Interplay between Stochastic PDEs and AQFT

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Curved spacetimes, field theory and beyond

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Outline of the Talk

1. Motivations
2. SPDEs and Renormalization
3. An algebraic viewpoint: The ϕ^4_d-model

Based on

- C. D., N. Drago, P. Rinaldi and L. Zambotti, Comm. Cont. Math. **24** (2022) 2150075
- A. Bonicelli, C. D. and P. Rinaldi, Ann. Henri Poinc. **24**, (2023) 2443
- A. Bonicelli, C. D. and N. Drago, [arXiv:2302.10579 [math-ph]].
- A. Bonicelli, B. Costeri, C. D. and P. Rinaldi, [arXiv:2309.16376 [math-ph]]

The prototypical problem

Consider two Gaussian random variables $\xi(x,t)$, $\xi_{\mathbb{C}}(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$

$$\mathbb{E}(\xi) = 0, \quad \mathbb{E}(\xi(x,t)\xi(y,t')) = \delta(x-y)\delta(t-t').$$

Consider a **random distribution** u (real) or ψ (complex)

$$\partial_t u - \Delta u - \lambda u^n = \xi$$
$$\Delta u + \lambda u^n = \xi$$
$$i\partial_t \psi = \Delta\psi + \lambda|\psi|^2\psi + \xi_{\mathbb{C}}$$

with $n \geq 2$ and $\lambda \in \mathbb{R}$.

Question: How do you solve such kind of problems?

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Question: How do you solve such kind of problems?

A perturbative viewpoint

A first attempt to construct *solutions*:

- We call G the fundamental solution of $\partial_t - \Delta$
- We look for a *perturbative solution* $u \equiv u[[\lambda]] = \sum_{j\geq 0} \lambda^j u_j$

$$u_0 \equiv \varphi \doteq G \star_s \xi, \quad u_1 = -G \star_s \varphi^3, \quad u_j = -G \star_s \sum_{j_1+j_2+j_3=j-1} u_{j_1} u_{j_2} u_{j_3}$$

- There are divergences in defining φ^3 (need to renormalize)

Which kind of divergences?

$$\mathbb{E}(\varphi) = 0, \quad \mathbb{E}(\varphi(x)\varphi(y)) = (G \circ G^*)(x, y) \Longrightarrow \mathbb{E}(\varphi^2(f)) = (G \circ G^*)(f\delta_2)$$

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Renormalization in AQFT

The problem of divergences in SPDEs is structurally the same as in QFT

Which ingredients do we need?

- Epstein-Glaser renormalization
  - R. Brunetti, K. Fredenhangen, Comm. Math. Phys. **208** (2000), 623
- Pertrubative AQFT
  - R. Brunetti, M. Duetsch and K. Fredenhagen, Adv. Theor. Math. Phys. **13** (2009) no.5, 1541 – K. Rejzner, Math. Phys. Stud. (2016)
- Scaling Degree and Extension of distributions
  - Hörmander, Steinmann (1971), Brunetti & Fredenhangen (2000), Bahns & Wrochna (2014),...

Basic Ingredients: Data

We assign the following data:

- A *smooth Riemannian manifold* M and a *top-density* μ_M,
- E is a microhypoelliptic operator, for definiteness
  1. E is a second order elliptic PDE on M,
  2. $E = -\partial_t + K$ on $\mathbb{R} \times M$ with K, 2nd order elliptic on M.
- P (*resp.* P^*) is parametrix for E (*resp.* E^*),
- ξ is a Gaussian white noise on M (or on $\mathbb{R} \times M$).

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We call functional-valued distribution $\tau \in \mathcal{D}'(M; \mathrm{Fun})$

$$\tau : \mathcal{D}(M) \times \mathcal{E}(M) \to \mathbb{C}, \quad (f, \varphi) \mapsto \tau(f; \varphi)$$

which is linear in $\mathcal{D}(M)$ and continuous. We say

- $\tau^{(k)} \in \mathcal{D}'(M \times M^k; \mathrm{Fun})$ is the k-th derivative of τ if $\forall f \in \mathcal{D}(M), \psi_i \in \mathcal{E}(M)$,

$$\tau^{(k)}(f \otimes \psi_1 \otimes \ldots \otimes \psi_k; \varphi) \doteq \frac{\partial^k}{\partial s_1 \cdots \partial s_k} \tau(f; s_1\psi_1 + \ldots + s_k\psi_k + \varphi)\Big|_{s_1 = \ldots = s_k = 0},$$

- τ is **polynomial**, $\tau \in \mathcal{D}'(M; \mathrm{Pol})$ if $\exists \bar{k}$ such that $\tau^{(k)} = 0$ for all $k > \bar{k}$.

