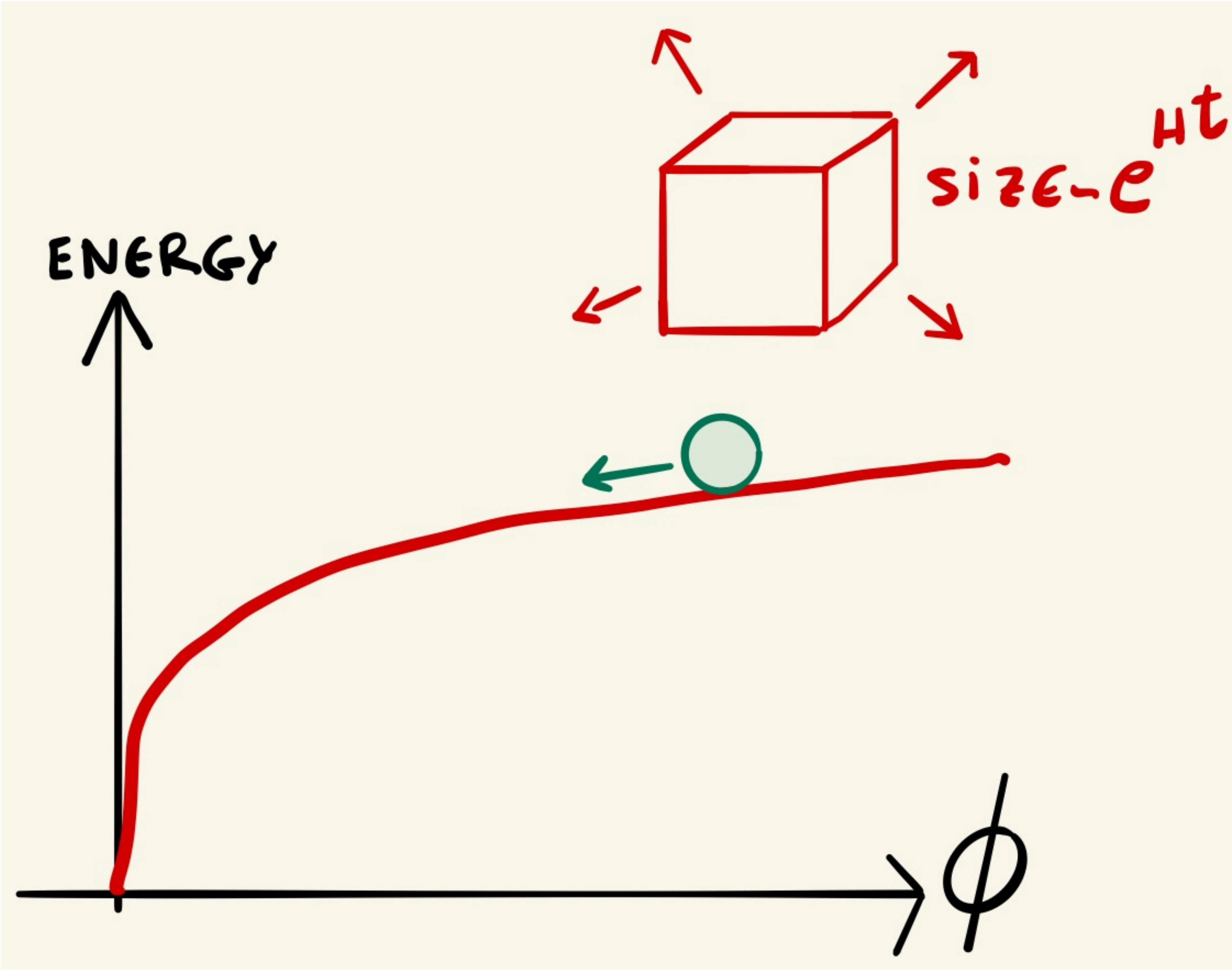
Leonardo Senatore (ETH)

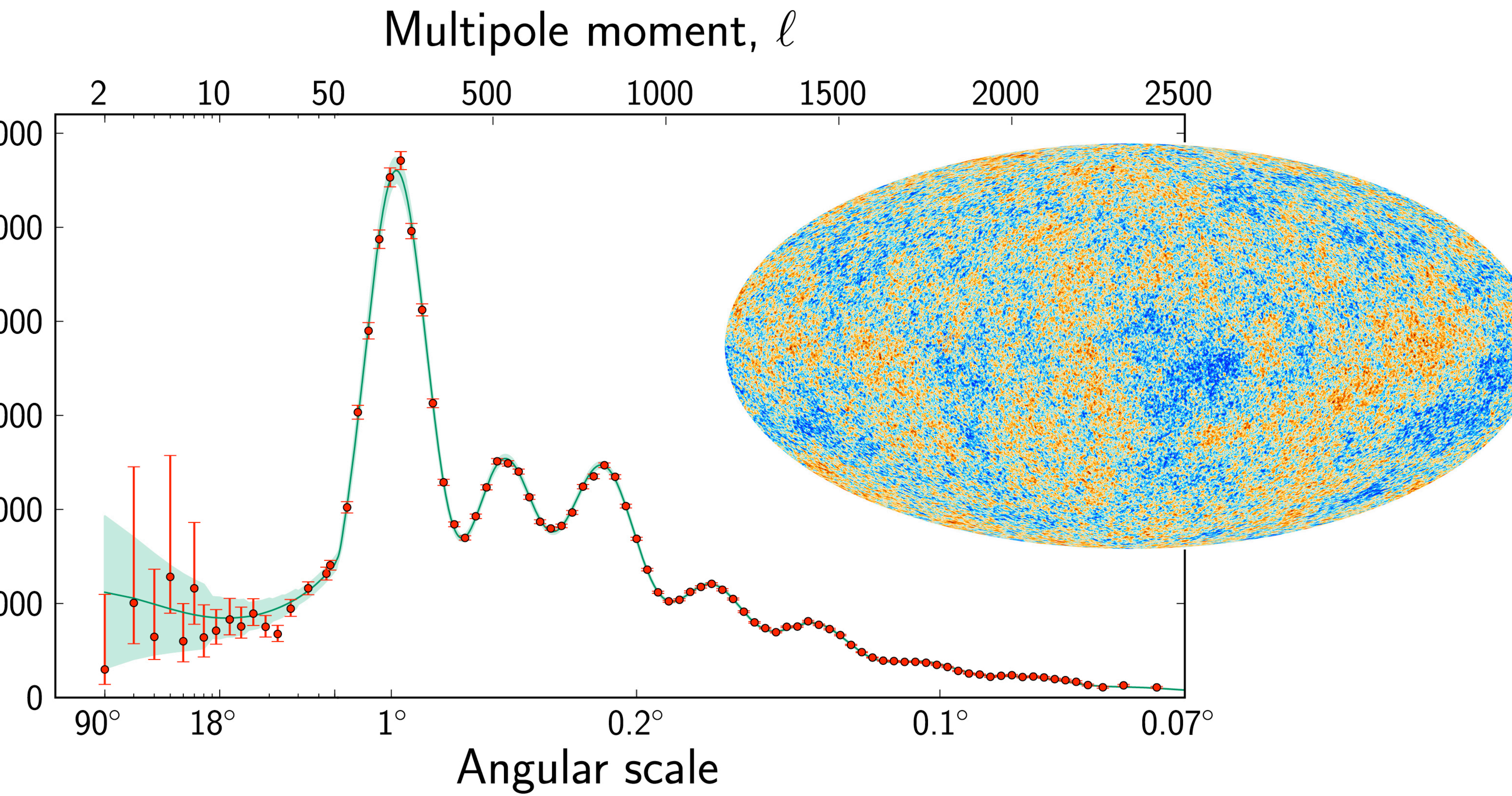
Starting Inflation
in inhomogeneous universe
using a novel connection to Mathematics

with East, Linde and Kleban **JCAP 2016**
with Kleban **JCAP 2016**
with Creminelli, Vasy, **Comm Math Phys 2020**
with Creminelli, Hershkovits, Vasy **Advances in Math. 2023**
with Hershkovits **2307**

