Bounds for the local entropy

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Tomita-Takesaki modular theory

$\mathcal{M}$ a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$S_0 : X\Omega \mapsto X^*\Omega, \quad X \in \mathcal{M}$$

$S_{\mathcal{M}} = \bar{S}_0 = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{1/2}$, polar decomposition, $\Delta_{\mathcal{M}}$ and $J_{\mathcal{M}}$ modular operator and conjugation

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta_{\mathcal{M}}^{it} X \Delta_{\mathcal{M}}^{-it}, \quad X \in \mathcal{M}$$

modular automorphisms **intrinsic evolution associated with φ!**

$$J_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}} = \mathcal{M}' \quad \text{on } \mathcal{H}$$

$\log \Delta_{\mathcal{M}}$ is called the modular Hamiltonian of φ

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exist; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi\|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi\|\psi) \geq 0$$

positivity of the relative entropy

Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a *closed, real linear subspace*.
Symplectic complement:

$$H' = \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is a standard subspace if it is $\mathcal{H}$ *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* $H \cap iH = \{0\}$

$$H \textit{ standard subspace} \rightarrow \textit{anti-linear operator } S_H$$

$$S_H : \xi + i\eta \rightarrow \xi - i\eta, \ \ \xi, \eta \in H$$

$S_H^2 = 1|_{D(S_H)}$, $D(S_H) = H + iH$. S_H is closed, densely defined, $S_H^* = S_{H'}$

Modular theory for standard subspaces

Conversely, S densely defined, closed, anti-linear involution on $\mathcal{H} \to H_S = \{\xi \in D(S) : S\xi = \xi\}$ is a standard subspace:

$H \leftrightarrow S$ is a bijection

Set $S_H = J_H \Delta_H^{1/2}$, polar decomposition. Then J_H is an anti-unitary involution, $\Delta_H > 0$ is non-singular called the modular conjugation and the modular operator; they satisfy $J_H \Delta_H J_H = \Delta_H^{-1}$ and

$H \leftrightarrow (J, \Delta)$ is a bijection.

Main relations:

$$\Delta_H^{it} H = H, \quad J_H H = H'$$

Every closed, real linear H is

$$\text{standard} \oplus (0 \subset \mathcal{H}) \oplus (\mathcal{H} \subset \mathcal{H})$$

Examples

Example 1: $\mathcal{M}$ von Neumann algebra on $\mathcal{H}$, Ω cyclic separating vector

$$H = \overline{\mathcal{M}_{\mathrm{s.a.}}\Omega} \text{ is a standard subspace of } \mathcal{H}$$

$$\Delta_H = \Delta_{\mathcal{M}}, \quad J_H = J_{\mathcal{M}}$$

Example 2: $\mathcal{H}$ (one-particle) Hilbert space, $H \subset \mathcal{H}$ real Hilbert space (of vectors localized in a region O)

$$\Gamma(\Delta_H) = \Delta_{\mathcal{A}(H)} \quad \Gamma(J_H) = J_{\mathcal{A}(H)}$$

$\mathcal{A}(H)$ von Neumann algebra on the Fock space $e^{\mathcal{H}}$

$$\mathcal{A}(H) = \{V(\xi) : \xi \in H\}''$$

$V(\xi)$ Weyl unitary

Passivity

$\log \Delta_H$ is characterised by complete passivity, following Pusz and Woronowicz in the von Neumann algebra case

$\mathcal{H}$ a complex Hilbert space, $H \subset \mathcal{H}$ a standard subspace and A a selfadjoint linear operator on $\mathcal{H}$ such that $e^{isA}H = H$, $s \in \mathbb{R}$.

A is passive with respect to H if

$$-(\xi, A\xi) \geq 0\,, \quad \xi \in D(A) \cap H\,.$$

A is completely passive w.r.t. H if the generator of $e^{itA} \otimes e^{itA} \cdots \otimes e^{itA}$ is passive with respect to the n-fold tensor product $H \otimes H \otimes \cdots \otimes H$, all $n \in \mathbb{N}$.

