UNIVERSITÄT
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Metric Reconstruction

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Outline

MOTIVATION

DOF of gravity

Local gauge theories like GR give a **redundant** description of dymamics $\Longrightarrow$ number of DOF $<$ number of field components!

$$\begin{aligned} \text{DOF} = &(10 \text{ components of } g_{ab}) \\ &- (4 \text{ components of lapse/shift}) \\ &- (4 \text{ constraints}) = 2 \end{aligned} \tag{1}$$

DOF of gravity

Therefore, **in principle**, a field Φ with two real (or one complex) component should suffice to describe the dynamics of GR!

However, **in practice**, I am not aware of a useful formulation involving just one such Φ, i.e., a suitably equivalent and fully general formulation of GR in terms of a partial differential equation for Φ.

Problem: DOF somehow cannot be identified **locally**.

Metric reconstruction

While not possible in general, finding a formulation of GR in terms of a single complex Φ may still be possible in a more restricted setting e.g., in **perturbation theory** around **specific backgrounds** (Minkowski, (A)dS, Schwarzschild, Kerr,...).

Metric reconstruction

Identify such Φ, its PDE, and its correspondence to the metric. Ideally this should be $1:1$ **up to gauge**. The formulation in terms of Φ should be **practical**.

Minkowski space

In Minkowski space, we can write a mode solution to the linearized EE in the form

Gravitational wave in Minkowski spacetime

$$h_{ab} = h_+ \varepsilon^+_{ab}(p) \sin(px) + h_\times \varepsilon^\times_{ab}(p) \sin(px)$$

- p_a = wave vector ($p_a p^a = 0$),
- $\varepsilon^+_{ab}, \varepsilon^\times_{ab}$ = polarization tensors ($\cong$ **two** DOFs)
- $h_+, h_\times$ = amplitudes

Minkowski space

A specific choice of the polarization tensors is (Re/Im = real/imaginary part)

$$\varepsilon_{ab}(p) = \mathrm{Re}/\mathrm{Im}\, Z_{acbd} p^c p^d, \quad Z_{abcd} = Z_{ab} Z_{cd}$$

where

- p_a = wave vector ($p_a p^a = 0$),
- $Z = m \wedge l$
- l^a, n^a, m^a = complex null (NP) tetrad of Minkowski

Minkowski space

With this, our gravitational wave can be rewritten as

Metric reconstruction in Minkowski spacetime

$$h_{ab} = \mathrm{Re}(\mathcal{S}^{\dagger}_{ab}\Phi)$$

where

- $\Phi = A \times$ sinusoidal *px*-dependence $=$ complex "Hertz potential"
- $\partial^a \partial_a \Phi = 0$
- $\mathcal{S}^{\dagger}_{ab} = Z_{acbd}\partial^c \partial^d =$ "reconstruction operator"

Minkowski space

The **metric reconstruction procedure** is a far-reaching generalization of this idea and its variants to **Kerr**, aimed at solving

Sourced linearized EE

$$(\mathcal{E}h)_{ab} = T_{ab},$$

where:

$$(\mathcal{E}h)_{ab} = -\frac{1}{2}\nabla^c\nabla_c h_{ab} + \cdots = \text{linearized Einstein operator in } \textbf{Kerr}.$$

Some references

- **Origins:** [Chrzanowski; Cohen & Kegeles; Wald; Teukolsky, ...] 70s & 80s: $T_{ab} = 0$
- **Further developments:** [Ori; Friedman, Keidl, Shah, L Price; Merlin, Pound & Barack; L Price & Whiting, Van de Meent, ... (many)] 00s & 10s: $T_{ab} = 0$ partially removed (point sources), better formalism, application to gravitational SF, ...
- **Recent:** [Green, SH & Zimmerman; SH & Toomani; Toomani, Spiers, Green, Hollands, Pound & Zimmerman; Casals, SH, Pound & Toomani; Bourgh, Leather, Casals, Pound & Wardell; Dolan, Kavanagh & Wardell; Dolan, Durkan, Kavanagh & Wardell; Green & Toomani; Aksteiner, L Anderson & Bäckdahl; ...]: General T_{ab}, other gauges, non-linearities,...
- **Connections with mathGR:** All approaches involve the Teukolsky equation, so mathGR results on this equation are highly relevant [Dafermos, Holzegel, Rodnianski, Shlapentokh-Rothman; Häffner; L Anderson, Bäckdahl, Blue & Ma; ...]