Inflation

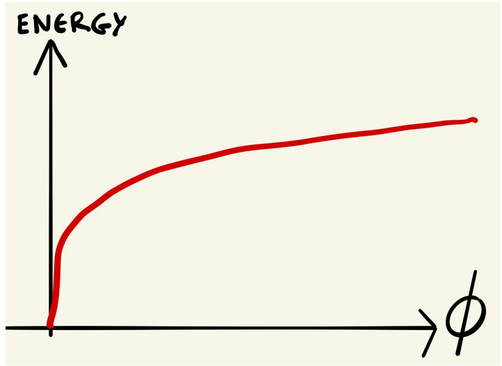


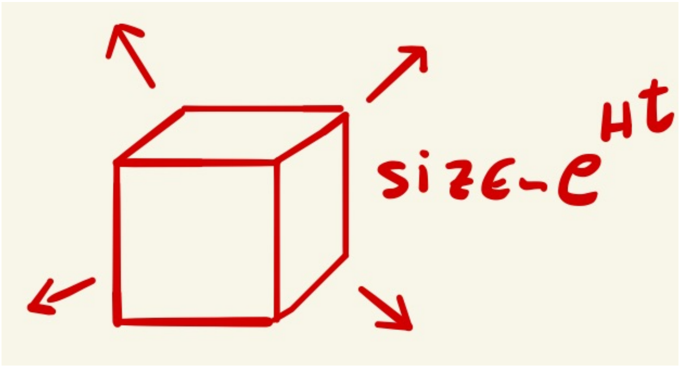
Cosmic Fluctuations



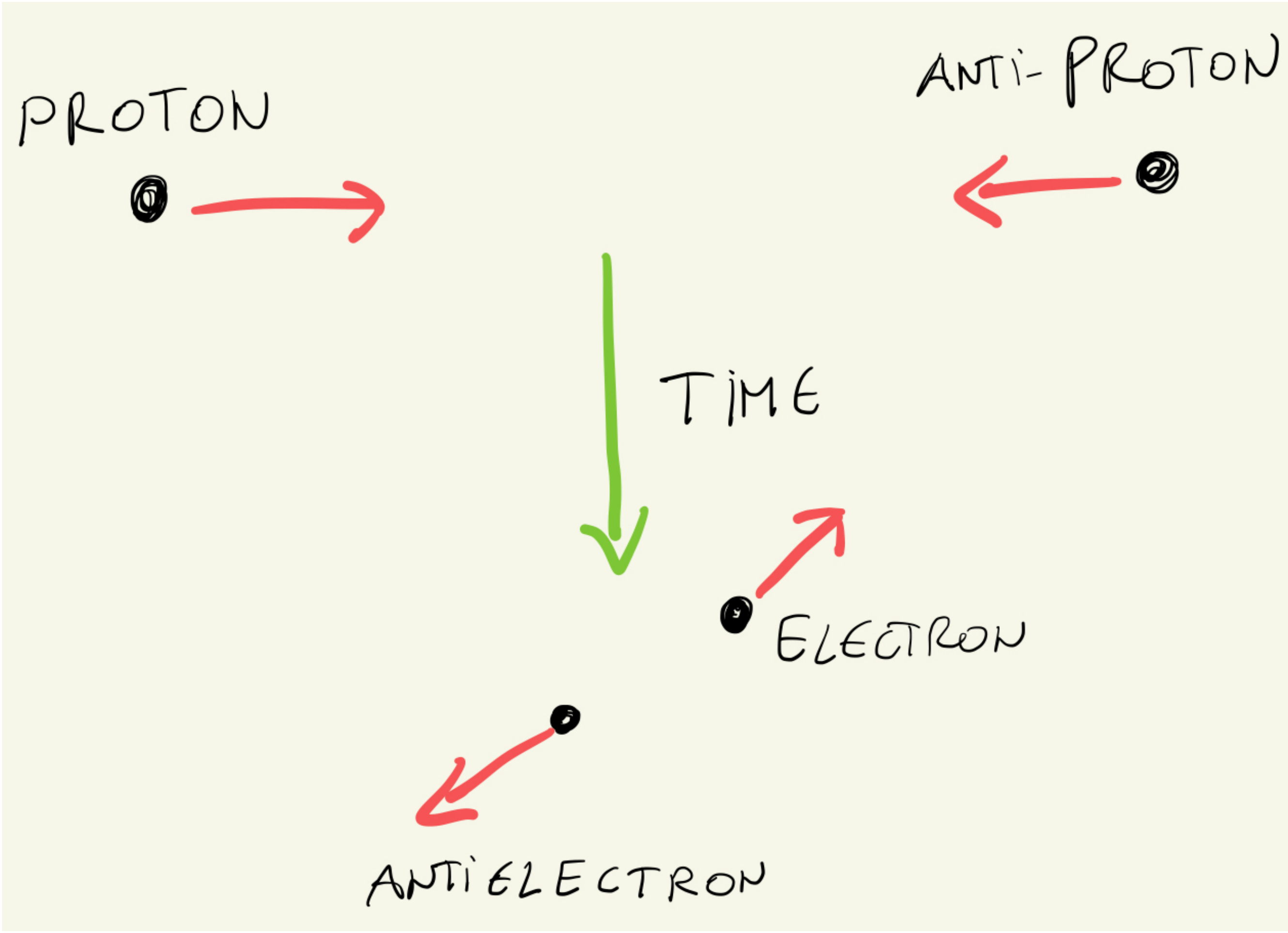
Planck 2018

Starting Inflation





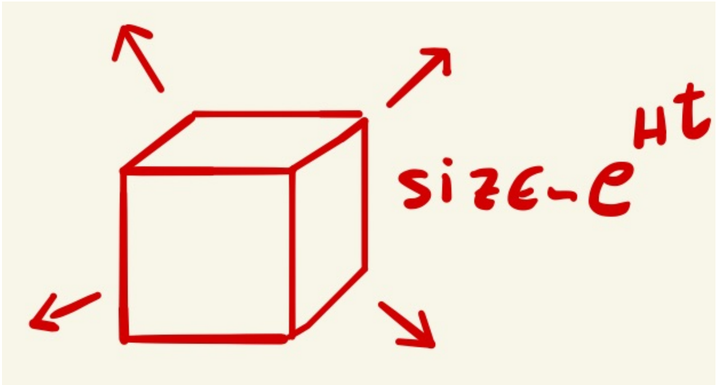
- For long time, unclear how inflation starts.
- Two challenges:
  - Philosophical challenge: unusual in Physics:
    - normally: choose initial state and *predict* evolution

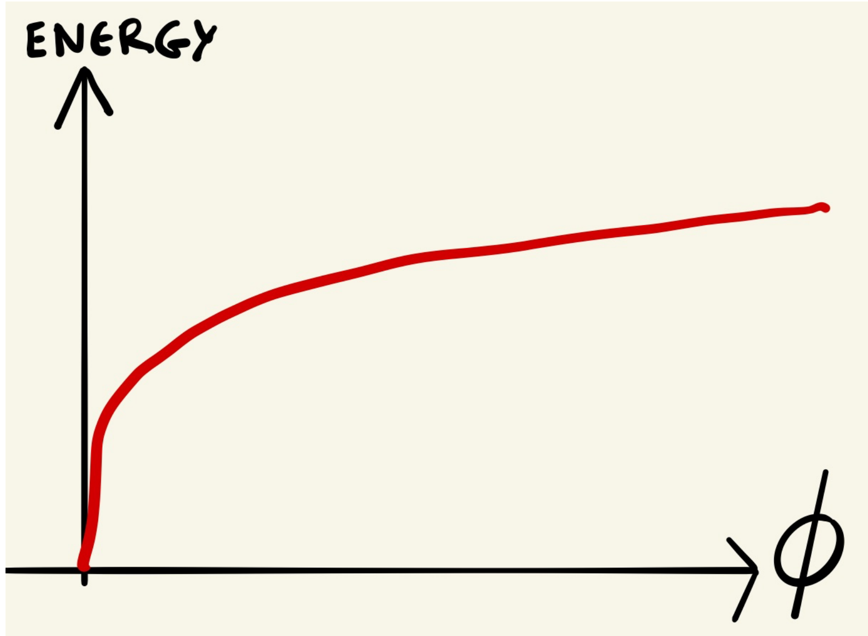


    - here: viability of some initial states
- High inhomogeneity $\Rightarrow$ Lack of Control