A is completely passive with respect to H iff $\log \Delta_H = \lambda A$ *for some* $\lambda \geq 0$.

positivity of energy $\leftrightsquigarrow$ *comp. passivity of modular Hamiltonian* (equivalence in principle)

Entropy of a vector relative to a real linear subspace

Let $\mathcal{H}$ be a complex Hilbert space and $H \subset \mathcal{H}$ a standard subspace

The entropy of a vector $h \in \mathcal{H}$ with respect to $H \subset \mathcal{H}$ is defined by

$$S(h\|H) = -\Im(h, P_H i \log \Delta_H\, h) = \Re(h, iP_H i \log \Delta_H\, h)$$

(in a quadratic form sense), where P_H is the cutting projection; if H is factorial

$$P_H : H + H' \to H\,, \quad h + h' \mapsto h$$

We have $P_H^* = -iP_H i$ and the formula

$$P_H = (1 + S_H)(1 - \Delta_H)^{-1}$$
$$= (1 - \Delta_H)^{-1} + J_H \Delta_H^{1/2}(1 - \Delta_H)^{-1}\,;$$

(P_H is the closure of the right-hand side).

In QFT, the cutting projection P_H is geometric.

Properties of the entropy of a vector

Some of the main properties of the entropy of a vector are:

- $S(h\|H) \geq 0$ or $S(h\|H) = +\infty$ positivity
- If $K \subset H$, then $S(h\|K) \leq S(h\|H)$ monotonicity
- If $h_n \to h$, then $S(h\|H) \leq \liminf_n S(h_n|H)$ lower semicontinuity
- If $H_n \subset H$ is an increasing sequence with $\overline{\bigcup_n H_n} = H$, then $S(h\|H_n) \nearrow S(h\|H)$ monotone continuity
- If $h \in D(\log \Delta_H)$ then $S(h\|H) < \infty$ finiteness on smooth vectors
- $S(h\|H) = S(k\|H)$ if $k - h \in H'$ locality

By locality, we may talk of the entropy of a class of vectors

Entropy of coherent sectors

Given $\xi \in \mathcal{H}$ consider coherent state φ_ξ on Weyl von Neumann algebra $\mathcal{A}(H)$ on the Bose Fock space $e^{\mathcal{H}}$:

The vacuum relative entropy of φ_ξ on $\mathcal{A}(H)$ is given by

$$S(\varphi_\xi \| \varphi_0) = S(\xi \| H)$$

Araki's relative entropy

Entropy of vector

Ω vacuum vector, $\varphi_\xi = (V(\xi)\Omega, \cdot V(\xi)\Omega)$, $V(\xi)$ Weyl unitary

Fermi case (Galanda, Much, Verch): Similar formula for $\varphi_\xi = (\Phi(\xi)\Omega, \cdot \Phi(\xi)\Omega)$, $\Phi(\xi)$ selfadjoint (unitary) Fermi free field

Entropy operator

The entropy operator $\mathcal{E}_H$ is defined by

$$\mathcal{E}_H = A(\Delta_H) + J_H B(\Delta_H)\,,$$

$A(\lambda) \equiv -a(\lambda)\log\lambda,\ B(\lambda) \equiv b(\lambda)\log\lambda$

In the factorial case

$$\mathcal{E}_H = iP_H i \log\Delta_H$$

(closure of the right-hand side). We have

$$S(h\|H) = \Re(h, \mathcal{E}_H h)\,, \quad k \in \mathcal{H}\,.$$

real quadratic form sense.

The entropy operator $\mathcal{E}_H$ is *real linear, positive, and selfadjoint* w.r.t. to the real part of the scalar product.

First and second quantisation

First quantisation: map

$$O \subset \mathbb{R}^d \mapsto H(O) \text{ real linear space of } \mathcal{H}$$

local, covariant, etc.