Example: for all $k \geq 1$, we call $\Phi^k \in \mathcal{D}'(M; \mathrm{Pol})$

$$\Phi^k(f; \varphi) = \int_M \varphi^k(x) f_\mu(x), \quad f_\mu \doteq f\mu_M$$

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Basic Ingredients: WF constraints

Long Term Goal: codify the correlations of ξ in the functionals

Let us introduce $\widehat{x}_k = (x_1, \ldots, x_k)$

$$C_1 \doteq \emptyset, \quad C_2 = WF(\delta_2), \ldots$$

$$\begin{aligned} C_k := \{(\widehat{x}_k, \widehat{\xi}_k) \in T^*M^k \setminus \{0\} \mid \\ \exists \ell \in \{1, \ldots, k-1\}, \{1 \ldots, k\} = I_1 \uplus \ldots \uplus I_\ell \text{, such that} \\ \forall i \neq j, \forall (a,b) \in I_i \times I_j \text{, then } x_a \neq x_b \text{,} \\ \text{and } \forall j \in \{1, \ldots, \ell\}, (\widehat{x}_{I_j}, \widehat{\xi}_{I_j}) \in \mathrm{WF}(\delta_{\mathrm{Diag}_{|I_j|}})\}, \end{aligned}$$

Definition

We call $\mathcal{D}'_C(M; \mathrm{Pol}) \doteq \{\tau \in \mathcal{D}'(M; \mathrm{Pol}) \mid WF(\tau^{(k)}) \subseteq C_{k+1}, \forall k \geq 0\}$.

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Basic Ingredients: Algebra Structure

Goal: endow the functionals with an algebra structure

Let $\tau \in \mathcal{D}'(M; \mathrm{Pol})$. We call

$$[P \star_s \tau](f;\varphi) := \tau(P \star_s f;\varphi)\,, \qquad \forall f \in \mathcal{D}(M)\,, \forall \varphi \in \mathcal{E}(M)\,.$$

Definition

Let $\mathbf{1}, \Phi \in \mathcal{D}'(M; \mathrm{Pol})$ be

$$\Phi(f;\varphi) := \int_M f_\mu(x)\varphi(x)\,, \qquad \mathbf{1}(f;\varphi) = \int_M f_\mu(x)\,.$$

We set recursively the $\mathcal{E}(M)$-modules

$$\mathcal{A}_0 := \mathcal{E}[\mathbf{1}, \Phi]\,, \qquad \mathcal{A}_j := \mathcal{E}[\mathcal{A}_{j-1} \cup P \star_s \mathcal{A}_{j-1}]\,, \qquad \forall j \in \mathbb{N}\,,$$

where $P \star_s \mathcal{A}_{j-1} := \{P \star_s \tau \,|\, \tau \in \mathcal{A}_{j-1}\}$. Since $\mathcal{A}_{j_1} \subseteq \mathcal{A}_{j_2}$ if $j_1 \leq j_2$, let

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The Strategy

Our plan is the following:

1. We wish to encode in $\mathcal{D}'_C(M; \mathrm{Pol})$ that actually φ should be read as

$$\varphi = P \star_S \xi, \quad \mathbb{E}(\varphi) = 0, \quad \mathbb{E}(\varphi(x)\varphi(y)) = Q(x,y) = (P \circ P^*)(x,y).$$

2. This can be obtained deforming the algebra product,
3. Computing expectation values is like evaluating at $\varphi = 0$,
4. Warning: divergences occur if one wishes to compute

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Encoding the correlations of ξ - I

We proceed in steps:

Step 1: Observe that

$$\mathcal{A} = \varinjlim \mathcal{M}_j,$$

where $\mathcal{M}_j$ is the elements of $\mathcal{A}$ with at most j fields Φ

Step 2: Let $P_\epsilon \in \mathcal{E}(M^2)$ be such that $w - \lim_{\epsilon \to 0^+} P_\epsilon = P$ and $Q_\epsilon = P_\epsilon \circ P_\epsilon$.