Starting Inflation

- High inhomogeneity $\Rightarrow$ Lack of Control

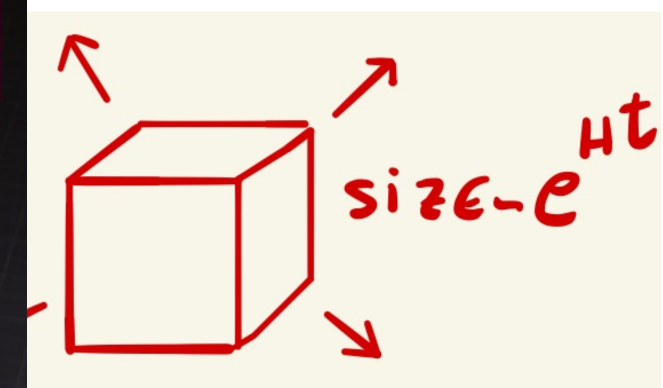


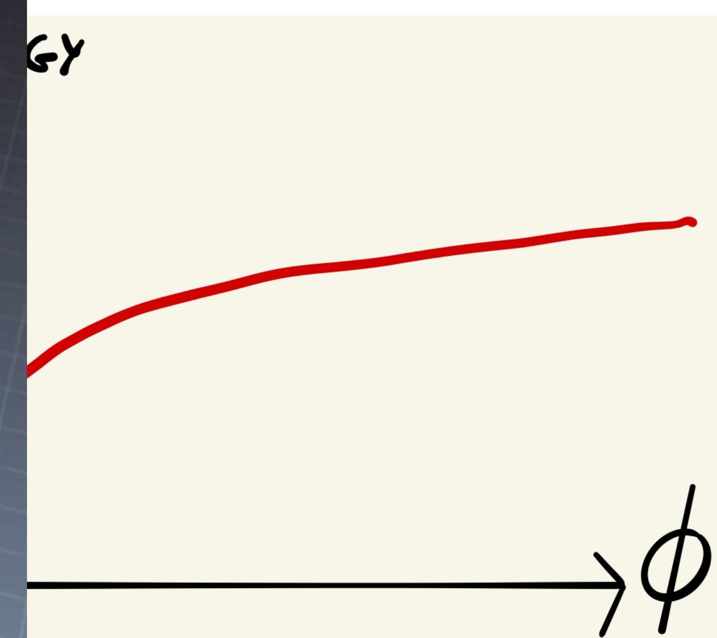


with East, Linde and Kleban **2016**

Starting Inflation

- High inhomogeneity =

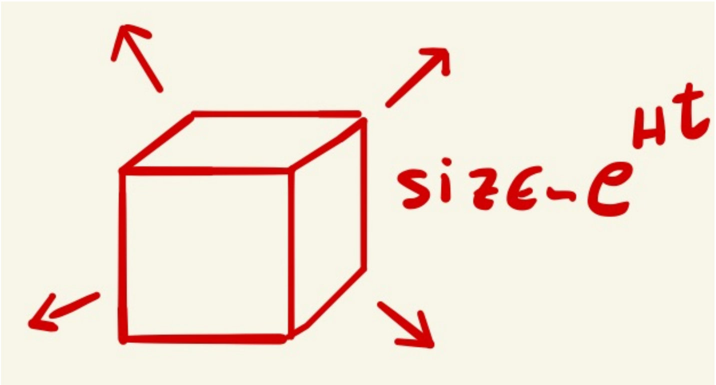


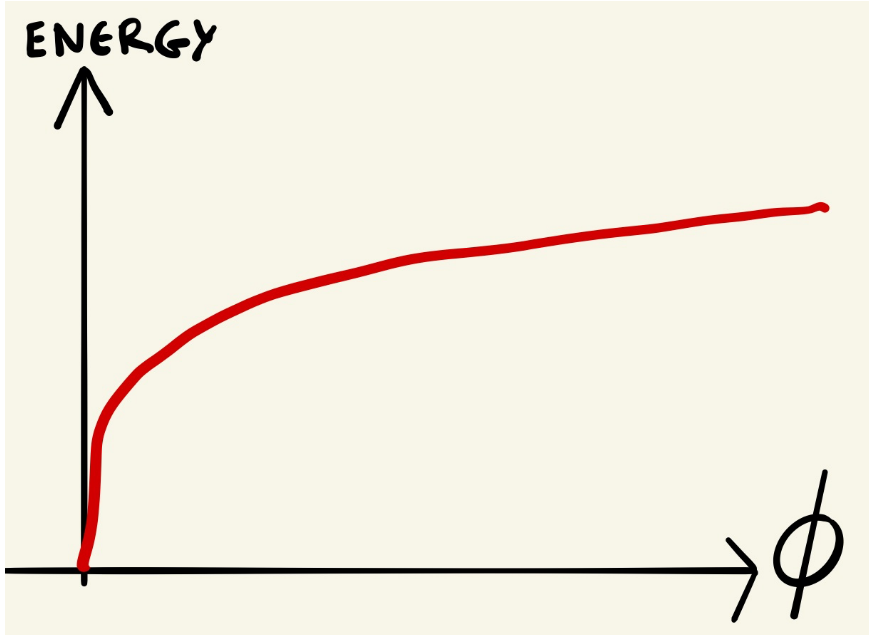


with East, Linde and Kleban **2016**

Starting Inflation

• High inhomogeneity $\Longrightarrow$ Lack of Control

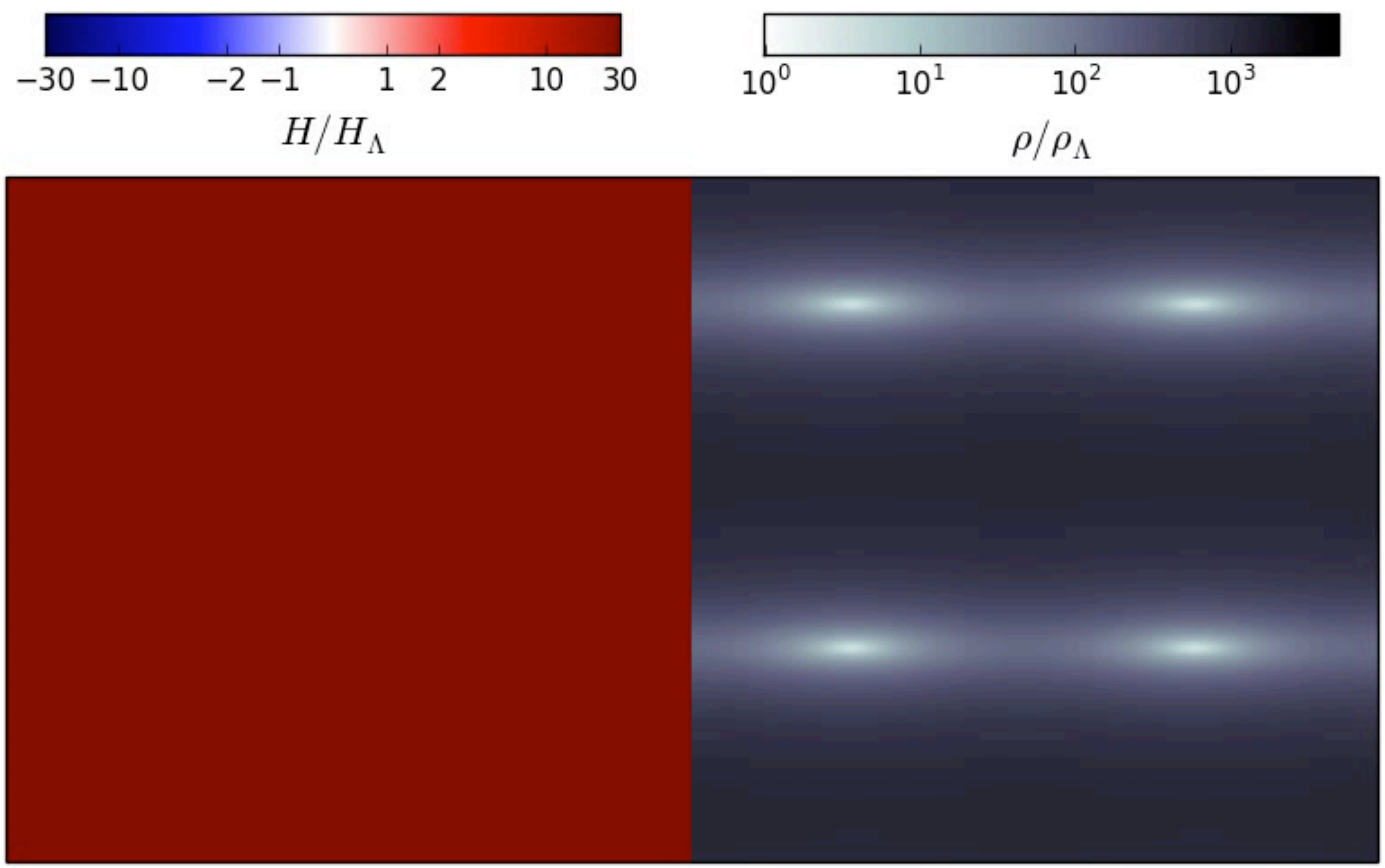


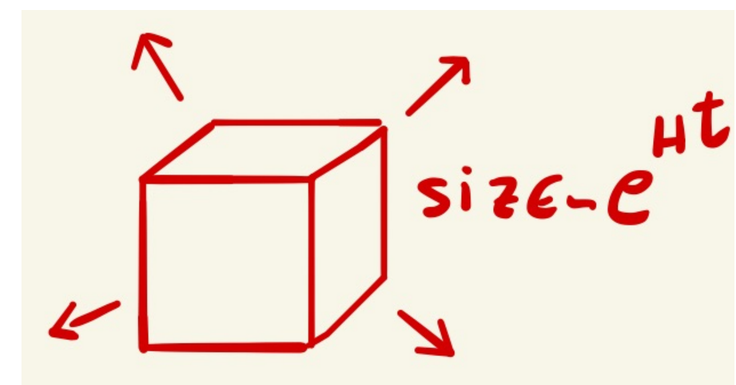


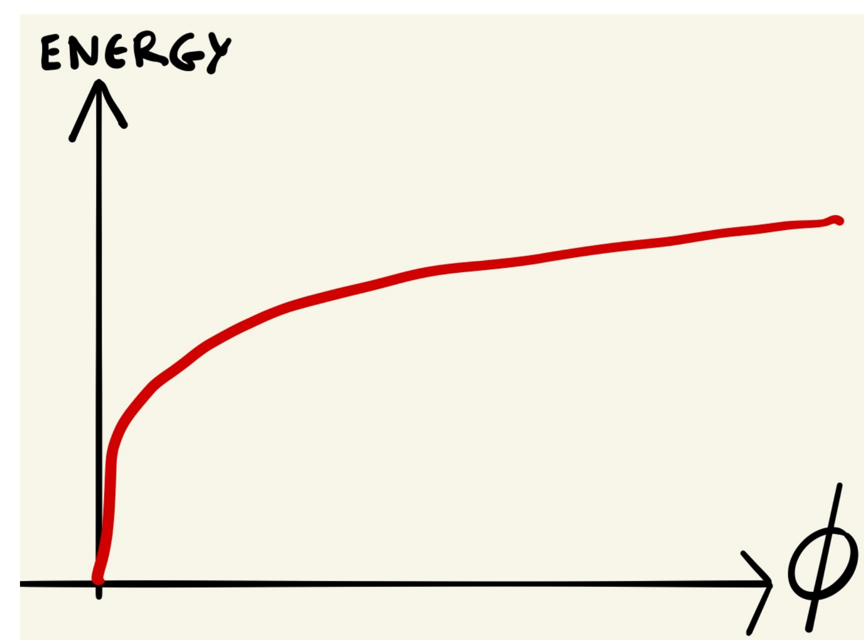
with East, Linde and Kleban **2016**

Starting Inflation

- High inhomogeneity $\Rightarrow$ Lack of Control
- Advanced numerical techniques *(same codes as LIGO)*