Second quantisation: map

$$O \subset \mathbb{R}^d \mapsto \mathcal{A}(O) \text{ v.N. algebra on } e^{\mathcal{H}}$$

$\mathcal{A}(O) = \mathcal{A}(H(O))$

In our case $H(O)$ is generated by the waves with Cauchy data in B (O double cone with time-zero basis B)

Wave packets

By a Klein-Gordon wave (or wave packet), we mean a real solution of the wave equation

$$(\Box + m^2)\Phi = 0 \ ,$$

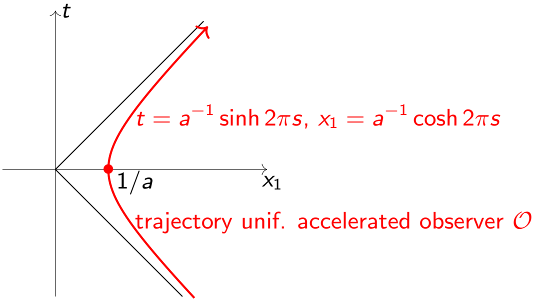
with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes Φ by the stress-energy tensor $T_{\mu\nu}$, that provides the energy-momentum density of Φ at any time.

But, how to define the information, or entropy, carried by Φ in a given region at a given time?

Bisognano-Wichmann theorem '75

Rindler spacetime (wedge $W = \{x_1 > |t|\}$), vacuum modular group



a : uniform acceleration of O
s/a : proper time of O
$\beta = 2\pi/a$: inverse KMS temperature of O

Hawking-Unruh effect!

Entropy of a wave

Let Φ be a real Klein-Gordon wave and $H = H(W)$.

The entropy $S_\Phi(\lambda)$ of Φ w.r.t. the wedge region W_λ is the entropy of the vector Φ w.r.t. the standard subspace $H(W_\lambda)$.

$$S_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1\geq\lambda} (x^1 - \lambda)\, T_{00}(x)\, dx$$

then

$$S''_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1=\lambda} \langle v, Tv\rangle dx \geq 0 \ ,$$

where v is the light-like vector $v = (1, 1, 0\dots, 0)$.

Here the energy density is $T_{00} = \frac{1}{2}\big(\Phi'^2 + |\nabla\Phi|^2 + m^2\Phi^2\big)$

The second derivative of $S''_\Phi(\lambda)$ gives the QNEC inequality for coherent states and constant null translations

$$S''_\Phi(\lambda) \geq 0$$

(F. Ciolli, G. Ruzzi, R. L.)

Borchers' theome one-article analogue

Let $H \subset \mathcal{H}$ be a standard subspace and $T(t) = e^{iAt}$ a one-parameter unitary group on $\mathcal{H}$ such that

- $A \geq 0$
- $T(t)H \subset H$, $t \geq 0$

Then

$$\Delta_H^{is} T(t) \Delta_H^{-is} = T(e^{-2\pi s)} t) \; , \; J_H T(t) J_H = T(-t)$$

$T(t)$ and Δ_H^{is} generates a 2-dimensional Lie group!

Abstract result

Let $H \subset \mathcal{H}$ be a standard subspace and $T(t) = e^{iAt}$ a one-parameter unitary group on $\mathcal{H}$ such that

- $A \geq 0$
- $T(t)H \subset H$, $t \geq 0$

Define $H_\lambda = T(\lambda)H\,,\ \lambda \in \mathbb{R}$, translated subspaces. Then the entropy function

$$\lambda \mapsto S(\lambda) = S(\psi \| H_\lambda) \text{ is convex for all } \psi$$

and finite for a dense set of vectors. If $S(\lambda_0) < \infty$, then

(i) $S(\lambda)$ is finite and C^1 on $[\lambda_0, \infty)$;

(ii) $S'(\lambda)$ is absolutely continuous in $[\lambda_0, \infty)$ with almost everywhere non-negative derivative $S''(\lambda) \geq 0$.

Entropy of localised states: $U(1)$-current model

One-dimensional case.

$U(1)$-current j: ℓ real function in $S(\mathbb{R})$, $L(x) \equiv \int_{-x}^{\infty} \ell(t)dt$.

$$S(\lambda) \equiv S(L \| H(\lambda, \infty)) = \pi \int_{\lambda}^{+\infty} (x - \lambda)\ell^2(x)\mathrm{d}x \ ,$$

$S(\lambda)$ vacuum relative entropy of excited state by $j \mapsto j + \ell$ (BMT sector with charge $q = \int \ell$)

$$S'(\lambda) = -\pi \int_{\lambda}^{+\infty} \ell^2(x)\mathrm{d}x \leq 0 \ ,$$

$$S''(\lambda) = \pi\ell^2(\lambda) \geq 0$$

positivity of S''