Nonlinearities & applications

Non-linearities are treated in a **naive** perturbation theory around Kerr:

$$g_{ab}(\epsilon) = \underbrace{g_{ab}}_{\text{Kerr}} + \epsilon h^{(1)}_{ab} + \epsilon^2 h^{(2)}_{ab} + \dots$$

At each order, we have a linear EE with source,

$$(\mathcal{E} h^{(n)})_{ab} = T^{(n-1)}_{ab} + \text{matter}.$$

Applications mostly to **gravitational self-force** (EMRI's, LISA physics,...)

Review: [Pound & Wardell, "Black Hole Perturbation Theory and Gravitational Self-Force"]

KERR PERTURBATION THEORY

NP tetrad $l^a, n^a, m^a \Longrightarrow$ **Weyl components** & **optical scalars:**

$$\Psi_0 = -C_{abcd} l^a m^b l^c m^d, \quad \text{etc.}$$

Optical scalars (e.g., $\rho = m^a \bar{m}^b \nabla_b l_a$)

scalar	interpretation
Re ρ	expansion
Im ρ	twist
σ	shear
...	...

Kerr: $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = \kappa = \kappa' = \sigma = \sigma' = 0$

Geroch-Held-Penrose (GHP)

NP frame has scaling ambiguity (local boosts + rotations):

$$l^a \to \lambda\bar{\lambda} l^a, \quad n^a \to (\lambda\bar{\lambda})^{-1} n^a, \quad m^a \to \lambda\bar{\lambda}^{-1} m^a \tag{2}$$

$\Longrightarrow$ Weyl components and optical scalars **transform** by some power $\lambda^p\bar{\lambda}^q$.

$\Longrightarrow$ Like "matter fields" in some representation $\{p, q\}$ of **local gauge group** $\mathbb{C} \setminus 0$.

$\Longrightarrow$ Require **gauge connection**, Θ_a.

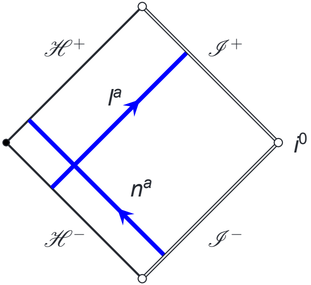
GHP weights

scalar	GHP weights
ρ	$\{1, 1\}$
ρ'	$\{-1, -1\}$
τ	$\{-1, 1\}$
τ'	$\{1, -1\}$
...	...
Ψ_i	$\{4 - 2i, 0\}$

Maths: GHP "scalars" $=$ sections in a complex line bundle $\mathscr{L}_{\{p,q\}}$

Invariant viewpoint: avoids "gauge" singularities

Principal null directions l^a and n^a



Teukolsky equation

If homogeneous EE $(\mathcal{E}h)_{ab} = 0$ holds, then $\psi_0 = \delta\Psi_0$ satisfies:

Teukolsky equation

$$\mathcal{O}\psi_0 \equiv [g^{ab}(\Theta_a + 4B_a)(\Theta_b + 4B_b) - 16\Psi_2]\psi_0 = 0$$

$B_a = -\rho n_a + \tau \bar{m}_a$ gravito-magnetic field

Sourced Teukolsky equation

If inhomogeneous EE $(\mathcal{E}h)_{ab} = T_{ab}$ holds, then $\psi_0 = \delta\Psi_0$ satisfies