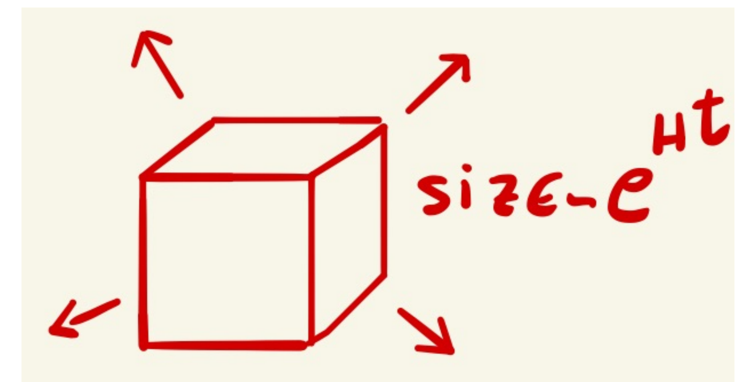


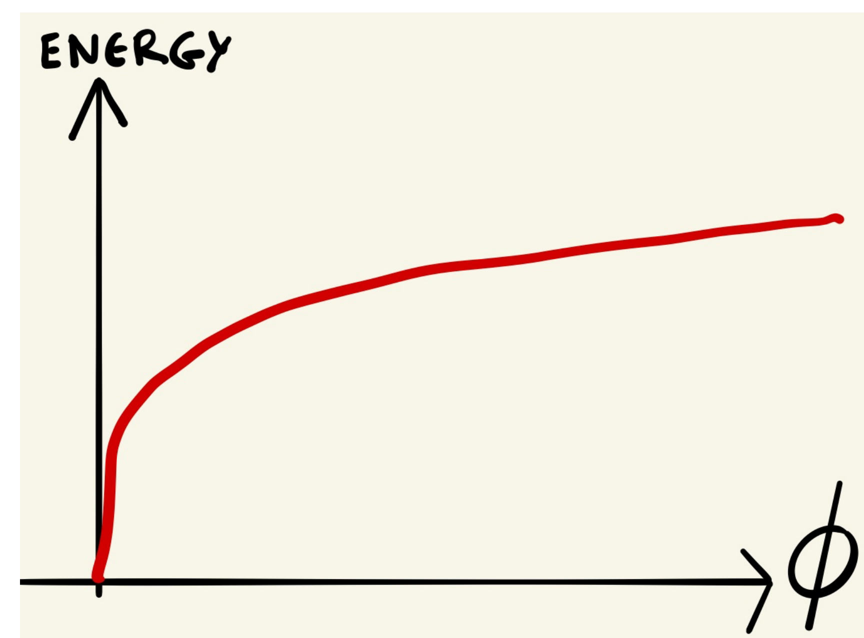


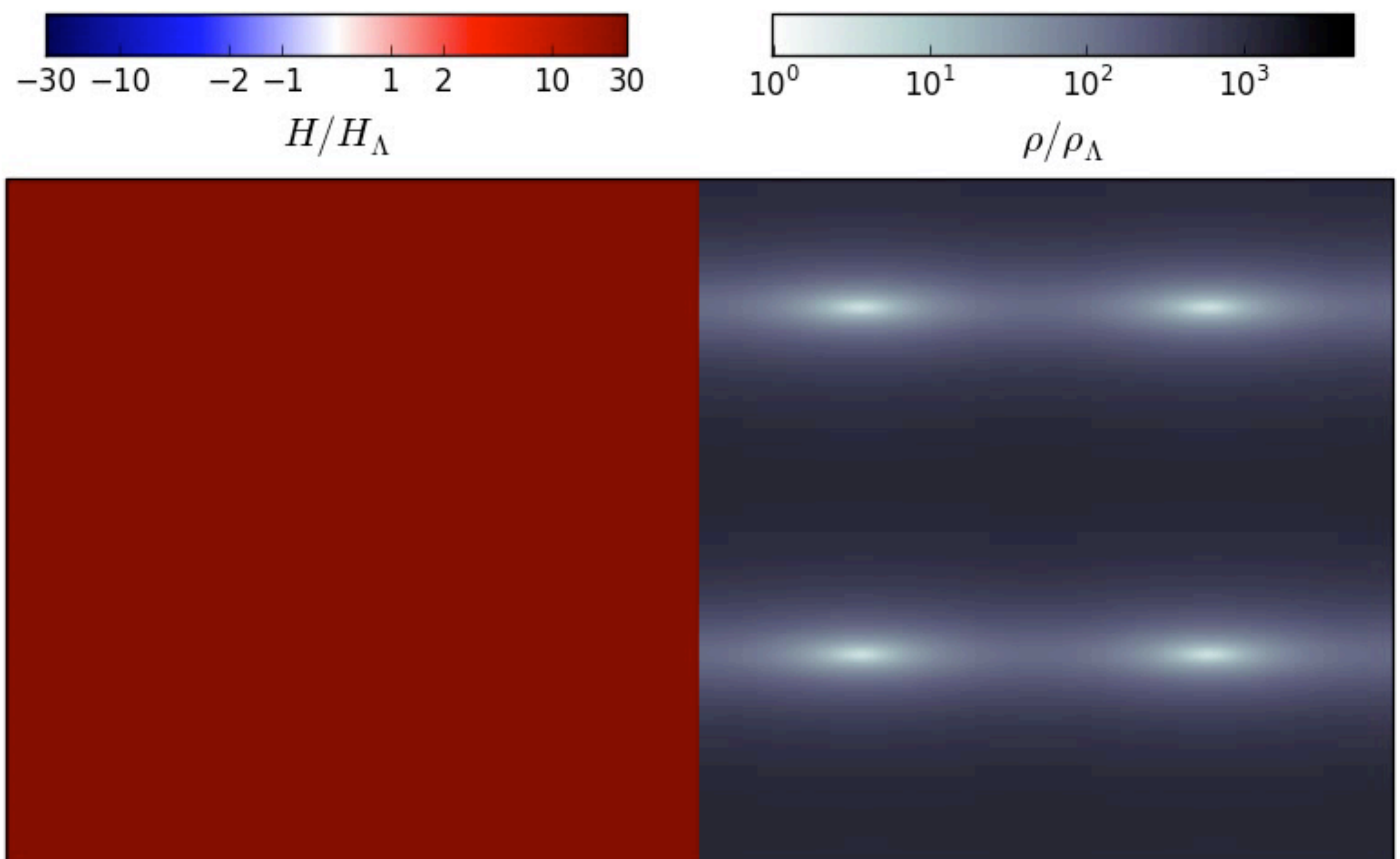
with East, Linde and Kleban **2016**

Starting Inflation

- High inhomogeneity $\Rightarrow$ Lack of Control
- Advanced numerical techniques *(same codes as LIGO)*
- Establish that inflation starts $\mathcal{O}(1)$ times out of inhomogenous initial conditions







with East, Linde and Kleban **2016**

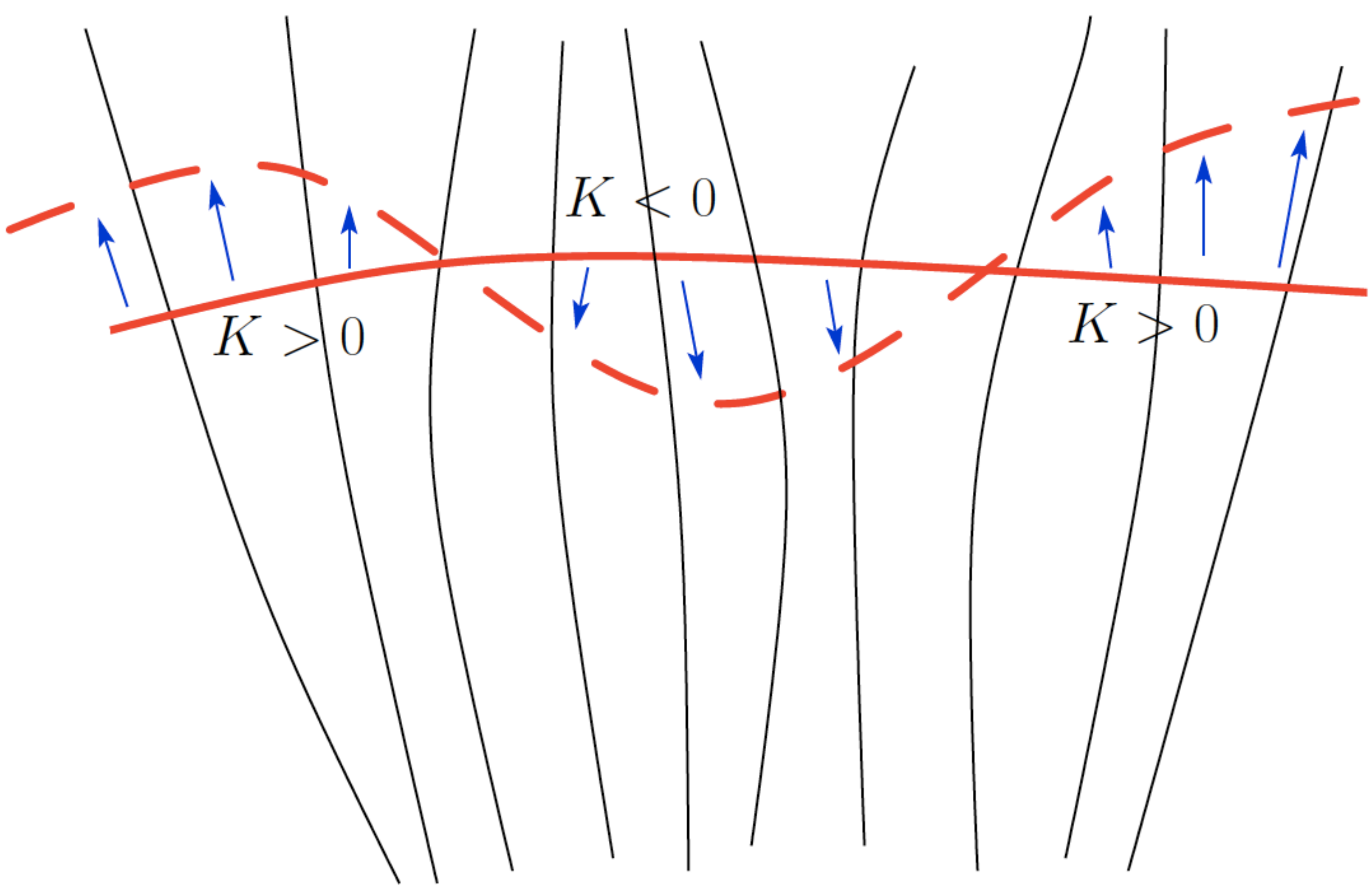
de Sitter no-hair Theorem

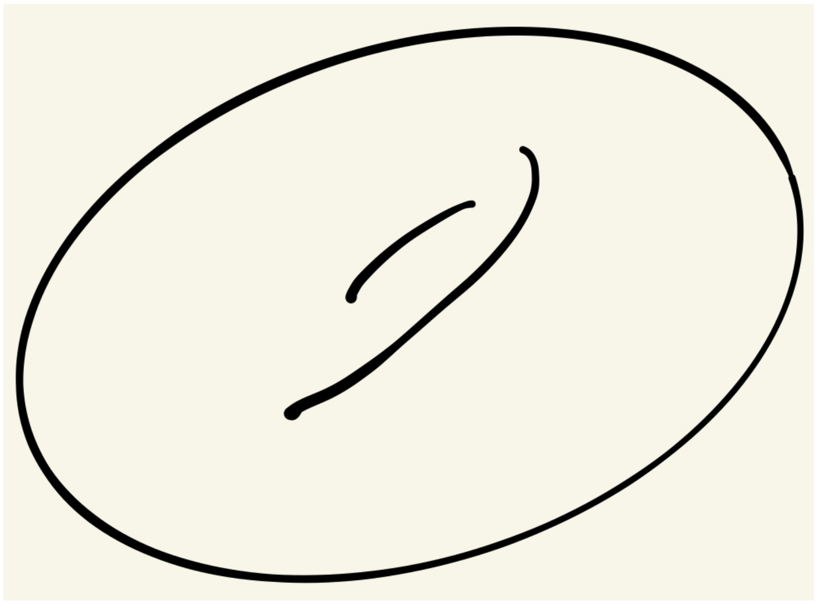
– Conjecture (~Hawking, …, Kleban & I):

all initial expanding universes with $\Lambda > 0$ *and with the right topology will reach de Sitter space*

