N}(\theta_1, \theta_2, r, \tau_-)$ a positive scalar function depending on $r$, $\tau_-$, and the spherical variables $\theta_1, \theta_2$.

When coupled to the Einstein equations in the most general settings, the energy-momentum tensor $T^{\mu\nu}$ obeys those loose decay laws. No symmetry nor other restrictions imposed.

In particular, we do not have stationarity outside a compact set, but instead a distribution of neutrinos decaying very slowly towards infinity.

"Geometric terms": same growth rate as in EV case.

"T" terms: growing at rate $\sqrt{|u|}$.

"Geometric terms": same growth rate as in EV case.

"T" terms: growing at rate $\sqrt{|u|}$.

In particular:

For data as in (B) as well as in (B*), there is a contribution from the neutrinos to the electric memory growing at rate $\sqrt{|u|}$.

For data as in (B*), in addition, we find the following contribution from the neutrinos to the magnetic memory: Fix $u_0$, then the integral $\int_{u_0}^{u} \left( T \right)^{*}_{34_3} \, du$ diverges like $\sqrt{|u|}$ as $|u| \to \infty$.

Similarly as before, solve the corresponding Hodge system on $S^2$ to derive the full changes of the spacetime.

# Summary

## Summary

- Spacetimes decaying like $O(r^{-1+\alpha})$ for $0 < \alpha \leqslant \frac{1}{2}$ cause magnetic memory of the above types diverging at $|u|^{+\alpha}$. Respectively, this holds for $0 < \alpha < 1$.

- The corresponding electric memories diverge at the same rate.

- Neutrinos contribute to the electric memory growing at rate $\sqrt{|u|}$.

- A non-trivial curl of neutrino stress-energy starts occurring at $O(r^{-\frac{1}{2}})$.

- The integral $\int_u \frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}})\, du$ as well as $\int_u \frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}})\, du$ generates finite electric (former), respectively finite magnetic (latter) memory.

# Homogeneous of Degree $-1$, Non-Isotropic Mass

Recall the spacetimes of type (H) above.

The spacetimes where the initial data includes a term that is homogeneous of degree $-1$. In particular, this may include a non-isotropic mass term.

$$\begin{aligned} \bar{g}_{ij} - \delta_{ij} &= l_{ij} + O\left(r^{-\frac{3}{2}}\right) \\ k_{ij} &= O\left(r^{-\frac{5}{2}}\right), \end{aligned}$$

with $l_{ij}$ being homogeneous of degree $-1$.

(B 2022)

# Peeling Stops

Solve initial value problem for the EV equations for (H) initial data to obtain **spacetimes of type (H)**. The Weyl curvature components have the following behavior towards future null infinity.

$$\underline{\alpha} = O\left(r^{-1}\,\tau_-^{-\frac{5}{2}}\right) \quad (57)$$

$$\underline{\beta} = O\left(r^{-2}\,\tau_-^{-\frac{3}{2}}\right) \quad (58)$$

$$\rho = O\left(r^{-3}\right) \quad (59)$$

$$\rho - \bar{\rho} = O\left(r^{-3}\right) \quad (60)$$

$$\sigma = O\left(r^{-3}\,\tau_-^{-\frac{1}{2}}\right) \quad (61)$$

$$\sigma - \bar{\sigma} = O\left(r^{-3}\,\tau_-^{-\frac{1}{2}}\right) \quad (62)$$

$$\beta = o\left(r^{-\frac{7}{2}}\right) \quad (63)$$

$$\alpha = o\left(r^{-\frac{7}{2}}\right) \quad (64)$$

Here $\tau_- := \sqrt{1+u^2}$ for retarded time $u$.

(63)-(64) hold under smallness assumptions, whereas for large data the behavior becomes $O(r^{-3})$.

# Limits

## Limits at Spacelike and Future Null Infinity for (H) Spacetimes

Consider $\rho$.

Denote by $P_{H_0}(\theta, \phi)$ the limit of $r^3 \rho$ at spacelike infinity.

Denote by $P(u, \theta, \phi)$ the following limit at future null infinity:

$$\lim_{C_u, t \to \infty} r^3 \rho \quad = \quad P(u, \theta, \phi)$$

Moreover, let

$$\lim_{u \to +\infty} P(u, \theta, \phi) \; = \; P^+(\theta, \phi)$$

$\mathbf{P_{H_0}}(\theta, \phi)$, respectively $\mathbf{P}^+(\theta, \phi)$, do not have any $l = 1$ modes, but they have all the other modes $l = 0$ and $l \geqslant 2$.