L is not a vector in the Hilbert space, L but gives a *class* vectors: $\{f \in S(\mathbb{R}^d) : f|_{[\lambda,\infty)} = L|_{[\lambda,\infty)}\}$

Entropy and Klein–Gordon field on a globally hyperbolic spacetime

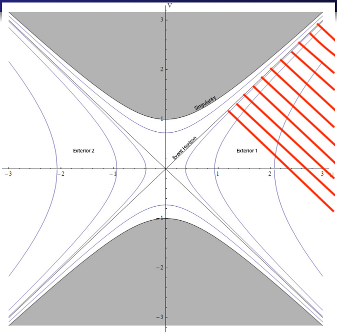
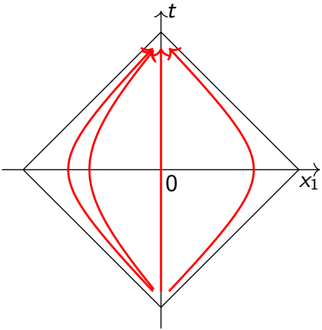


Figure: Schwarzschild-Kruskal spacetime. The red area is a null translated wedge

The convexity of the entropy w.r.t. to the null translation parameter holds for a Klein–Gordon field on a globally hyperbolic spacetime for coherent states (Ciolli, Ranallo, Ruzzi, L.) (cf. also R.L. and Holland, Ishibashi in untranslated case)

Double cone, conformal case

For a bounded region O (double cone, causal envelop of a space ball B), in the conformal case the modular group is given by the geometric transformation (Hislop, L. '81)



local modular trajectories

$$(u, v) \mapsto ((Z(u, s), Z(v, s))$$

$$Z(z, s) = \frac{(1+z)+e^{-s}(1-z)}{(1+z)-e^{-s}(1-z)}$$

$$u = x_0 + r, \quad v = x_0 - r, \quad r = |\mathbf{x}| \equiv \sqrt{x_1^2 + \cdots + x_d^2}$$

The local entropy of a massless wave

The modular Hamiltonian $\log \Delta_B$ associated with the unit ball B in the free scalar, massless QFT is (on Cauchy data)

$$-2\pi A = \log \Delta_B \, .$$

$$\log \Delta_B = 2\pi\imath_0 \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D & 0 \end{bmatrix}$$

with L_0 the higher dimensional Legendre operator

$$L_D = \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D$$

(Work with G. Morsella)

Local information in a wave packet

With $S_\Phi(R)$ the entropy of Φ in the radius R ball cantered at $\bar{\mathbf{x}}$, we have

$$S_\Phi(R) = \pi \int_{B_R(\bar{\mathbf{x}})} \frac{R^2 - r^2}{R} \langle T_{00}(t, \mathbf{x}) \rangle_\Phi d\mathbf{x} \quad \text{stress-energy tensor term}$$
$$+ \pi \frac{d-1}{2R} \int_{B_R(\bar{\mathbf{x}})} \Phi^2(t, \mathbf{x}) d\mathbf{x} \quad \text{Born type term}$$

with $r = |\mathbf{x} - \bar{\mathbf{x}}|$

Nets of standard subspaces

$\mathcal{H}$ complex Hilbert space, $\mathcal{O}$ the family of double cones of the Minkowski spacetime $\mathbb{R}^{d+1}$.

A local Poincaré covariant *net of real linear subspaces* is a map

$$O \in \mathcal{O} \mapsto H(O) \subset \mathcal{H}\,,$$

with $H(O)$ real linear, closed subspace of $\mathcal{H}$, s.t.

- $O_1 \subset O_2 \implies H(O_1) \subset H(O_2)$ (*isotony*);
- $O_1 \subset O_2' \implies H(O_1) \subset H(O_2)'$ (*locality*);
- $\exists$ a unitary, positive energy representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ s.t. $U(g)H(O) = H(gO)$ (*Poincaré covariance*);
- $\overline{\sum_{x\in\mathbb{R}^{d+1}} H(O+x)} = \mathcal{H}$ (*non-degeneracy*).