Proposition

We call $\mathcal{A}_{\cdot Q_\epsilon}$ the unital, commutative and associative algebra such that, for all $f \in \mathcal{D}(M)$ and for all $\varphi \in \mathcal{E}(M)$,

$$[\tau \cdot_{Q_\epsilon} \tau'](f;\varphi) = \sum_{k \geq 0} \frac{1}{k!}[(\delta_2 \circ Q_\epsilon^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})](f \otimes 1_{1+2k};\varphi).$$

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Notice (I mean it!)

Obs. 1: The product is well defined because we control

- $WF(\tau^{(k)}) \subseteq C_{k+1}$,
- $WF(\delta_2 \otimes Q_\epsilon^{\otimes k})$.

Obs. 2: If we compute

$$[\Phi \cdot_{Q_\epsilon} \Phi](f;\varphi) = \int_M f_\mu(x)[\varphi^2(x) + Q_\epsilon(x,x)] = \Phi^2(f;\varphi) + Q_\epsilon(f\delta_2),$$

hence

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Can we get rid of ϵ? Can we compute also correlations?

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Encoding the correlations of ξ - III

Theorem (First Key Result)

There exists a linear map $\Gamma_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'_{\mathrm{C}}(M; \mathrm{Pol})$ *such that*

1. *for all* $\tau \in \mathcal{M}_1$, $\Gamma_{\cdot_Q}(\tau) = \tau$.
2. *for all* $\tau \in \mathcal{A}$ *it holds* $\Gamma_{\cdot_Q}(P \star_S \tau) = P \star_S \Gamma_{\cdot_Q}(\tau)$.
3. *for all* $\psi \in \mathcal{E}(M)$ *it holds*
$$\Gamma_{\cdot_Q} \circ \delta_\psi = \delta_\psi \circ \Gamma_{\cdot_Q} \,, \qquad \Gamma_{\cdot_Q}(\psi\tau) = \psi\Gamma_{\cdot_Q}(\tau) \,.$$
4. *For all* $\tau \in \mathcal{M}_k$
$$\sigma_p(\Gamma_{\cdot_Q}(\tau)) \leq pd + \frac{k-p}{2} \max\{0, d-4\} \,,$$
where $\sigma_p(\tau) = \mathrm{sd}_{\mathrm{Diag}_{p+1}}(\tau^{(p)})$ *and* $\mathrm{Diag}_{p+1} \subset M^{p+1}$ *is the total diagonal of* M^{p+1}.

Key aspects of the proof - I

The proof is inductive and divided in several cases. Observe

- Main idea: If $\tau = \tau_1 \dots \tau_n \in \mathcal{A}$, we set

$$\Gamma_{\cdot_Q}(\tau) = \Gamma_{\cdot_Q}(\tau_1) \cdot_Q \cdots \cdot_Q \Gamma_{\cdot_Q}(\tau_n)$$

- We focus on E elliptic, self-adjoint for simplicity
- with $\dim M = d = 2, 3$ the product is well defined.
- If we construct $\Gamma_{\cdot_Q}(\tau)$, $\tau \in \mathcal{A}$, then $\Gamma_{\cdot_Q}(P \star_S \tau)$ is completely determined

$$\Gamma_{\cdot_Q}(P \star_S \tau) = P \star_S \Gamma_{\cdot_Q}(\tau).$$

All conditions 1.-4. are met by direct inspection

Key aspects of the proof - I

The proof is inductive and divided in several cases. Observe

- Main idea: If $\tau = \tau_1 \dots \tau_n \in \mathcal{A}$, we set

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Recall $\mathcal{A} = \varinjlim \mathcal{M}_j$

Step 0: If $j = 0, 1$, there is nothing to do

Step 1: If $j = 2$, it suffices to consider $\mathcal{M}_2^0 = \mathrm{span}_{\mathcal{E}(M)}\left(1, \Phi, \Phi^2\right)$

Only unknown $\Gamma_{\cdot_Q}(\Phi^2)(f;\varphi) = [\Gamma_{\cdot_Q}(\Phi) \cdot_Q \Gamma_{\cdot_Q}](f;\varphi) = \Phi^2(f;\varphi) + P^2(f \otimes 1)$