# Limits at Future Null Infinity $\mathcal{I}^+$

## Limits at Future Null Infinity $\mathcal{I}^+$ for (A) Spacetimes

$$
\begin{aligned}
\lim_{C_u, t \to \infty} r^3 \rho &= P(u, \theta, \phi) \\
\bar{P} &= \bar{P}(u) \\
(P - \bar{P})(u, \theta, \phi) &: \text{ does not decay in } |\, u \,| \text{ as } |\, u \,| \to \infty, \\
& \quad \text{leading order term is dynamical, i.e. depends on } u, \\
& \quad \text{and also depends on the angles } \theta, \phi \\
\lim_{u \to +\infty} P(u, \theta, \phi) &= P^+(\theta, \phi)
\end{aligned}
$$

We see that $P = P(u, \theta, \phi)$ is a function on $R \times S^2$, and $P^+ = P^+(\theta, \phi)$ is a function on $S^2$. Thus, in particular, as $u \to +\infty$, the quantity $P(u, \theta, \phi)$ tends to a function $P^+(\theta, \phi)$ on $S^2$, not a constant.

For (CK) spacetimes it is

$$
\begin{aligned}
P - \bar{P} &= O(|\, u \,|^{-\frac{1}{2}}) \\
\lim_{u \to +\infty} P &= P^+ = \lim_{u \to +\infty} \bar{P} = \bar{P}^+ = -2M^+_{ADM} = constant
\end{aligned}
$$

# Summary of Results for (H) Spacetimes

For (H) spacetimes the following hold:

• There are natural contributions from $(P - P_{[1]})$ and $F$ to the gravitational wave memory effect.

• Peeling of the Weyl curvature components at future null infinity stops at the order $r^{-3}$ for large data. For small data, this limit is of the dynamical order $r^{-\frac{7}{2}}$. These orders are achieved by the curvature component $\beta$ for large, respectively small data.

• The limit $\lim_{C_u, t \to \infty} r^3 \rho = P(u, \theta, \phi)$ tends to a function $P^+(\theta, \phi)$ on $S^2$ when the retarded time $u \to +\infty$. In (CK) the corresponding limit is a constant.

• $\rho - \bar{\rho}$, respectively $P - \bar{P}$, does not decay in retarded time $u$. (Here, $\bar{\rho}$ means the mean value of $\rho$ on $S_{t,u}$, and $\bar{P}$ the mean value of $P$ on $S^2$.)

• Energy and momenta at future null infinity are well-defined. In particular, angular momentum can be defined and is finite despite the slow decay for $\beta$ and its derivatives.

# Angular Momentum

## Angular Momentum at $\mathcal{I}^+$

Classical definition of angular momentum at $\mathcal{I}^+$:

$$J^k \quad := \quad \int_{S^2} \varepsilon^{AB} \nabla_B \tilde{X}^k (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB}) \ , \quad k = 1, 2, 3.$$

Bondi-Sachs coordinates.

$\tilde{X}^k$ for $k = 1, 2, 3$: standard coordinate functions in $\mathbb{R}^3$ restricted to $S^2$,

$N_A$: angular momentum aspect,

$C_{AB}$: shear tensor,

$\varepsilon_{AB}$: volume form of the standard round metric $\sigma_{AB}$ of $S^2$.

Further, in the Bondi-Sachs notation, $N_{AB}$ is the news tensor and $m$ the mass aspect.

# Christodoulou-Klainerman and Bondi-Sachs

We use (CK) notation.

Relate the Christodoulou-Klainerman notation to the Bondi-Sachs coordinate system. The left hand side is given in the (CK) notation:

$$\begin{aligned} B_A &= -N_A \\ \underline{B}_A &= \nabla\!\!\!\!/^{\,B} N_{AB} \\ \underline{A}_{AB} &= -2\partial_u N_{AB} \\ \Sigma_{AB} &= -\frac{1}{2} C_{AB} \\ \Xi_{AB} &= -\frac{1}{2} N_{AB}\ . \end{aligned}$$

In (H) spacetimes the limit $B_A$ may not exist. Nevertheless, we can define angular momentum, because the involved $l = 1$ modes behave better.