Starting Inflation: Connections to Math

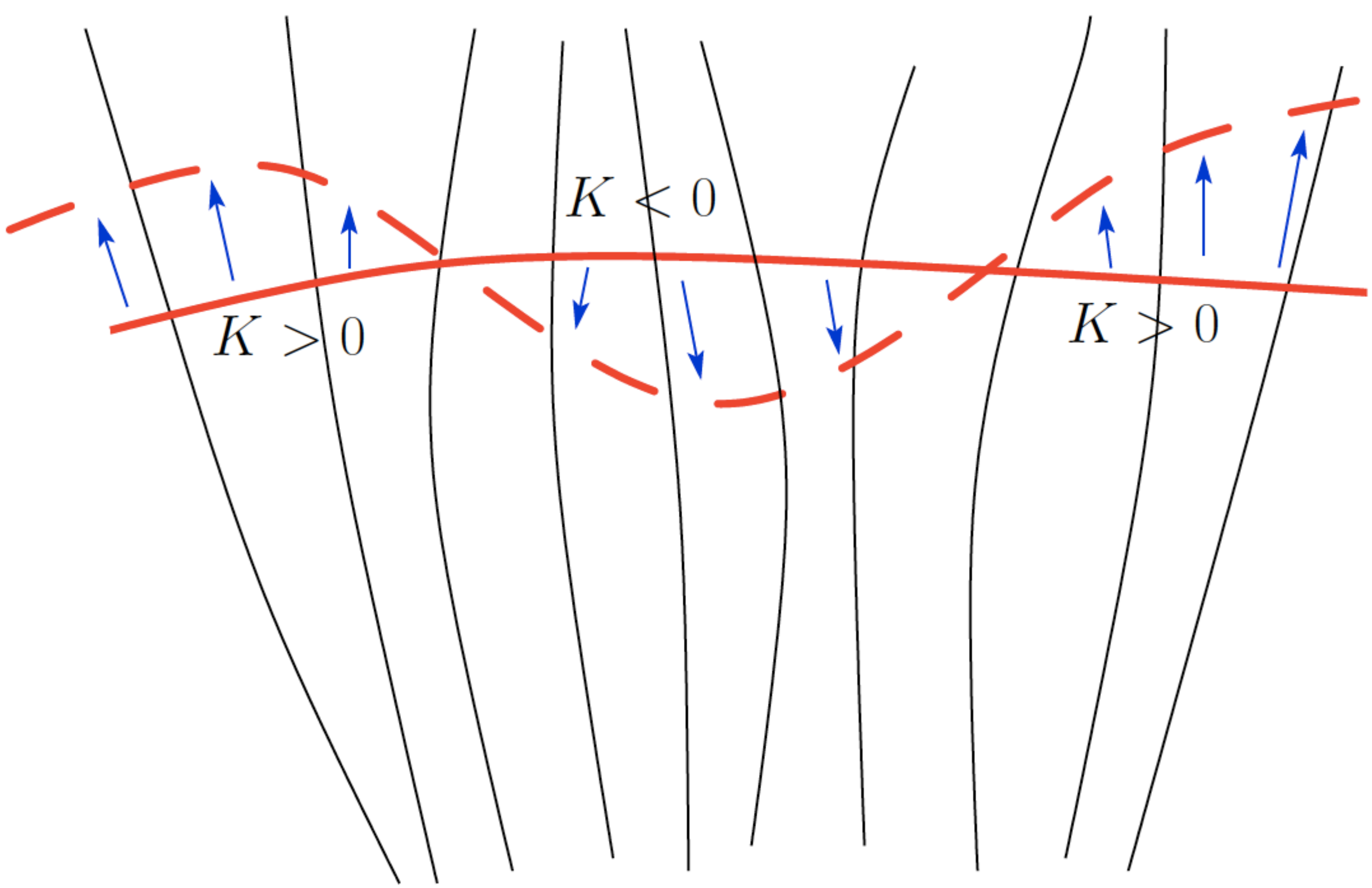
- Handle analytically spacetimes with no symmetries and singularities
    - ~Hawking-Penrose, Christodoulou, Huiskin, …

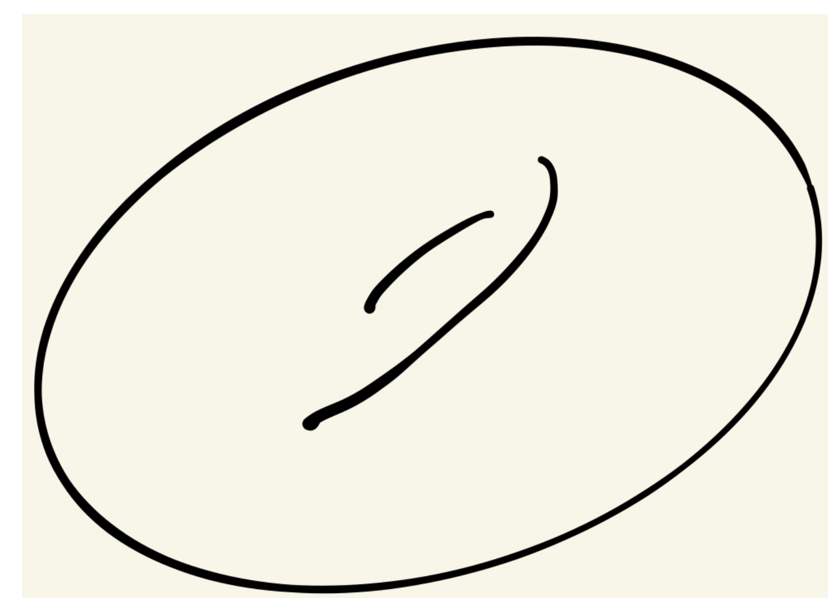




Starting Inflation: Connections to Math

- Handle analytically spacetimes with no symmetries and singularities
    - ~Hawking-Penrose, Christodoulou, Huiskin, …
    - Thurston Geometrization Classification (Poincarè Hypothesis)
  - Mean Curvature Flow





Thurston Geometrization Conjecture

Thurston, Hamilton, Perelman

– All compact oriented 3-manifold fall into one of these three classes

–(i) ``Closed'': $R^{(3)}$ can be anything

- ex: $S^3,\ S^2 \times S^1,\ S^3/\Gamma(\text{with } \Gamma \in SO(4)), RP^3$
  - and connected sums

–(ii) ``Flat'': $R^{(3)}$ must be either negative somewhere or zero everywhere

- ex: $R^3/\Gamma(\text{ with } \Gamma \text{ an isometryof } R^3)$
  - and connected sums

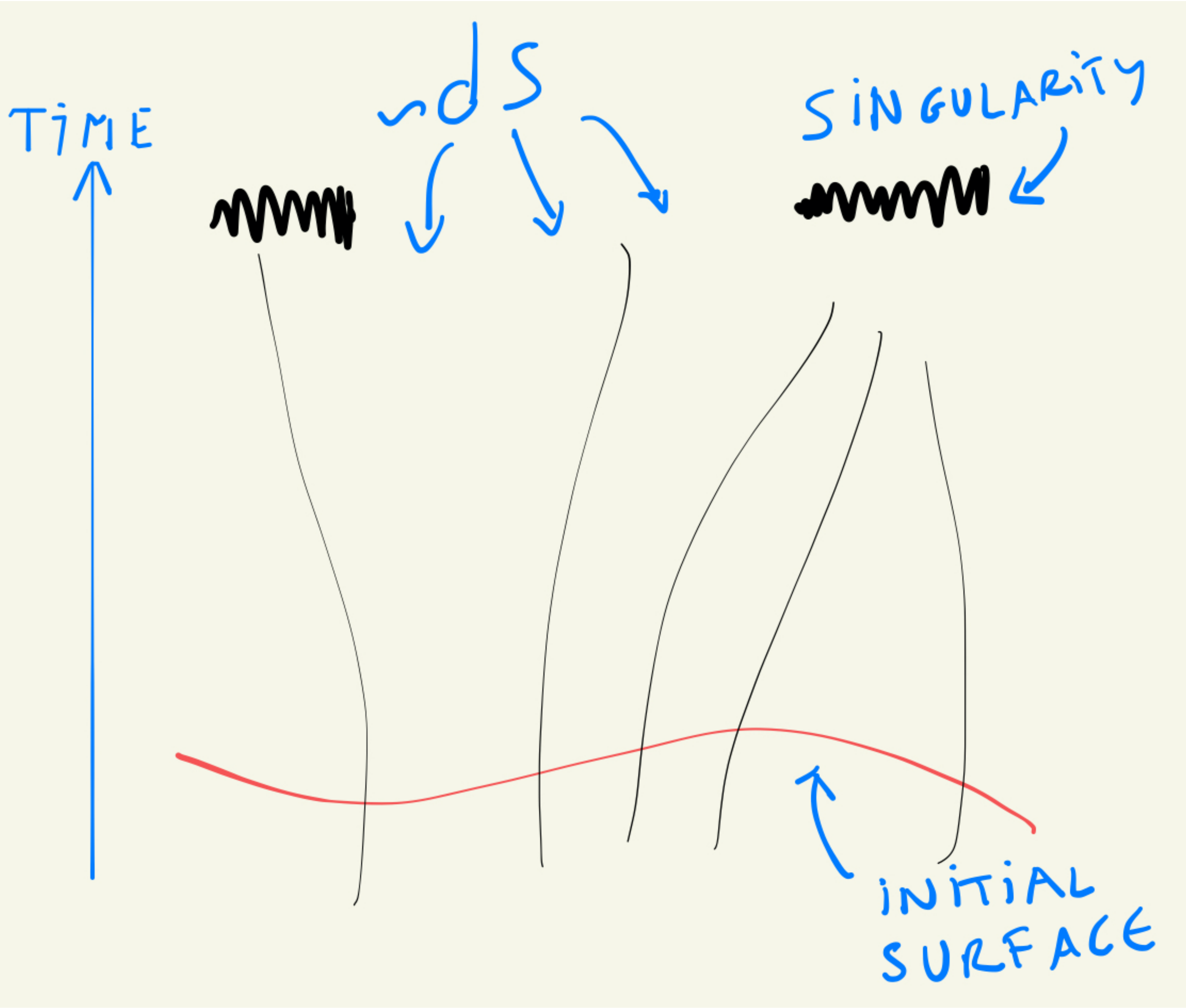
–(iii) ``Open'': $R^{(3)}$ must be negative somewhere

