Gluing variations

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based on joint work with Wan Cong
arXiv:2302.06928 [gr-qc]

Questions

Throughout this work: vacuum spacetimes with $\Lambda \in \mathbb{R}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \, .$$

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2. glue together vacuum spacetimes?
3. realise *data on lower dimensional submanifolds* by embedding in a vacuum spacetime?
4. ??? data on lower dimensional submanifolds ???

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Data on lower dimensional manifolds?

The Aretakis-Czimek-Rodnianski question

Question (Aretakis, Czimek and Rodnianski (2021))

Can you find vacuum characteristic initial data interpolating between two **sphere data sets***?*

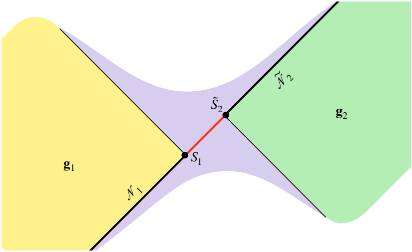

Figure: Gluing construction of Aretakis-Czimek-Rodnianski

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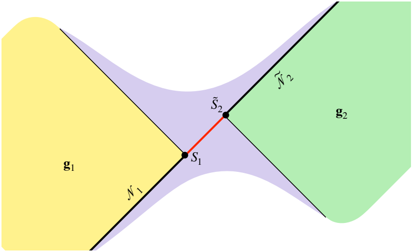

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Gluing method

Nonlinear “superpositions”

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Corvino-Schoen gluing

Theorem (Corvino & Schoen, 2010)

An asymptotically flat initial data set can be glued to a Kerr one, with nearby global charges across a sufficiently distant coordinate annulus.

Theorem (Peter Hintz, arXiv:2210.13960)

Let (Σ, g, K) be a vacuum initial data set and suppose that there are no Killing vectors near $p \in \Sigma$. For every $\epsilon > 0$ sufficiently small there exists a vacuum initial data set which coincides with (g, K) outside an ϵ-neighborhood of p and coincides with a small Kerr black hole inside the neighborhood.

This can be done all over the place

State-of-the-art variations on Corvino-Schoen

Gluing-in small black holes with $\Lambda = 0$; the Hintz black hole sprinkler (compare Anderson, Corvino, Pasqualotto arXiv:2301.08238)

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Theorem 1.1. *Let $N \in \mathbb{N}$. For $i = 1, \ldots, N$, fix points $p_i \in \partial M = \mathbb{S}^3 \subset \mathbb{R}^4$ and (subextremal) masses $0 < \mathfrak{m}_i < (3\Lambda)^{-1/2}$ such that the balance condition*

$$\sum_{i=1}^{N} \mathfrak{m}_i p_i = 0 \in \mathbb{R}^4. \tag{1.2}$$

holds. Then there exists a metric g solving the Einstein vacuum equation (1.1) in a neighborhood of ∂M with the following properties:

(1) *in a neighborhood of p_i, g is isometric to a Schwarzschild–de Sitter black hole metric with mass $\mathfrak{m}_i$, containing future affine complete event and cosmological horizons;*
(2) *outside a small neighborhood of $\{p_1, \ldots, p_N\}$, $\cos^2(s)g$ is smooth down to $s = \pi/2$, and asymptotic to the rescaled de Sitter metric $\cos^2(s)g_{\mathrm{dS}}$ at the rate $\cos^3(s)$.*

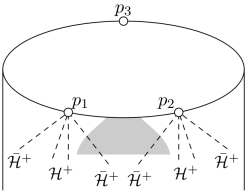

FIGURE 1.2. Illustration of Theorem 1.1. We glue SdS black holes into neighborhoods of the points p_i; only two black holes are shown here. The

Mao-Tao's simplification of the Corvino-Schoen gluing

- The linearised prescribed scalar constraint equation at the Euclidean metric is

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• this can be used to give a simple and sharper version of the Corvino-Schoen gluing:

Theorem (Mao, Oh, Tao, arXiv:2308.13031)

An asymptotically flat initial data set can be glued to any Kerr (possibly Schwarzschild) with longer energy-momentum vector

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• and an alternative proof of the Carlotto-Schoen gluing:

Theorem (Mao & Tao, arXiv:2210.09437)

Simple proof of the Carlotto-Schoen theorem, including optimal decay, using spacelike gluing.