Sourced TE

$$\mathcal{O}\psi_0 = J_0,$$

where $J_0 =$ (2 derivatives on T_{ab}) is **Teukolsky's source**

$$\mathcal{S}T \equiv \frac{1}{4} Z^{bcda} \zeta^{-4} \nabla_a (\zeta^4 \nabla_b T_{cd}),$$
$$\mathcal{T}h \equiv -\frac{1}{2} Z^{bcda} \nabla_a \nabla_b h_{cd}$$

These tensorial forms of $\mathcal{S}, \mathcal{T}$: [Araneda]

SEOT [Wald]

$$\text{Teukolsky} \implies \mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$$

- $\mathcal{T}$ gives ψ_0 from h_{ab}
- $\mathcal{S}$ gives J_0 from T_{ab}
- $\mathcal{E}$ operator in EE

$$\Longrightarrow \mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger \tag{3}$$

Metric reconstruction [Chrzanowski, Cohen & Kegeles, Wald]

If $\mathcal{O}^\dagger\Phi = 0$ then

$$h_{ab} = \mathrm{Re}(\mathcal{S}^\dagger_{ab}\Phi)$$

solves homogeneous EE!

- "Same" $\mathcal{S}^\dagger_{ab}$ as in Minkowski, produces polarization tensors
- Reconstructed metric is in **TIRG** $h_{ab}l^a = 0 = h$
- Only works for **homogeneous** EE
- Unclear if **every** solution to homogeneous EE can be represented this way

Can we write a generic perturbation in the form

$$h_{ab} = \mathrm{Re}(\mathcal{S}^{\dagger}_{ab}\Phi) \qquad ?$$

- h_{ab} cannot have an ingoing energy-momentum $T_{ab}l^b \neq 0$, no matter what is Φ!
- h_{ab} cannot represent algebraically special perturbation $\dot{g}_{ab}$!
- h_{ab} cannot represent perturbation not in TIRG!

GHZ metric reconstruction

GHZ theorem [Green, SH & Zimmerman; SH & Toomani]

If h_{ab} asymptotically flat at $\mathscr{I}^+$, decaying at i^0, $(\mathcal{E}h)_{ab} = T_{ab}$, $\Longrightarrow \exists X^a, \dot{g}_{ab}, x_{ab}, \Phi$ such that

$$h_{ab} = \underbrace{\mathrm{Re}(\mathcal{S}^\dagger_{ab}\Phi)}_{\text{reconstructed}} + \underbrace{(\mathcal{L}_X g)_{ab}}_{\text{gauge}} + \underbrace{\dot{g}_{ab}}_{\text{zero mode}} + \underbrace{x_{ab}}_{\text{corrector}}$$

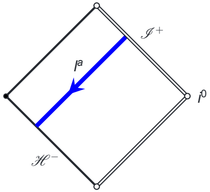
– x_{ab} **uniquely** determined from T_{ab},

– $\dot{g}_{ab}$ algebraically special perturbation **uniquely** determined by ADM quantities

– X^a some gauge vector field **unique** up to Killing VF

– Φ **uniquely** determined from ψ_0 and Cauchy data

Transport equations for corrector x_{ab}



$$\rho^2 \partial_l \left(\frac{\bar\rho}{\rho^3} \partial_l \left[\frac{\rho}{\bar\rho} x_{m\bar m} \right] \right) = T_{ll}$$

$$\frac{\rho}{2(\rho+\bar\rho)} \partial_l \left((\rho+\bar\rho)^2 \partial_l \left[\frac{1}{\rho(\rho+\bar\rho)} x_{nm} \right] \right) = T_{lm} + \text{(terms involving } x_{m\bar m}\text{)},$$

$$\tfrac{1}{2}(\rho+\bar\rho)^2 \partial_l \left(\frac{1}{\rho+\bar\rho} x_{nn} \right) = T_{ln} + \text{(terms involving } x_{m\bar m}, x_{nm}\text{)},$$