- Here $Q = P \circ P^* = P^2$ since $E = E^*$
- $P^2 \in \mathcal{D}'(M^2 \setminus \mathrm{Diag}_2)$ and $\mathrm{sd}(P^2) \leq 2(d-2)$

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Step 1b: Check that all hypothesis are met $(WF(P^2) = WF(\delta_2))$

Step 2: Proceed inductively to $\mathcal{M}^0_{k+1} = \mathrm{span}_{\mathcal{E}(M)}(1, \Phi, \ldots \Phi^{k+1})$

$$\Gamma_{\cdot_Q}(\Phi^{k+1}) = \underbrace{\Gamma_{\cdot_Q}(\Phi) \cdot_Q \ldots \cdot_Q \Gamma_{\cdot_Q}(\Phi)}_{k+1}(f;\varphi) =$$

$$= \sum_{\ell=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1}{2\ell} (Q_{2\ell} \cdot \Gamma_{\cdot_Q}(\Phi)^{k+1-2\ell})(f;\varphi)$$

where $Q_{2\ell}(f) = (P^2)^{\otimes \ell} \cdot (\delta_{\mathrm{Diag}_\ell} \otimes 1_\ell)(f \otimes 1_{2\ell-1})$.

$$Q_{2l}(f) \mapsto \widehat{Q}_{2l}(f) \doteq \widehat{P}_2^{\otimes \ell} \cdot (\delta_{\mathrm{Diag}_\ell} \otimes 1_\ell)(f \otimes 1_{2\ell-1})$$

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Consequences

Observe that

- $\mathcal{A}_{\cdot_Q}$ is a unital, commutative and associative algebra

$$\tau \cdot_Q \tau' = \Gamma_{\cdot Q}[\Gamma_{\cdot Q}^{-1}(\tau)\Gamma_{\cdot Q}^{-1}(\tau')], \quad \forall \tau, \tau' \in \mathcal{A}_{\cdot Q}.$$

- We are still not able to compute correlations such as

$$\mathbb{E}[\Phi^2(x)\Phi^2(y)]$$

More precisely, formally we have to deal with

$$[\Phi^2 \bullet_Q \Phi^2](f_1 \otimes f_2; \varphi) =$$

$$\int_{M\times M} f_{1,\mu}(x_1) f_{2,\mu}(x_2)\left[\varphi(x_1)^2\varphi(x_2)^2 + 4\varphi(x_1)Q(x_1,x_2)\varphi(x_2) + 2Q(x_1,x_2)^2\right],$$

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Correlations and the $\bullet_Q$-product

Consider $\mathcal{A}_{\cdot Q} = \Gamma_{\cdot_Q}[\mathcal{A}]$ and

$$\mathcal{T}[\mathcal{A}_{\cdot Q}] \doteq \mathcal{E}(M) \oplus \bigoplus_{l>0} \mathcal{A}_{\cdot Q}^{\otimes l} \quad \text{Universal Tensor Module}$$

together with

$$\mathcal{T}'_C(M;\mathrm{Pol}) = \mathbb{C} \oplus \bigoplus_{n>0} \mathcal{D}'_C(M;\mathrm{Pol})^{\otimes n}$$

endowed with the product

$$(\tau_1 \bullet_Q \tau_2)(f_1 \otimes f_2; \varphi) = \sum_{k\geq 0} \frac{1}{k!}[(1_{n_1+n_2} \otimes Q^{\otimes k}) \cdot (\tau_1^{(k)} \widetilde{\otimes} \tau_2^{(k)})](f_1 \otimes f_2 \otimes 1_{2k}; \varphi)\,,$$

with $\tau_j \in \mathcal{D}'_C(M^{n_j})$ and $f_j \in \mathcal{D}(M^{n_j})$.