Set $H(C) \equiv \mathrm{lin.span.}\{H(O) : O \subset C\}$ for any region C

Reeh-Schlieder theorem: $H(C)$ is cyclic for every $C \subset \mathbb{R}^{d+1}$ with non-empty interior. Therefore, $H(C)$ is standard if both C and C' have a non-empty interior.
Then, we may consider the modular operator and the modular conjugation

$$\Delta_C = \Delta_{H(C)}\,, \qquad J_C = J_{H(C)}\,.$$

The following property plays a crucial role:

- For every wedge region $W \subset \mathbb{R}^{d+1}$,

$$\Delta_W^{-is} = U\big(\Lambda_W(2\pi s)\big)\,,\ s \in \mathbb{R}\,,$$

(Bisognano-Wichmann property).

$\Lambda_W =$ boost subgroup of $\mathcal{P}_+^\uparrow$ leavings W globally invariant.

Consequences (cf. D. Guido, R.L.)

H net with the Bisognano-Wichmann property. Then:

(i) The representation U of $\mathcal{P}_+^\uparrow$ is unique;

(ii) For every wedge W,

$$H(W)' = H(W')$$

($Wedge\ duality$);

(iii) If U does not contain the identity representation, then $H(W)$ is factorial for every wedge W, namely $H(W) \cap H(W)' = \{0\}$;

(iv) $\Theta \equiv J_W U(R_W)$ is independent of W, $(0 \in \partial W)$; $\Theta^2 = 1$ and

$$\Theta H(O) = H(-O) \quad \Theta U(g)\Theta = U(rgr)$$

(PCT). Here, R_W is the space π-rotation preserving W and r is the spacetime reflection $r : x \mapsto -x$.

$\implies$ U of $\mathcal{P}_+^\uparrow$ extends canonically to an anti-unitary representation $\tilde{U}$ of $\mathcal{P}_+$ acting covariantly on H.

The dual net

H a local Poincaré covariant net the Bisognano-Wichmann property. The *dual net* H^d is

$$H^d(O) = H(O')' , \quad O \in \mathcal{O} ,$$

$$H^d(O) = \bigcap_{W \supset O} H(W), \qquad W \text{ wedge}$$

By locality, $H(O') \subset H(O)'$, therefore

$$H(O) \subset H^d(O) , \quad O \in \mathcal{O}, \quad H^d(W) = H(W).$$

H^d is local, Poincaré covariant, with BW property and satisfies Haag duality

$$H^d(O)' = H^d(O') , \quad O \in \mathcal{O} .$$

H^d is the maximal extension of H on $\mathcal{H}$ that is relatively local with respect to H

$\mathcal{H}$ complex Hilbert space, $\exists$ a one-to-one correspondence between:

(*a*) *Anti-unitary, positive energy, representations of $\tilde{U}$ of $\mathcal{P}_+$ on $\mathcal{H}$* such that $U = \tilde{U}|_{\mathcal{P}_+^\uparrow}$ does not contain infinite spin subrepresentations;

(*b*) *Poincaré covariant, Haag dual nets H of real linear subspaces on $\mathcal{H}$* with the Bisognano-Wichmann property

Therefore the dual net H^d depends only on the anti-unitary $\tilde{U}$ of $\mathcal{P}_+$ and not on H

$\mathcal{A}$ local, $\mathcal{P}_+^\uparrow$-cov. net of von Neumann algebras, with Haag duality

$$\mathcal{A}(O) = \mathcal{A}(O')' , \quad O \in \mathcal{O} .$$

Set

$$H_{\mathcal{A}}(O) = \overline{\mathcal{A}(O)_{\mathrm{sa}}\Omega} , \quad O \in \mathcal{O}$$

In general (possibly always), $H_{\mathcal{A}}$ is not a dual net, namely

$$H_{\mathcal{A}}(O) \subsetneq H^d_{\mathcal{A}}(O) , \quad O \in \mathcal{O}$$

Universal bound (V. Morinelli, R. L.)