Derive a conservation of angular momentum for (H) spacetimes.

# Velocity-Coded Memory

B and A. Polnarev 2024:
Recent New Results on Velocity-Coded Memory

Scenario: A supermassive black hole surrounded by a large accretion disk. A less massive black hole moves perpendicular to the plane of the disk and intersects it.

After crossing the disk $\Rightarrow$ smaller black hole experiences a jump of acceleration.
$\Rightarrow$ Acceleration jump is seen as a jump in curvature, which happens in a very short time interval.

At the detector, this burst arrives and lasts for the short time $\triangle u$. After this short time $\triangle u$, the velocity of the test masses stays constant over a very long time interval $\delta u$. $\Rightarrow$ Velocity-Coded Memory

B and D. Garfinkle 2023:
Experiment to Measure Electromagnetic Memory

The electromagnetic memory is a residual velocity (i.e. kick) of test charges. (Bieri-Garfinkle 2013)

Electromagnetic memory $\Rightarrow$ requires a source whose charges are not confined to any bounded spatial region.

Experiment: Apply a short microwave pulse to the center of a long dipole antenna. $\Rightarrow$ Create a situation of unbound charges for a short time in the antenna, until the pulse of charges reach the end of the antenna.

To measure the memory, use another dipole antenna in the far field region. Before the charges reach the ends of the dipole antenna of the source, they induce a current in the receiver antenna due to the fact that the integral of the electric field over time is nonzero.

Measurement has to be done within a short time before the pulse reaches the end of the antenna.

# Electromagnetic (EM) Memory

Motion of a charge in the presence of an electromagnetic wave.

Charged test masses $\Rightarrow$ Measure residual velocity (= kick).

For a charge $q$ with mass $m$ the equation of motion is

$$m\frac{d^2\vec{x}}{dt^2} = q\vec{E} \tag{65}$$

It follows that once the wave has passed the charge has received a kick given by

$$\Delta\vec{v} = \frac{q}{m}\int_{-\infty}^{\infty}\vec{E}dt \tag{66}$$

# Null Memory Continued

Maxwell's equations in spherical coordinates.

Spherical coordinate indices: write $r$ for radial direction and capital latin letters for two-sphere direction.

$$\partial_r E_r + 2r^{-1}E_r + r^{-2}D_A E^A = 4\pi\rho \tag{67}$$

$$\partial_r B_r + 2r^{-1}B_r + r^{-2}D_A B^A = 0 \tag{68}$$

$$\partial_t B_r + r^{-2}\varepsilon^{AB}D_A E_B = 0 \tag{69}$$

$$\partial_t E_r - r^{-2}\varepsilon^{AB}D_A B_B = -4\pi j_r \tag{70}$$

$$\partial_t B_A + \varepsilon_A{}^B(D_B E_r - \partial_r E_B) = 0 \tag{71}$$

$$\partial_t E_A - \varepsilon_A{}^B(D_B B_r - \partial_r B_B) = -4\pi j_A \tag{72}$$

Here $D_A$ and $\varepsilon_{AB}$ are respectively the derivative operator and volume element of the unit two-sphere, and all indicies are raised and lowered with the unit two-sphere metric.

# Null Memory Continued

Expand all quantities in inverse powers of $r$ with expansion coefficients that are functions of retarded time $u = t - r$ and the angular coordinates.

For an electromagnetic field that is smooth at null infinity it follows that

$$E_A = X_A + \dots \quad (73)$$

$$B_A = Y_A + \dots \quad (74)$$

$$E_r = W r^{-2} + \dots \quad (75)$$

$$B_r = Z r^{-2} + \dots \quad (76)$$

$$\rho = j_r = r^{-2} L + \dots \quad (77)$$

where $\dots$ means "terms higher order in $r^{-1}$" and we also assume that at large $r$ the angular components of $j_a$ are negligible compared to the radial component.