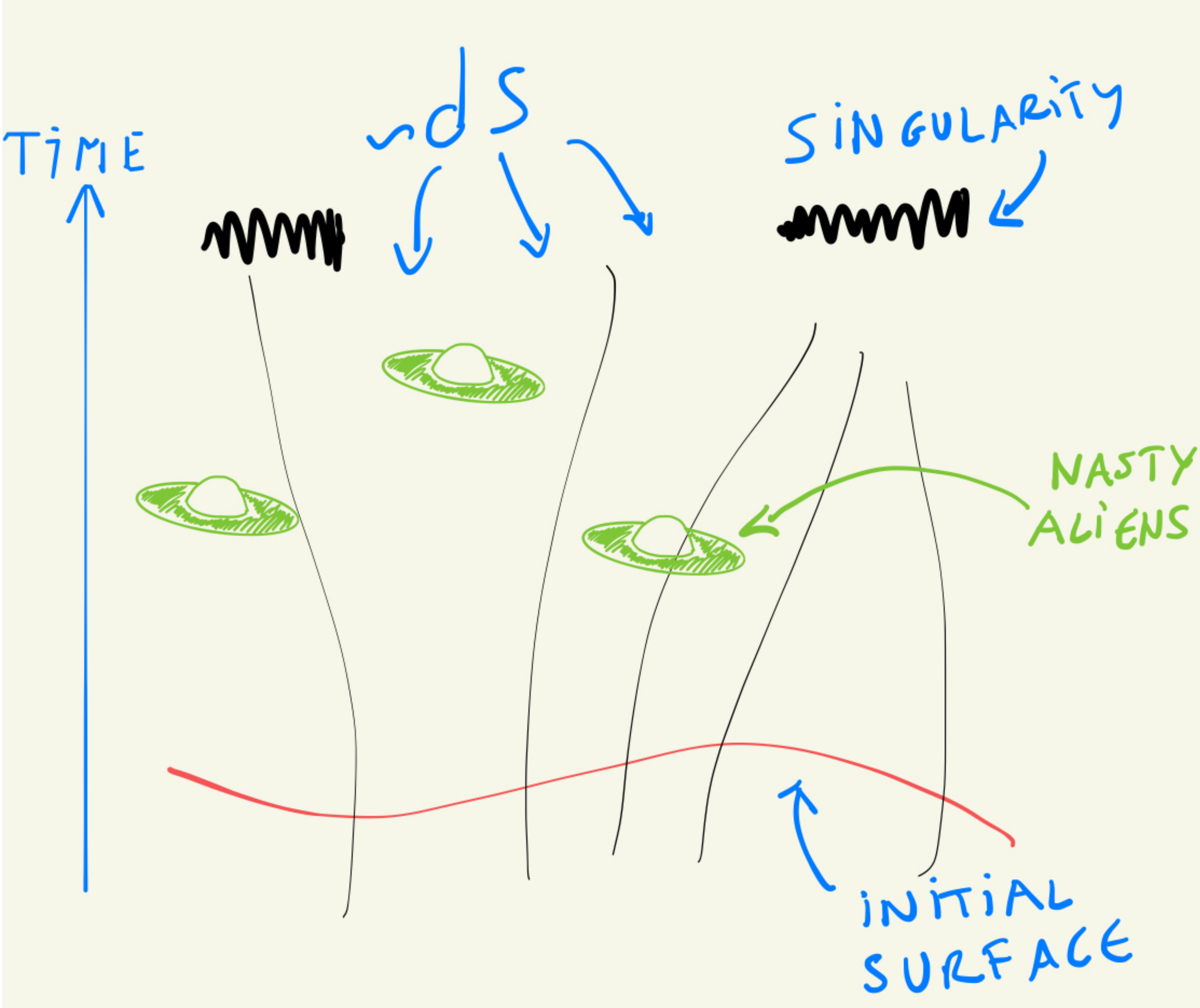
- ex: $H^3/\Gamma,\ H^2 \times R, nil, sol, \widetilde{SL}(2, R)$

–Any connected sum of (i) and (ii) with a factor of (iii) is of kind (iii)

Personal comments on the connection to Math

- GR/Diff. Geometry is very active field of Mathematics
  - often: deal with stability of spacetimes: so, they know what they have at hand
  - often: focus on the fact that bad things must happen
  - sometimes: more interested in geometry conclusions than physics conclusions
- In Cosmology: almost always deal with small fluctuations: we know the asymptotic regime of the universe, up to small corrections.
  - Exceptions: *eternal inflation* and *prior to inflation*
- Here we are dealing with something different:
  - the spacetime is quite unknown, and we wish to explore it
    - to answer pressing physics question


TIME
dS
SINGULARITY
INITIAL SURFACE


TIME
AdS
SINGULARITY
NASTY ALIENS
INITIAL SURFACE

Homogenous Cosmology

– Already Wald (1983) had shown that if the DEC is preserved, all homogeneous but inosotropic universe (Bianchi universes) that are not `closed' (that is non-Bianchi-Type-IX universes) and $\Lambda > 0$, asymptote to de Sitter.

– DEC: $-T^{\mu}{}_{\nu}k^{\nu}$ is future-directed timelike or null for any timelike k^{μ}

– WEC: $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$ (i.e. "$\rho \geq 0, \rho + p > 0$") , for any t^{μ} timelike

– SEC: $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)k^{\mu}k^{\nu} \geq 0$

– But inhomogeneities are more challenging.

  – diff. equations become partial diff., and singularities form, geodesic cross, etc.. It is a hugely less symmetric situation.

    • we will however see that a sort of similar conclusion holds

• Let us therefore consider general `cosmologies'.

A Cosmology

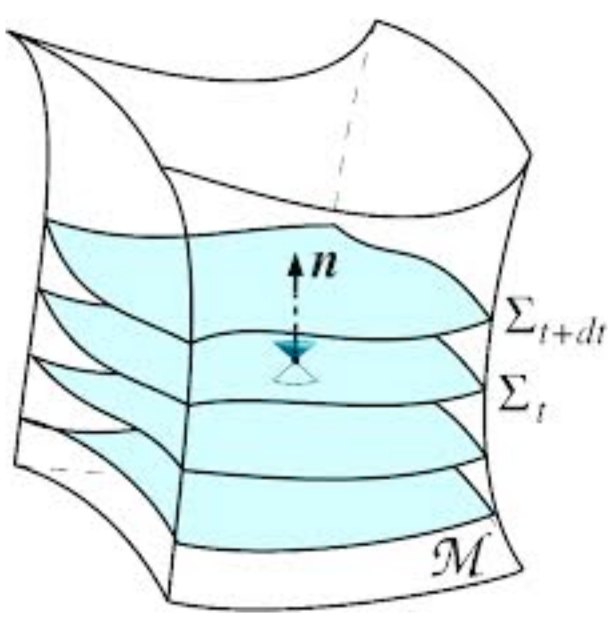
–*First Assumption*: we consider a cosmology:

– a connected 3+1 dimensional spacetime with a compact Cauchy surface

- This implies (Geroch 1970):

–the spacetime is topologically $R \times M$ where M is a 3-manifold

–it can be foliated by a family of topologically identical Cauchy surfaces M_t



Hypothesis

– We will prove a theorem under the following assumptions:

– *First Assumption:* A cosmology $M^{(3+1)}$

– *Second Assumption:* Matter satisfies Dominant and Strong energy condition and there is also $\Lambda > 0$

– *Third Assumption:* The spatial topology of M_t must not be `closed', i.e. it must not be of type (i) that we defined earlier (roughly, M_t must not have topology of sphere)

2+1 dimensions

Proved

– de Sitter no hair theorem proved in with Creminelli, Vasy, **Comm Math Phys 2020**

– Here the Gauss-Bonnet theorem and the fact that the Riemann is known in terms of the Ricci played a hugely simplifying role.

– Though, quite non trivial statement.

3+1 dimensions

Anisotropic Inhomogenous Cosmology

with Kleban **JCAP 2016**

Theorem

with Kleban **JCAP2016**

There cannot exist a non-singular spacelike hypersurface with maximum volume: given any time slice, there is another with larger spatial volume. Furthermore, in an initially expanding universe there must be at least one expanding region on every timeslice, and if $\Lambda > 0$ the expansion rate in that region is bounded from below by that of de Sitter spacetime in the flat slicing.

For the first sentence, see also Barrow and Tippler **1985**

Theorem

with Kleban **JCAP2016**

– This implies that in a big bang cosmology, there cannot be a big crunch

  – *very strongly* suggesting cosmology reaches infinite volume, gradient energy will dilute, and inflation will start, no matter initial inhomogeneities and scale of inflation.