Screening the gravitational field

Carlotto-Schoen “exotic gluings” (2014)

- Remove a solid cone C_1 from Euclidean space; initial data $(\mathbb{R}^n, g = \delta, K_{ij} = 0)$

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Mao, Tao, arXiv:2210.09437: can be done with optimal $1/r$ decay using a Green function for δR supported in a cone.

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CONJECTURE ("third law of black hole dynamics", Bardeen, Carter & Hawking (1973))

A black hole with zero surface-gravity cannot be formed in a dynamical process.

zero surface-gravity $\approx$ zero temperature

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The third law is wrong for spherically symmetric solutions of the Einstein-Maxwell-charged-scalar-field equations.

Proof: use null gluing to an extreme Reissner-Nordström black hole.

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ACR gluings: applications

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Black holes can be formed in vacuum by focusing of gravitational waves.

Proof: null gluing of a Minkowskian light-cone to a Kerr black hole

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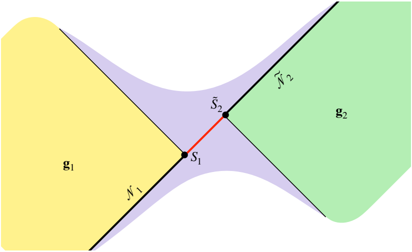

Figure: Gluing construction of ACR

General topologies, higher dimensions, differentiability

Characteristic gluing: implicit function theorem together with

Theorem (Aretakis, Czimek & Rodnianski, arXiv:2107.02449)

The C^2 linearised characteristic gluing at $(3+1)$-Minkowski is solvable up to a 10-dimensional space of obstructions.

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Theorem (Cong, PTC and Gray, arXiv:2401.04442)

The C^k linearised characteristic gluing at $(n+1)$-Birmingham - Kottler with $m \neq 0$ is solvable up to a space of obstructions of dimension $\leq n+1$.

$(n+1)$-Birmingham - Kottler: cross-section $\mathbf{S}$ compact Einstein spaces e.g. spheres, torus, higher genus; $\Lambda \in \mathbb{R}$, $m \in \mathbb{R}$.

General topologies, higher dimensions, differentiability

Wan Cong, PTC, and Finnian Gray, arXiv:2401.04442

Obstructions arise from kernels of linear elliptic operators on the cross-section **S** of the characteristic hypersurface; affected by *dimension* and *topology* of **S**, e.g.:

C^2-gluing with $m = 0$, $\Lambda = 0$	S^2	$\mathbb{T}^2$	S^4
dim. of obstruction space	10	7	30

Both a non-vanishing *mass m* and a non-zero *cosmological constant* Λ provide additional degrees of freedom to remove some of the obstructions, e.g.:

C^k-gluing	S^2, $m = 0$	S^2, $m = 0$	S^2, $m \neq 0$	**S**, $m \neq 0$
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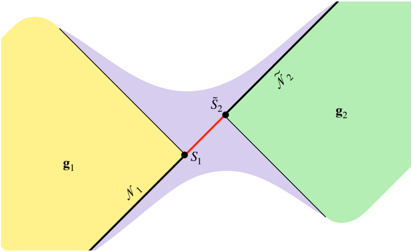

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Let P be a submanifold of M.
Let $k \in \mathbb{N} \cup \{\infty\}$.

Definition

Let g be any smoothly differentiable metric defined in a neighborhood of P. The collection

$$j^k g := \{\partial_{\alpha_1} \cdots \partial_{\alpha_\ell} g_{\mu\nu}|_P\,,\ 0 \le \ell \le k\}$$

will be called *jet of order k of g at P*.

Einstein equations
provide equations, differential and/or algebraic, for the jet of order at P.

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Einstein equations *and their derivatives up to order* $k-2$ provide equations, differential and/or algebraic, for the jet of order k at P.
A jet will be called vacuum if all such equations are satisfied.

Data on lower dimensional manifolds?

Spacelike/timelike/null vacuum submanifold data

Definition

The collection of all vacuum jets of order k will be called vacuum submanifold data of order k and will be denoted by $\Psi[P,k]$.