Correlations and the $\bullet_Q$-product - I

Theorem (Second Key Result)

There exists a linear map $\Gamma_{\bullet_Q} : \mathcal{T}(\mathcal{A}_{\cdot_Q}) \to \mathcal{T}'_{\mathrm{C}}(M; \mathrm{Pol})$ *such that*

(i) *for all* $\tau_1, \ldots, \tau_\ell \in \mathcal{A}_{\cdot_Q}$ *with* $\tau_1 \in \Gamma_{\cdot_Q}(\mathcal{M}_1)$ *it holds*

$$\Gamma_{\bullet_Q}(\tau_1 \otimes \ldots \otimes \tau_\ell) := \tau_1 \bullet_Q \Gamma_{\bullet_Q}(\tau_2 \otimes \ldots \otimes \tau_\ell)\,,$$

(ii) *Let* $\tau_1, \ldots, \tau_\ell \in \mathcal{A}_{\cdot_Q}$ *and* $f_1, \ldots, f_\ell \in \mathcal{D}(M)$*. If* $\exists I \subsetneq \{1, \ldots, \ell\}$

$$\bigcup_{i \in I} \mathrm{spt}(f_i) \cap \bigcup_{j \notin I} \mathrm{spt}(f_j) = \emptyset\,,$$

then

$$\Gamma_{\bullet_Q}(\tau_1 \otimes \ldots \otimes \tau_\ell)(f_1 \otimes \ldots \otimes f_\ell) =$$
$$= \left[\Gamma_{\bullet_Q}\left(\bigotimes_{i \in I} \tau_i\right) \bullet_Q \Gamma_{\bullet_Q}\left(\bigotimes_{j \notin I} \tau_j\right)\right](f_1 \otimes \ldots \otimes f_\ell)\,.$$

Correlations and the $\bullet_Q$-product - II

In addition it holds

- for all $\ell \geq 0$, $\Gamma_{\bullet_Q} : \mathcal{A}_{\cdot_Q}^{\otimes \ell} \to \mathcal{T}'_{\mathrm{C}}(M; \mathrm{Pol})$ is a symmetric map,
- $\Gamma_{\bullet_Q}$ satisfies a set of identities, e.g.

$$\Gamma_{\bullet_Q}(\tau) = \tau\,, \qquad \forall \tau \in \mathcal{A}_{\cdot_Q}\,,$$
$$\Gamma_{\bullet_Q} \circ \delta_\psi = \delta_\psi \circ \Gamma_{\bullet_Q}\,, \qquad \forall \psi \in \mathcal{E}(M)\,.$$

Proposition

Given any map $\Gamma_{\bullet_Q}$ let

$$\mathcal{A}_{\bullet_Q} := \Gamma_{\bullet_Q}(\mathcal{A}_{\cdot_Q}) \subseteq \mathcal{T}'_{\mathrm{C}}(M; \mathrm{Pol})\,.$$

Then the bilinear map $\bullet_{\Gamma_{\bullet_Q}} : \mathcal{A}_{\bullet_Q} \times \mathcal{A}_{\bullet_Q} \to \mathcal{A}_{\bullet_Q}$ defined by

$$\tau \bullet_{\Gamma_{\bullet_Q}} \bar{\tau} := \Gamma_{\bullet_Q}(\Gamma_{\bullet_Q}^{-1}(\tau) \otimes \Gamma_{\bullet_Q}^{-1}(\bar{\tau}))\,, \qquad \forall \tau, \bar{\tau} \in \mathcal{A}_{\bullet_Q}\,,$$

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(Non-)Uniqueness Results

Question: Are the maps $\Gamma_{\cdot_Q}$ and $\Gamma_{\bullet_Q}$ unique?

Proposition

Let $\widetilde{\Gamma}_{\cdot_Q}, \Gamma_{\cdot_Q} : \mathcal{A} \to \mathcal{D}'(M; \mathrm{Pol})$ be two linear maps compatible with the existence theorem. Then the algebras $\mathcal{A}_{\cdot_Q} = \Gamma_{\cdot_Q}(\mathcal{A})$ and $\widetilde{\mathcal{A}}_{\cdot_Q} = \widetilde{\Gamma}_{\cdot_Q}(\mathcal{A})$ coincide and in particular there exists $\{c_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathcal{E}(M)$ a family of smooth functions, such that for all $k \in \mathbb{N}$

$$\widetilde{\Gamma}_{\cdot_Q}(\Phi^k) = \Gamma_{\cdot_Q}\left(\Phi^k + \sum_{\ell=0}^{k-2} \binom{k}{\ell} c_{k-\ell} \Phi^\ell\right).$$

Observe that

- A similar theorem holds true for $\Gamma_{\bullet_Q}$
- We do not have *local covariance* to further constraint $\{c_\ell\}_{\ell \in \mathbb{N}_0}$
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1^{st} Example: The Φ^3_d Model