There cannot exist a non-singular spacelike hypersurface with maximum volume: given any time slice, there is another with larger spatial volume. Furthermore, in an initially expanding universe there must be at least one expanding region on every timeslice, and if $\Lambda > 0$ *the expansion rate in that region is bounded from below by that of de Sitter spacetime in the flat slicing.*

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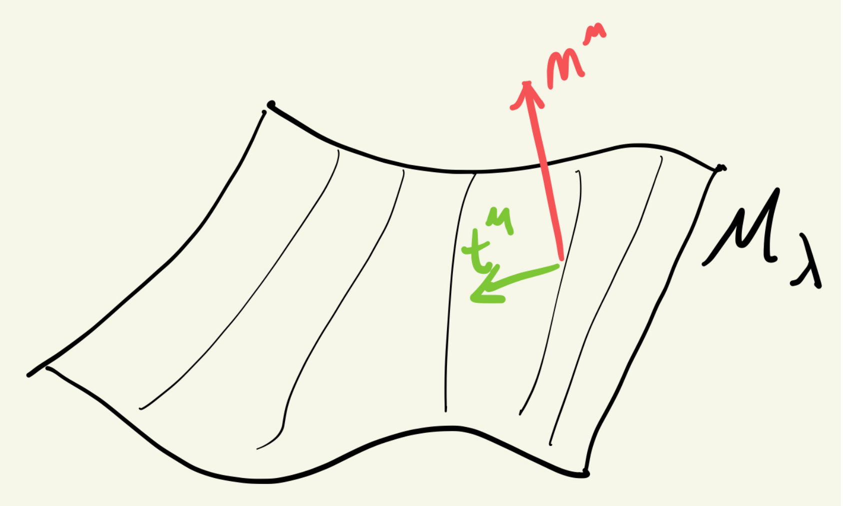
Notation



– n_μ is the orthonormal vector to M_t : $n_\mu n^\mu = -1$

– Spatial metric $g_{\mu\nu}$: $g^{(4)}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$

– Extrinsic curvature $K_{\mu\nu} := g_\mu{}^\alpha \nabla_\alpha n_\nu \equiv \frac{1}{3} K g_{\mu\nu} + \sigma_{\mu\nu}$

–how much the family of geodesics induced by n_μ deviates

–Notice $\mathcal{L}_n \log\sqrt{h} = K,$: rate of growth of volume

$$\Rightarrow \quad \sqrt{h} \sim \sqrt{h_0}\, e^{Kt}$$

Proof

– Consider (Λ reabsorbed in stress tensor)

$$n^\mu n^\nu \; 8\pi G \; T_{\mu\nu} = G_{\mu\nu} n^\mu n^\nu$$

– From Gauss-Codazzi

$$n^\mu n^\nu G_{\mu\nu} = \frac{1}{2}\left\{ R^{(3)} + \left(K^\mu_\mu\right)^2 - K_{\mu\nu}K^{\mu\nu} \right\} = \; 3 - \text{surface quantities}$$

– $\Longrightarrow$ we have

$$16\pi G_N T_{\mu\nu} n^\mu n^\nu = R^{(3)} + \frac{2}{3}K^2 - \sigma_{\mu\nu}\sigma^{\mu\nu}$$

– If a surface has extremal volume, the volume is stationary wrt any variations. Since

$$\mathcal{L}_n \log \sqrt{h} = K, \quad \Rightarrow \quad K = 0 \quad \text{everywhere}$$

– Then, if there exist an extremal surface, on that surface we must have,

$$16\pi G_N \, T_{\mu\nu} n^\mu n^\nu = R^{(3)} - \sigma^{\mu\nu}\sigma_{\mu\nu}$$

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– Then, if there exist an extremal surface, on that surface we must have,

$$\underbrace{16\pi G_N \, T_{\mu\nu} n^\mu n^\nu}_{\geq 0 \text{ by WEC}} = R^{(3)} \underbrace{-\sigma^{\mu\nu}\sigma_{\mu\nu}}_{\leq 0}$$

A no Big-Crunch theorem

- Impose Weak Energy Condition

$$\underbrace{16\pi G_N T_{\mu\nu} n^\mu n^\nu}_{\geq 0 \text{ by WEC}} = R^{(3)} \underbrace{- \sigma^{\mu\nu}\sigma_{\mu\nu}}_{\leq 0}$$

$T_{\mu\nu} t^\mu t^\nu \geq 0$ (i.e.“$\rho \geq 0, \rho + p > 0$”) , for any t^μ timelike

- If there is a topological condition such that $R^{(3)} \leq 0$ at least at one point
- $\Rightarrow$ equation cannot be satisfied, $\Rightarrow$ extremal surface does not exists
- The by-now-proved Poincarè Hypothesis, indeed shows that “most” of 3-manifold must have $R^{(3)} \leq 0$ at least at one point
  - for these topologies, some regions of the universe keep expanding, notwithstanding the development of singularities
  - therefore they likely reach infinite volume, energy dilutes, and inflation start

A de Sitter no-hair Theorem
for Cosmologies
with isometry group forming 2-dimensional orbits

with Creminelli, Hershkovits, Vasy **Advances in Math. 2023**

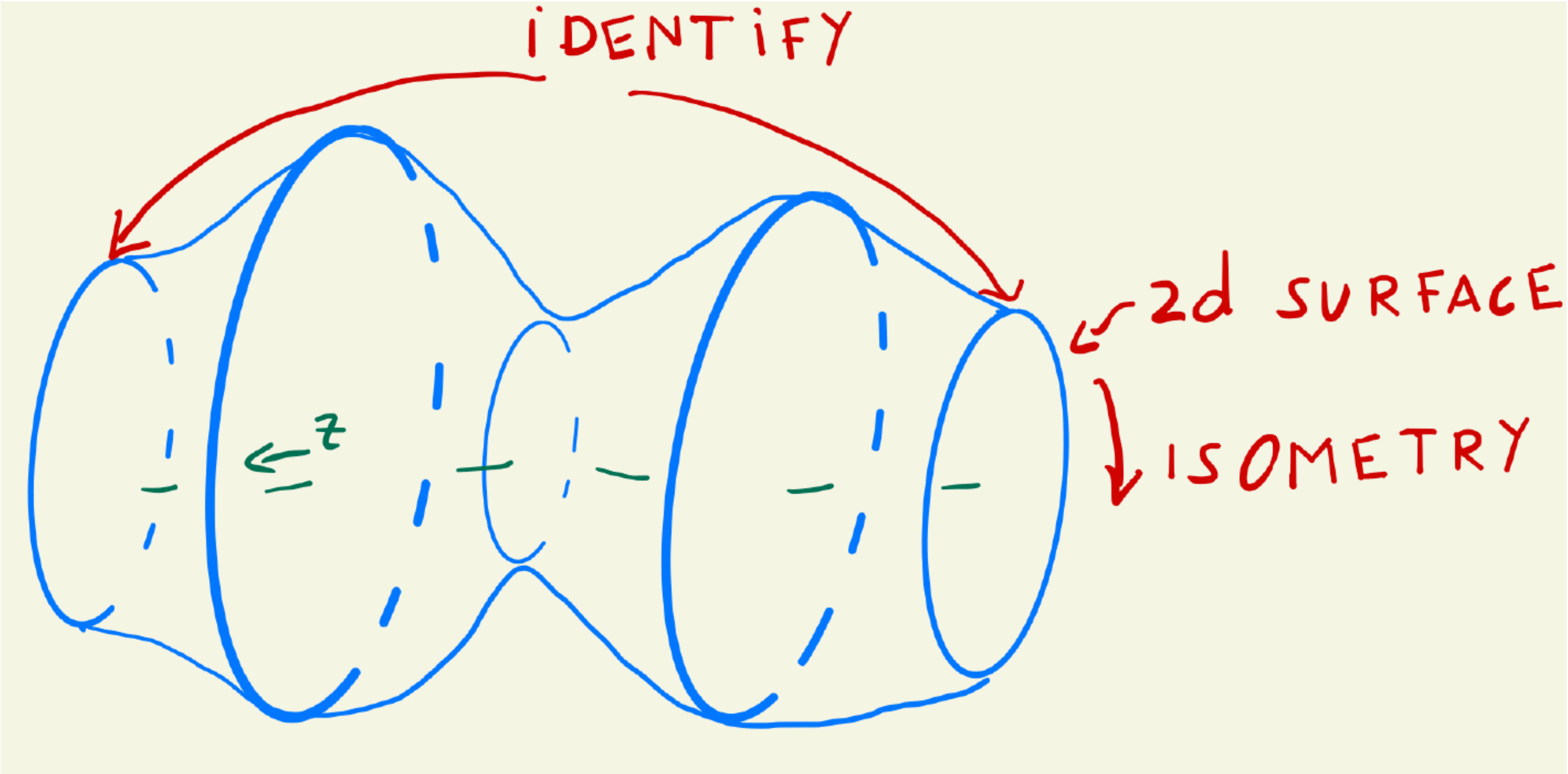
Hypothesis

–*Fourth Assumption:* there is a group G which acts on $M^{(3)}$ and such that the induced action on $M^{(3+1)}$ is by isometries, and such that the orbits under G are closed surfaces.