A jet of order k of a metric g will be called $\begin{cases} \textit{spacelike}, \\ \textit{timelike}, \\ \textit{null}, \end{cases}$ if the metric induced by g on P is.

We similarly define spacelike/timelike/null/characteristic vacuum submanifold data of order k at P.

Question

Given a submanifold $P \subset M$ and a vacuum jet $j^k g$ in $\Psi[P,k]$, is there a vacuum metric on M which realises $j^k g$?

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$j^k g =$ the coefficients of the Taylor series of a metric g at p.

vacuum $\equiv$ algebraic conditions on the Taylor coefficients

Now: *in normal coordinates the Taylor coefficients can be expressed in terms of the Riemann tensor and its covariant derivatives.*

For example, using normal coordinates,

$$j^2 g|_p \approx \{g_{\mu\nu}|_p, W^\alpha{}_{\beta\gamma\delta}|_p\} \, ,$$

where $W^\alpha{}_{\beta\gamma\delta}$ has the symmetries of the Weyl tensor.

$\Psi[\{p\}, 2] \approx$ all such pairs

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Is there a vacuum metric which realises $\Psi[\{p\},]$?

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Spacelike vacuum hypersurface data

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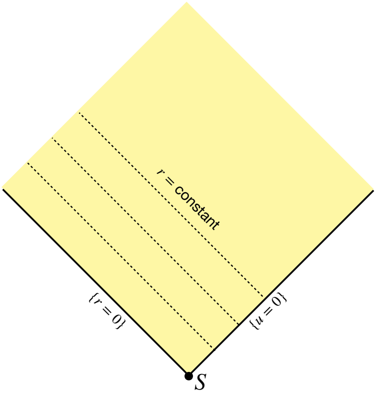
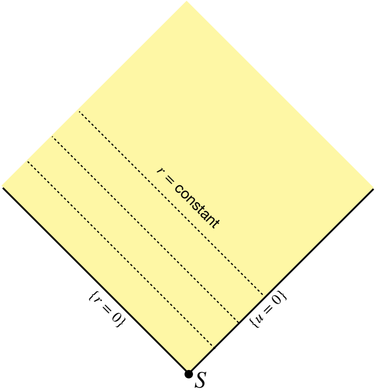
Characteristic Cauchy problem



Transversally intersecting null hypersurfaces,

Characteristic Cauchy problem



Transversally intersecting null hypersurfaces, or a light cone.

The hypersurfaces $\mathscr{N} = \{u = 0\}$ and $\underline{\mathscr{N}} = \{r = 0\}$ are characteristic for the metric

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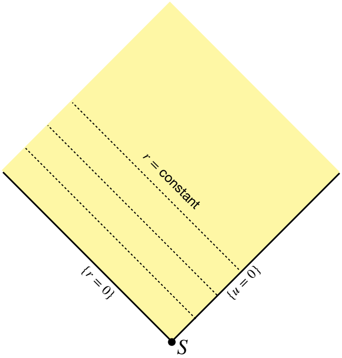

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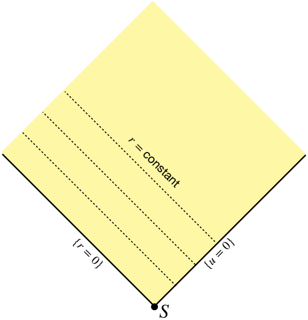
Embedding a null hypersurface in a vacuum spacetime?



Question

What about a vacuum metric in a whole neighborhood of $\mathcal{N} \cup \underline{\mathcal{N}}$*? What about a single characteristic hypersurface?*

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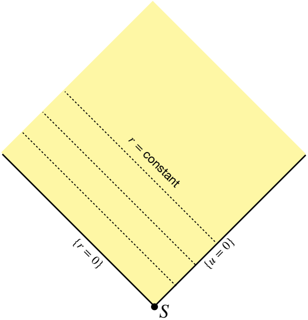

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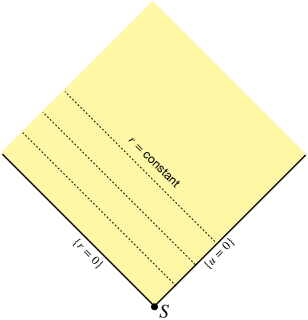

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In the Isenberg-Moncrief coordinate system the vacuum characteristic initial data $\Psi[\mathscr{N}, k]$ can be reduced to

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where **S** *is a cross-section of* $\mathscr{N}$*.* □

Proof: transverse derivatives of the metric on a characteristic hypersurface $\mathscr{N}$ are determined uniquely by the above data through ODEs along the null geodesics threading $\mathscr{N}$ or through algebraic equations.