Consider on $\mathbb{R} \times \mathbb{R}^d$

$$\partial_t u = \Delta u - \lambda u^3 + \xi$$

We consider $u[[\lambda]] = \sum_{j \geq 0} \lambda^j u_j$ where

$$u_0 = \Phi,\; u_1 = -P_\chi \star_S \Phi^3, \ldots \;\; u_j = -P_\chi \star_S \sum_{j_1+j_2+j_3=j-1} u_{j_1} u_{j_2} u_{j_3}$$

Next we interpret each term in $\mathcal{A}_{\cdot Q}$

$$u[[\lambda]] \mapsto \Gamma_{\cdot_Q}(u[[\lambda]]).$$

which entails that

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First order of Φ^3_d Model

At first order in perturbation theory

$$u[[\lambda]] = \Phi - \lambda P_\chi \star_S \Phi^3 + O(\lambda^2),$$

from which it descends

$$\Gamma_{\cdot_Q}(u[[\lambda]])(f;\varphi) = \Phi(f;\varphi) - \lambda P_\chi \star_S (\Phi^3 + 3C\Phi)(f;\varphi) + O(\lambda^2),$$

where $C \in \mathcal{E}(\mathbb{R} \times \mathbb{R}^d)$. Hence evaluating at $\varphi = 0$

$$\mathbb{E}(u[[\lambda]]) = O(\lambda^2).$$

What about the two-point correlation function?

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What about the two-point correlation function?

Correlation function at first order

Our approach tells that

$$\omega_2(f_1 \otimes f_2; \varphi) = \left(\Gamma_{\cdot_Q}(u[[\lambda]]) \bullet_{\Gamma_{\bullet_Q}} \Gamma_{\cdot_Q}(u[[\lambda]]) \right) (f_1 \otimes f_2; \varphi).$$

At first order in perturbation theory

$$\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi) \otimes \Gamma_{\cdot_Q}(P_\chi \star_s \Phi^3))(f_1 \otimes f_2; \varphi) =$$

$$= (\Phi \otimes (P_\chi \star_s (\Phi^3 + 3C\Phi))(f_1 \otimes f_2; \varphi) + Q \cdot (1 \otimes 3P_\chi \star_s (\Phi^2 + C1))(f_1 \otimes f_2; \varphi).$$

Evaluating once more at $\varphi = 0$

$$\mathbb{E}(\widehat{u}[[\lambda]] \otimes \widehat{u}[[\lambda]])(f_1 \otimes f_2) = \omega_2(f_1 \otimes f_2; 0) =$$

$$Q(f_1 \otimes f_2) + 3\lambda Q \cdot (1 \otimes (P_\chi \star_s C))(f_1 \otimes f_2) + O(\lambda^2).$$

- We can also construct the *renormalized equation* obeyed by $\Gamma_{\cdot_Q}(u)$.

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$$\omega_2(f_1 \otimes f_2; \varphi) = \Big(\Gamma_{\cdot_Q}(u[[\lambda]]) \bullet_{\Gamma_{\bullet_Q}} \Gamma_{\cdot_Q}(u[[\lambda]])\Big)(f_1 \otimes f_2; \varphi).$$

At first order in perturbation theory

$$\Gamma_{\bullet_Q}(\Gamma_{\cdot_Q}(\Phi) \otimes \Gamma_{\cdot_Q}(P_\chi \star_s \Phi^3))(f_1 \otimes f_2; \varphi) =$$
$$= (\Phi \otimes (P_\chi \star_s (\Phi^3 + 3C\Phi))(f_1 \otimes f_2; \varphi) + Q \cdot (1 \otimes 3P_\chi \star_s (\Phi^2 + C\mathbf{1}))(f_1 \otimes f_2; \varphi).$$

Evaluating once more at $\varphi = 0$

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The Stochastic Sine-Gordon model[1]

On $(\mathbb{R}^2, \eta)$

$$(\Box_\eta + m^2)u + \lambda g a \sin(au) = \xi, \quad a^2 < \frac{4\pi}{\hbar}, \text{ and } g \in \mathcal{D}(\mathbb{R}^2).$$

Main Data:

- $u \equiv u[[\lambda]] = \sum_{n=0}^{\infty} \lambda^n u_n \Longrightarrow u_0 = G_{ret} \star_S \xi$

  $$Q = G_{ret} \circ_\chi G_{adv} \in C^0(\mathbb{R}^2; [0,\infty)) \Longrightarrow Q(x,x) \in C^\infty(\mathbb{R}; [0,\infty)).$$

- for $\xi = 0$ we have the sine-Gordon model (see Bahns, & Rejzner 2018 + Pinamonti 2023)

[1] A. Bonicelli, C.D. and P. Rinaldi, arXiv:2311.01558 [math-ph]

The Sine-Gordon model in AQFT

For any $G \in \mathcal{D}'(\mathbb{R}^2; \mathrm{Fun}_{loc})$ the **quantum counterpart** is

$$R_V(G) = [S(\lambda V)^{\star_{\hbar\omega}}]^{-1} \star_{\hbar\omega} (S(\lambda V) \star_{\hbar\Delta_F} G),$$

where

- $V = \cos(au)$,
- ω is a Hadamard two-point correlation function,
- $\Delta_F = \omega + iG_{ret}$ and $S(\lambda V) = \exp_{\star_{\hbar\Delta_F}} \left(\frac{i}{\hbar}\lambda V\right)$.

In particular if we set $G = u_f$, $f \in \mathcal{D}(\mathbb{R}^2)$ we have the *interacting field*

The series $R_V[u_f]$ is convergent – [Bahns & Rejzner 2018]

The stochastic Sine-Gordon model from AQFT

We prove the following two key results:

Theorem

The series $\Gamma_Q[R_{\lambda V}[u](f,\varphi)]$ *is absolutely convergent for all* $(f,\varphi) \in \mathcal{D}(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$.

Theorem

The limit

$$\lim_{\hbar \to 0^+} \Gamma_Q[R_{\lambda V}[u](f,\varphi)]$$

exists and it converges to a solution of the stochastic sine-Gordon equation.

Outlook

We have

- Constructed a new framework to analyze perturbatively SPDEs
- extended it to cover the stochastic nonlinear Schrödinger equation
- connected **the microlocal world and the germs of distributions** [2]

[2] F. Caravenna and L. Zambotti – EMS Surv. Math. Sci. **7** (2020), 207
P. Rinaldi and F. Sclavi – J. Math. Anal. & Appl. **501** (2021), 125215
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What's Next

- Connect our framework to Hairer's regularity structures **and to Gubinelli's paracontrolled calculus**,
- Extend our framework to cover the stochastic wave equation,
- Explore Coleman's correspondence between the stochastic GN and the Sine-Gordon models,
- Tackle the problem of convergence of the perturbative series,

Trivia on random distributions - I

NOTATION: Given $z = (t, x) \in \mathbb{R}^{1+d}$, $\varphi \in C_c^\infty(\mathbb{R}^{1+d})$

$$\varphi_z^\lambda(s, y) = \lambda^{-d-2}\varphi(\lambda^{-2}(s-t), \lambda^{-1}(y-x)), \quad \lambda \in (0,1).$$

Definition (Negative Hölder Spaces)

Let $\eta \in \mathcal{S}'(\mathbb{R}^{1+d})$ and let $\alpha < 0$. We say $\eta \in \mathcal{C}^\alpha$ if

$$|\eta(\varphi_z^\lambda)| \lesssim \lambda^\alpha, \qquad \text{for} \quad \lambda \in (0,1], \quad \varphi \in \mathcal{B}_\alpha,$$

locally uniformly for $z \in \mathbb{R}^{1+d}$, with

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$$\mathbb{E}[\eta(\varphi)\eta(\psi)] = (C * \varphi, \psi)_{L^2}$$

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Space-Time White Noise is the Gaussian random distribution on $\mathbb{R}^{1+d}$ with covariance given by the delta distribution δ, **i.e.**, $\xi(\varphi)$ is centred Gaussian for every $\varphi \in C_c^\infty(\mathbb{R}^{1+d})$ and $\mathbb{E}[\xi(\varphi)\xi(\psi)] = (\varphi, \psi)_{L^2}$.

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$\tilde{\eta}$ is a version of η if $\forall \varphi \in C_c^\infty$, $\tilde{\eta}(\varphi) = \eta(\varphi)$ almost surely.

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