Algorithm for finding GHZ decomposition

1. Determine $\delta M, \delta a$ by standard ADM-type formulas from Cauchy data [Arnowitt, Deser & Misner; Iyer & Wald, ...] $\Longrightarrow \dot{g}_{ab}$
2. Solve $x_{ab} = X_{ab}{}^{a'b'} * T_{a'b'}$ with explicit Green's function [Casals, SH, Pound & Toomani]
3. Determine Cauchy data for ψ_0 from those of h_{ab} (algebraic in time-domain [e.g., Campanelli & Lousto])
4. Solve Cauchy problem $\mathcal{O}\psi_0 =$ Teukolsky source (standard in frequency-domain [e.g., BH perturbation toolkit])
5. Reconstruct Φ from ψ_0 via Teukolsky-Starobinski (algebraic in frequency-domain [Ori; SH & Toomani])

GHZ $\Longrightarrow$ metric reconstruction

Corollary of GHZ theorem [SH & Toomani]

If h_{ab} asymptotically flat at $\mathscr{I}^+$, decaying at i^0, $(\mathcal{E}h)_{ab} = 0 \Longrightarrow \exists X^a, \dot{g}_{ab}, \Phi$ such that

$$h_{ab} = \underbrace{\mathrm{Re}(\mathcal{S}^\dagger_{ab}\Phi)}_{\text{reconstructed}} + (\mathcal{L}_X g)_{ab} + \dot{g}_{ab}$$

- For modes [Ori]
- Reconstructed metrics “dense” [Prabhu & Wald]
- Widely used without proof!

OTHER GAUGES

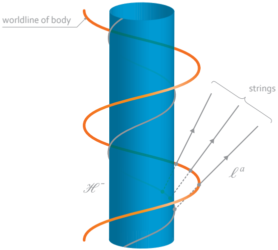
Disadvantages of IRG

- IRG metric grows near $\mathscr{I}^+$ [many people]
- IRG may have gauge singularities [Merlin, Pound & Barack]
- IRG has non-standard propagation of singularities in microlocal sense [Casals, SH, Pound & Toomani]

$\Longrightarrow$ practical problems at higher order in perturbation theory!

"Dirac"-string-like gauge singularities

$$T^{ab}(x) = m \int \mathrm{d}\tau \, \dot{\gamma}^a \dot{\gamma}^b \delta^4(\gamma(\tau) - x)/\sqrt{-g}$$



[Figure: Green, Hollands, Zimmerman, Class. Quant. Grav. 37 (2020) 7, 075001]

Resolution: Pass to a better-behaved gauge!

- **Lorenz gauge** [Dolan, Durkan, Kavanagh, Wardell; Green & Toomani; ...]
- **“No-string” gauge** [Toomani, Spiers, Green, Hollands & Pound; Bourg, Leather, Casals, Pound & Wardell, ... (many people)]
- ...
- To be **combined** with other ideas such as puncture scheme [Pound et al.] (for singular sources), Detweiler-Whiting Green’s function [Detweiler & Whiting], frequency domain techniques, ...

$\Longrightarrow$ practical usefulness at higher order in perturbation theory for gravitational self-force problem remains to be understood better

Lorenz gauge

- Gauge trafo $h_{ab} \to h_{ab} + (\mathcal{L}_\xi g)_{ab}$, find ξ^a imposing Lorenz gauge
- Can be reduced to solving a Maxwell equation $\nabla_a F^{ab} = J^b$
- GHZ for spin-1 or different method

Alternative reconstruction (spin-1) [Dolan et al.; Green et al.; see also: Aksteiner et al.]

$$A_a = \mathrm{Re}(\nabla^b H_{ab} + X_a)$$

where $H_{ab} = \zeta(Z_{ab}\Phi_2 + Z'_{ab}\Phi_0)$ with suitable Hertz potentials Φ_0, Φ_2

- Gives a solution **up to** term annihlated by ∂_t; requires **two** Hertz potentials
- Corrector X_a is **algebraically** constructed from J^a.

Summary

- Reconstruction $=$ parameterize GR by 2 DOF through a local field Φ,
- Possible for perturbation theory around Kerr if additional structure (corrector) is introduced
- Potentially useful for gravitational SF problems
- Relationship to other formalisms like double copy etc.?