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The “hand-crank construction”

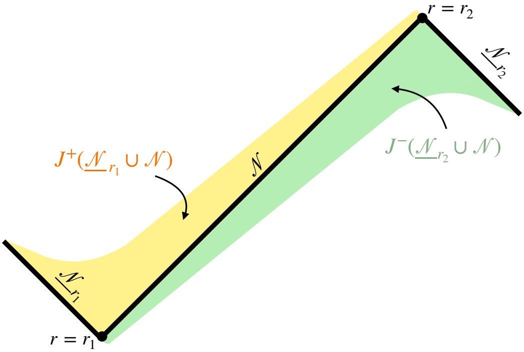

Figure: The “hand-crank construction”.

Corollary: Extending a characteristic future development

A Fledermaus = two cranks

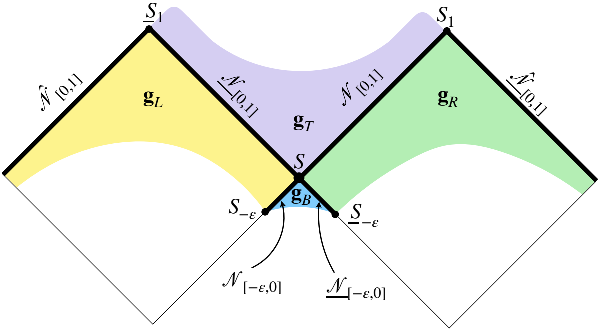

Figure: The Fledermaus.

Extension theorem

Theorem

Let (Σ, g, K) be spacelike codimension-1 vacuum initial data on a manifold with boundary $\partial\Sigma$.
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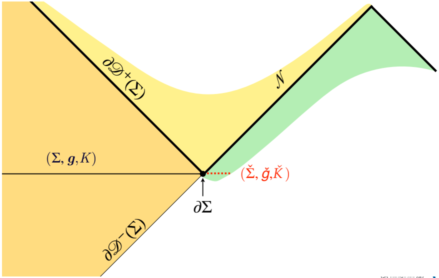

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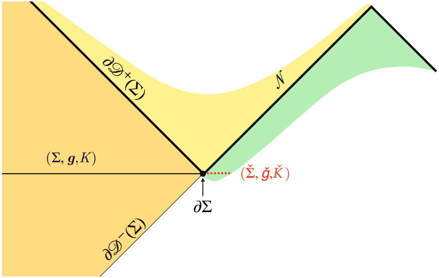
Proof (?):

use an ACR gluing to Kerr



Corollary: Embedding a truncated cone

Use the spacelike data extension

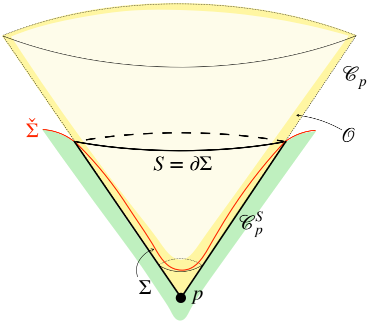


Figure: Extending a vacuum metric on a truncated future cone $J^+(p) \cap J^-(\mathscr{S})$ to a neighborhood thereof.

Corollary: Embedding data at a point

Use the embedding of a cone

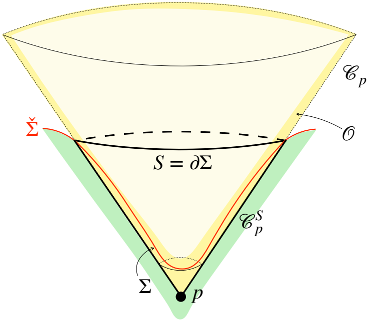


Figure: Extending data at a point