H a local, Poincaré covariant net of real linear subspaces on the complex Hilbert space $\mathcal{H}$, with covariance unitary representation U. Given $h \in \mathcal{H}$, we are interested in a bound for the *entropy of h with respect the region $C \subset \mathbb{R}^{d+1}$ relative to H* defined by

$$S_H(h\|C) \equiv S\big(h\|H(C)\big)\,.$$

Given an anti-unitary, positive energy representation V of $\mathcal{P}_+$, we then define *the entropy of φ with respect to C associated with V* as

$$S_V(\varphi\|C) \equiv S_K(\varphi\|C)\,,$$

where K is the local net of real linear spaces associated with V

With U the covariance unitary representation of $\mathcal{P}_+^\uparrow$ of H, let $\tilde{U}$ be the canonical extension of U to an anti-unitary rep. of $\mathcal{P}_+$.
For every region $C \subset \mathbb{R}^{d+1}$ and vector $\varphi \in \mathcal{H}$, the bound

$$S_H(h\|C) \leq S_{\tilde{U}}(h\|C)$$

holds and depends only on $\tilde{U}$, not on H.

Möbius covariant net

Let H be a local, Möbius covariant net of closed real linear subspaces on the complex Hilbert space $\mathcal{H}$ on $\mathbb{R}$. Let H^d be the dual net, and $\tilde{U}$ the covariance unitary representation of Möb associated with H^d.

For every interval $I \in \mathcal{I}_0$ of the real line, the bound

$$S_H(\varphi \| I) \leq S_{\tilde{U}}(\varphi \| I)$$

holds and depends only on $\tilde{U}$, not on H.

$\tilde{U}$ is quasi-equivalent to the positive energy unitary representation of Möb with lowest weight 1.

Nets associated with the $U(1)$-current and its derivatives

$H_{(1)}$ the (one-particle) $U(1)$ current j net on $\mathbb{R}$

$$[j(x_1), j(x_2)] = i\delta'(x_1 - x_2)$$

$C_0^\infty(\mathbb{R})$ densely embeds in the Hilbert space $\mathcal{H}_{(k)}$.

$H_{(k)}$ the net on $\mathbb{R}$ of standard subspaces associated with the k-derivative of j.

$H_{(k)}$ is the restriction to $\mathbb{R}$ of the net on S^1 associated with the unitary rep. of Möb with lowest weight k.

$$H_{(k)}(I) \subset H_{(1)}(I)$$

The dual net of $H_{(k)}$ is $H_{(1)}$

$$H^d_{(k)}(I) = H_{(1)}(I)$$

Modular hamiltonian and entropy, $U(1)$-case

$B = (-1,1)$. The modular Hamiltonian associated with $H^{(1)}(B)$ on $\mathcal{H}^{(1)}$ is given by

$$\iota_1 \log \Delta_{H^{(1)}(B)} f = \pi(1-x^2)f' \,, \;\; f \in C_0^\infty(\mathbb{R})\,;$$

$C_0^\infty(\mathbb{R})$ is a core for $\log \Delta_{H^{(1)}(B)}$.

The entropy of $f \in C_0^\infty(\mathbb{R})$ w.r.t. $H^{(1)}(B)$ is

$$S\big(f \| H^{(1)}(B)\big) = \pi \int_B (1-x^2) f'(x)^2 dx \,.$$

Modular hamiltonian and entropy, higher derivative case

Let $f \in C_0^\infty(\mathbb{R})$. Then f belongs to the domain of the modular Hamiltonian $\log \Delta_{H^{(k)}(B)}$ on $\mathcal{H}^{(k)}$ associated with $H^{(k)}(B)$ and

$$\big(\iota_k \log \Delta_{H^{(k)}(B)} f\big)(x) = 2\pi\big(k-1\big)xf(x) + \pi(1-x^2)f'(x)$$

The space $C_0^\infty(\mathbb{R})$ is core for $\log \Delta_{H^{(k)}(B)}$.

The entropy of f w.r.t. $H_{(k)}(B)$ on $\mathcal{H}^{(1)}$ is given by

$$S\big(f\|H_{(k)}(B)\big) = \pi \int_B (1-x^2)f'(x)^2 dx - \pi k(k-1) \int_B f(x)^2 dx\,;$$

if $\int_B x^n f(x)dx = 0$, $n = 0, 1, \ldots, k-2$.

Hence, the expression on the right-hand side is non-negative, and

$$S\big(f\|H_{(k)}(B)\big) \leq S\big(f\|H_{(1)}(B)\big)\,,\ k = 1, 2, \ldots$$

Local entropy of a wave packet (Ciolli, Morsella, Ruzzi, R.L.)