# Null Memory Continued

Consider a field that is both charged and massless. This is the analog for electromagnetism of fields whose stress-energy gets out to null infinity. Closer look at equation (77):

$$\rho = j_r = r^{-2}L + \dots$$

Introduce the current density four-vector $J^\mu$ given by $J^t = \rho$ and $J^a = j^a$. We also introduce the advanced time $v = t + r$. It then follows that $J_u = -\frac{1}{2}(j_r + \rho)$ and $J_v = \frac{1}{2}(j_r - \rho)$. Thus the behavior given in (77) is equivalent to

$$J_u = -r^{-2}L + \dots \quad (78)$$

$$J_v = O(r^{-3}) \quad , \quad J_A = O(r^{-3}) \quad (79)$$

Substitute in the above equations ⇒ Some of the equations yield identical results and the full set of independent equations becomes

$$-\partial_u W + D_A X^A = 4\pi L$$

$$\partial_u Z + \varepsilon^{AB} D_A X_B = 0$$

# Null Memory Continued

Define the quantity $S_A$ by

$$S_A = \int_{-\infty}^{\infty} X_A du \tag{80}$$

Then it follows that $S_A$ satisfies the equations

$$D_A S^A = (W(\infty) - W(-\infty)) + 4\pi F \tag{81}$$

$$\varepsilon^{AB} D_A S_B = Z(-\infty) - Z(\infty) \tag{82}$$

where the quantity $F$ is defined by

$$F = \int_{-\infty}^{\infty} L du \tag{83}$$

$\Rightarrow$ It follows that the kick points in the direction of $S^A$ and has a magnitude of

$$\Delta v = \frac{q}{mr} |S^A| \tag{84}$$

EM Memory consists of ordinary kick and null kick .

# EM Memory : Analyzed

EM Memory consists of ordinary kick and null kick .

$$\Delta v = \frac{q}{mr}|S^A| \tag{85}$$

- ordinary kick due to difference between the early and late time values of the radial component of the electric field $E_r$
- null kick due to charge radiated to infinity, that is $F$ giving the amount of charge radiated to infinity per unit solid angle.

Consider systems which at large positive and negative times consist of widely separated charges each moving at constant velocity.

For a single charge moving at constant velocity
$\Rightarrow$ the $r^{-2}$ piece of $B_r$ vanishes.

By superposition it follows that the same is true for a collection of such charges.

$\Rightarrow$ It follows that both $Z(-\infty)$ and $Z(\infty)$ vanish.

$\Rightarrow$ It follows that there is a scalar $\Phi$ such that

$$S_A = D_A \Phi \tag{86}$$

Then we find

$$D_A D^A \Phi = (W(\infty) - W(-\infty)) + 4\pi F \tag{87}$$

Note that it is required for the consistency of this equation that the right hand side integrated over all solid angle vanishes.

In physical terms: it follows that the integral over all solid angle of $W$ is $4\pi$ times the charge enclosed.

It then follows that the

Integral over all solid angle of $W(-\infty) - W(\infty)$ is $4\pi$ times the amount of charge lost by being radiated to null infinity.

But since $F$ is the charge radiated per unit solid angle

$\Rightarrow$ The integral over all solid angle of $4\pi F$ is also $4\pi$ times the lost charge.

From eqn. (87) it follows that $\Phi$ consists of
two pieces $\Phi = \Phi_1 + \Phi_2$ satisfying the following equations:

$$D_A D^A \Phi_1 = (W(\infty) - W(-\infty)) - (W(\infty) - W(-\infty))_{[0]} \qquad (88)$$

$$D_A D^A \Phi_2 = 4\pi(F - F_{[0]}) \qquad (89)$$

and that $S_A = S_{1A} + S_{2A}$ with $S_{1A} = D_A \Phi_1$ and correspondingly for $S_{2A}$. Subscript $[0]$ means the average value of that quantity on the 2-sphere.

[B, Garfinkle 2013] There are two electromagnetic analogs of gravitational wave memory.

Namely:

- due to fields that do and do not reach null infinity.

What other fields behave like that?

**The stress-energy tensor of the fields gets out to null infinity for**

- a field that is both **charged and massless** being the analog for electromagnetism of fields whose stress-energy gets out to null infinity (**Maxwell equations with massless charge, linear**),
- Maxwell-Klein-Gordon system for a charged, massless scalar field (**nonlinear**),
- **charged null dust** (**nonlinear**, can be derived from [BG] result on null fluids).

Outlook

- Measure memory, gravitational, electromagnetic, ....
- Gravitational wave sources where an extended neutrino halo is present: Expect to see the new structures.
- Dark matter of certain types may behave as described here. Investigate dark matter, including dark matter halos of galaxies.
- Couple Einstein equations to other types of matter-energy to investigate similar questions.
- Many more fascinating questions....

# Thank you!