The real linear wave's space $\mathcal{T} = \{\Phi : (\Box + m^2)\Phi = 0\}$ is given in Cauchy data

$$\Phi \leftrightarrow \langle f, g\rangle \in S(\mathbb{R}^d) \times S(\mathbb{R}^d)$$

• The complex structure on $\mathcal{T}$ is then

$$\imath_m = \begin{bmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{bmatrix}, \quad \mu = \sqrt{-\nabla^2 + m^2}$$

• The scalar product on $\mathcal{T}$ is the unique Poincaré covariant one

• Local structure: Waves with Cauchy data supported in region O (causal envelop of a space region B) form a real linear subspace $H(O) \equiv H(B)$.

• The information $S(\Phi \| O)$ carried by the wave Φ in the region O is the entropy $S(\Phi \| H(O))$ of the vector Φ w.r.t. $H(O)$

Entropy density of a wave packet (in progress)

The classical stress-energy tensor gives the energy

$$\langle T_{00}^{(0)} \rangle_\Phi = \frac{1}{2}\big((\partial_0 \Phi)^2 + |\nabla_{\mathbf{x}} \Phi|^2\big)$$

we then have

$$-(\Phi, \log \Delta_B \Phi) = 2\pi \int_{x_0=0} \frac{1-r^2}{2} \langle T_{00}^{(0)} \rangle_\Phi(x)dx + \pi D \int_{x_0=0} \Phi^2 dx$$

Recall: the entropy of a massless wave Φ in the unit ball B is

$$S(\Phi \| B) = 2\pi \int_B \frac{1-r^2}{2} \langle T_{00}^{(0)} \rangle_\Phi(x)dx + \pi D \int_B \Phi^2 dx$$

Local entropy density of a massive wave packet

- Describe the local, massive modular Hamiltonian: old problem.

$$\log \Delta_B = -\pi \imath_m \begin{bmatrix} 0 & M_m \\ L_m & 0 \end{bmatrix}$$

Bostelmann, Cadamuro, Mintz: computer numerical analysis

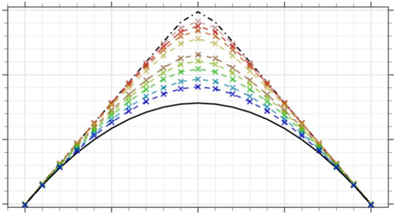


Figure: M_m as m varies

- Get rigorous bound on the local entropy in the massive case

New strategy: use the new notion of entropy operator and compare with the half-space entropy

Entropy bounds. A variational problem

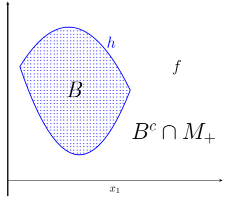
M_+ the *half-space* $x_1 \geq 0$ of $\mathbb{R}^d$, $m \geq 0$, then

$$\log \Delta_{m,M_+} = -\pi \imath_m \begin{bmatrix} 0 & x_1 \\ x_1[(\nabla^2 - m^2) - \partial_{x_1})] & 0 \end{bmatrix}$$

Problem. B a bounded region in M_+ with regular boundary (say B a ball), and h a smooth function on ∂B of B. Set

$$\mathfrak{I}_h \equiv \inf_{f|_{\partial B}=h} \int_{B^c \cap M_+} x_1 (m^2 f^2 + |\nabla f|^2) dx$$

$$f \in S(\mathbb{R}^d)$$



- Estimate $\mathfrak{I}_h$ in terms of h

A preliminary problem is the following:

- *Problem 2.* Is the minimum attained (within some Sobolev space)?

Assuming the answer to Problem 2 to be affirmative (I think this the case), let f_h minimize the functional. Then ($m = 1$)

$$x_1(f_h - \nabla^2 f_h) - \partial_{x_1} f_h = 0\,,$$

so

$$\mathfrak{I}_h = \frac{1}{2}\int_{\partial B} x_1 \partial_{\mathbf{n}}(f_h^2)dS = \int_{\partial B} x_1 h\, \partial_{\mathbf{n}}(f_h)dS$$

(normal derivative) and the problem is to estimate this integral in terms of h.