–In reality, G acts on the universal cover of $M^{(3)}$

–*Homogenous anisotropic surfaces*

Hypothesis



–Examples: $\mathbb{T}^3 = S^1 \times S^1 \times S^1$

$S^2 \times S^1$

$(\mathbb{H}^2/\Gamma) \times S^1$

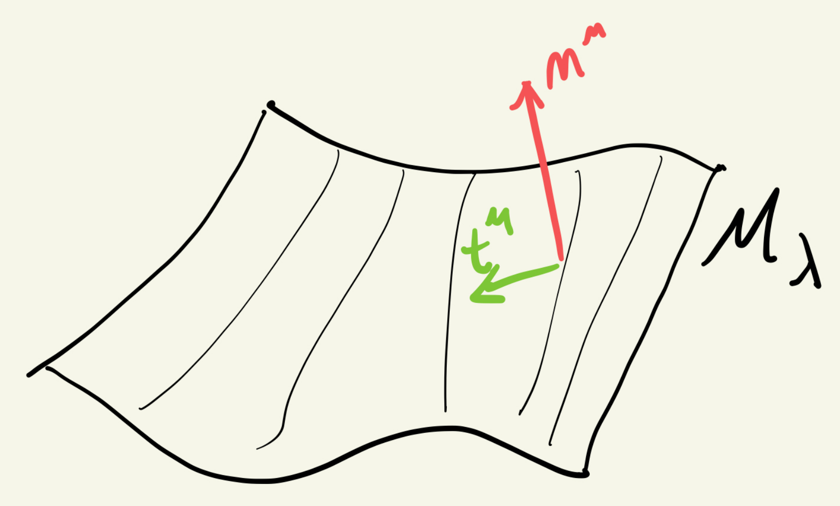
$\ldots$

Theorem

If the 2-surfaces have non-positive Euler characteristic (or in the case of 2-spheres, if the initial 2-spheres are large enough) and also if the initial spatial slice is expanding everywhere, then, asymptotically, the spacetime becomes physically indistinguishable from de Sitter space on arbitrarily large regions of spacetime. This holds true notwithstanding the presence of initial arbitrarily-large density fluctuations and potential singularities.

with Creminelli, Hershkovits, Vasy **Advances in Math. 2023**

Notation



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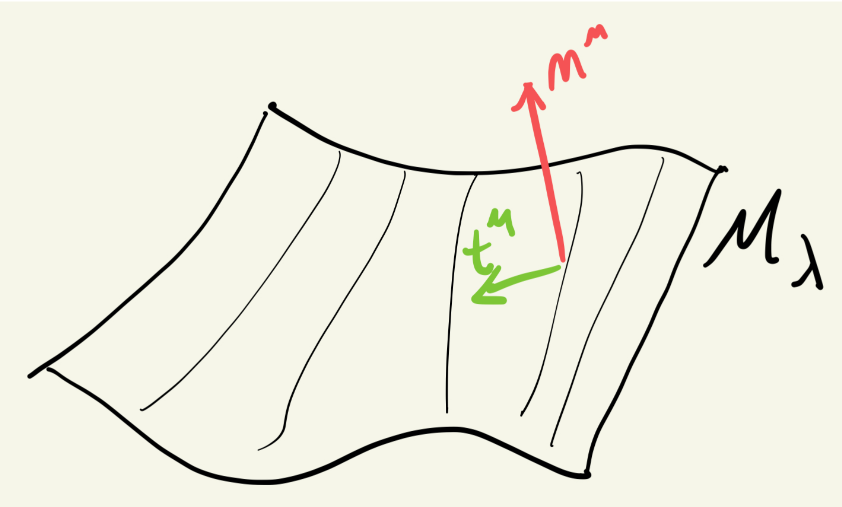


– t^μ is the orthonormal vector to 2-surface inside M_t :

$$t_\mu t^\mu = 1$$

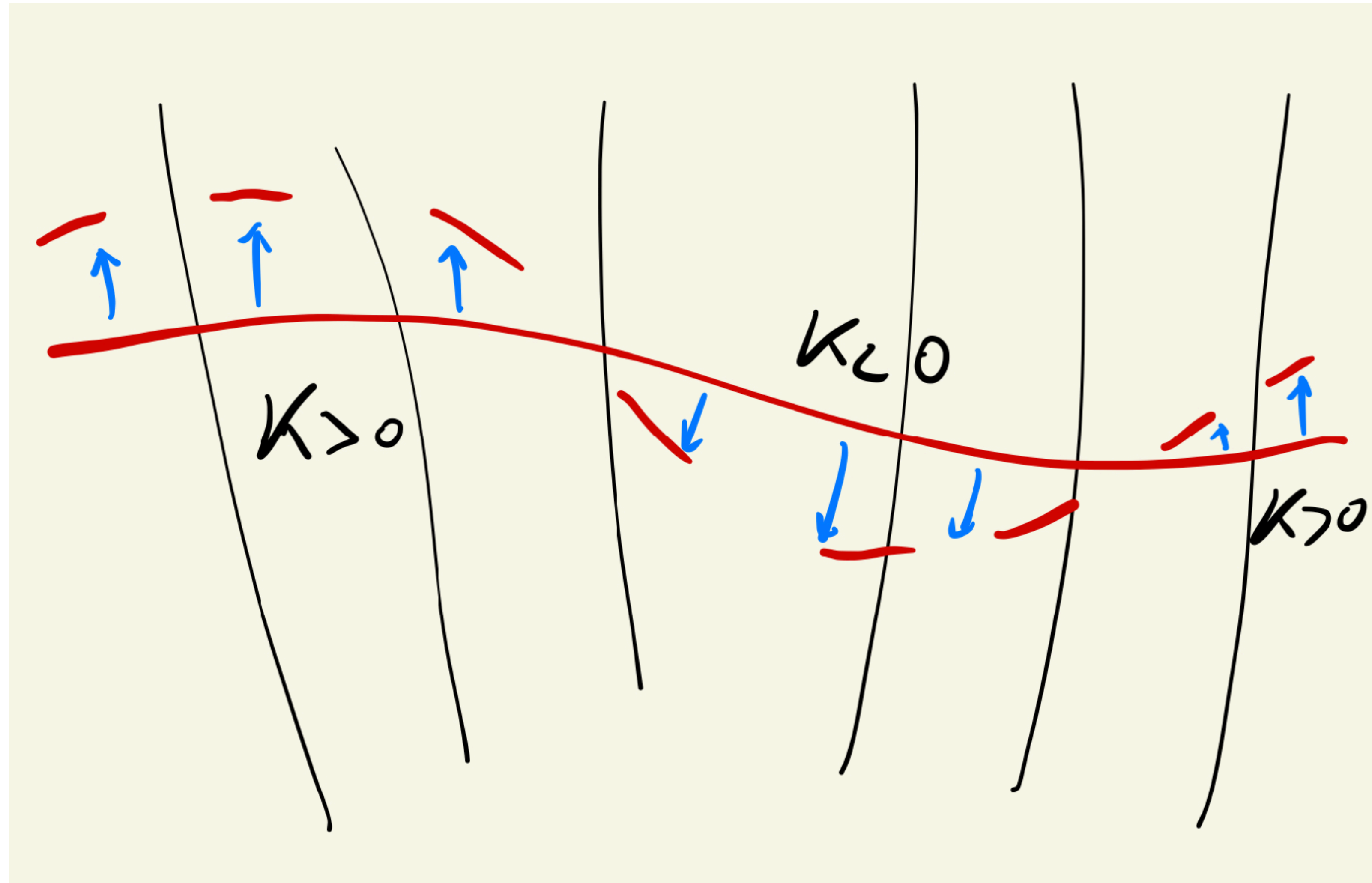
– Spatial 2-metric $h_{\mu\nu}$: $g_{\mu\nu} = h_{\mu\nu} + t_\mu t_\nu$

– Extrinsic curvature $A_{\mu\nu} := h_\mu^{\ \alpha} \nabla_\alpha t_\nu \qquad H := h^{\mu\nu} A_{\mu\nu}$

Method:
Mean Curvature Flow

Mean Curvature Flow

$$\frac{d}{d\lambda} y^\mu(x,\lambda) = K n^\mu(y^\alpha)$$



– We probe the geometry of the manifold, solution to Einstein equations, using mean curvature flow:

$$\mathcal{L}_n \log\sqrt{h} = K \ , \quad \Rightarrow \quad \frac{dV}{d\lambda} = \int_{\mathcal{M}_\lambda} d^4x\sqrt{h}\,K^2 \geq 0$$

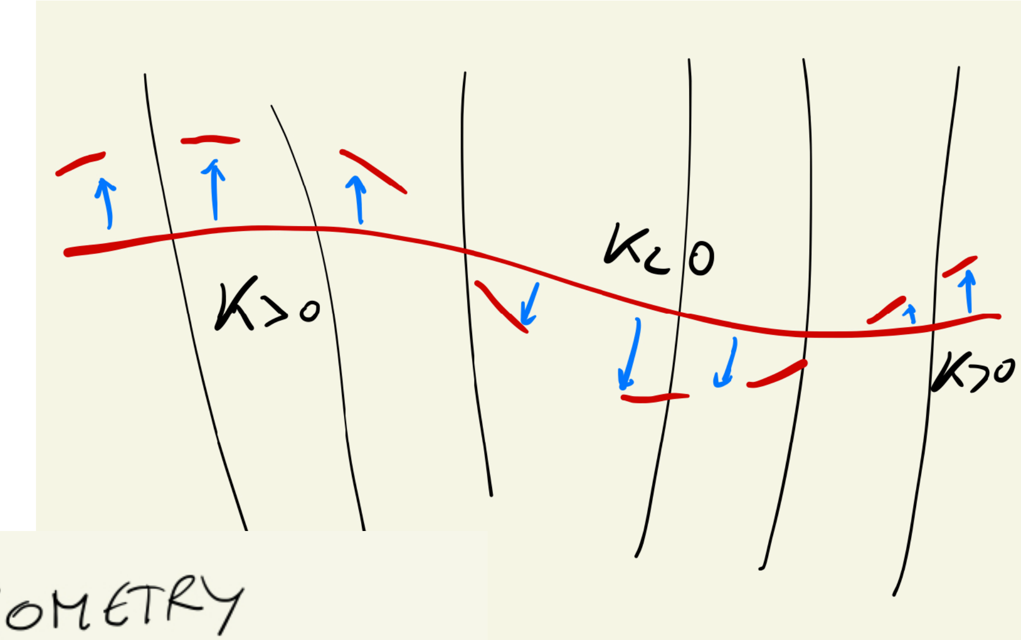
– We reconstruct the spacetime geometry from the one of the flow surfaces.

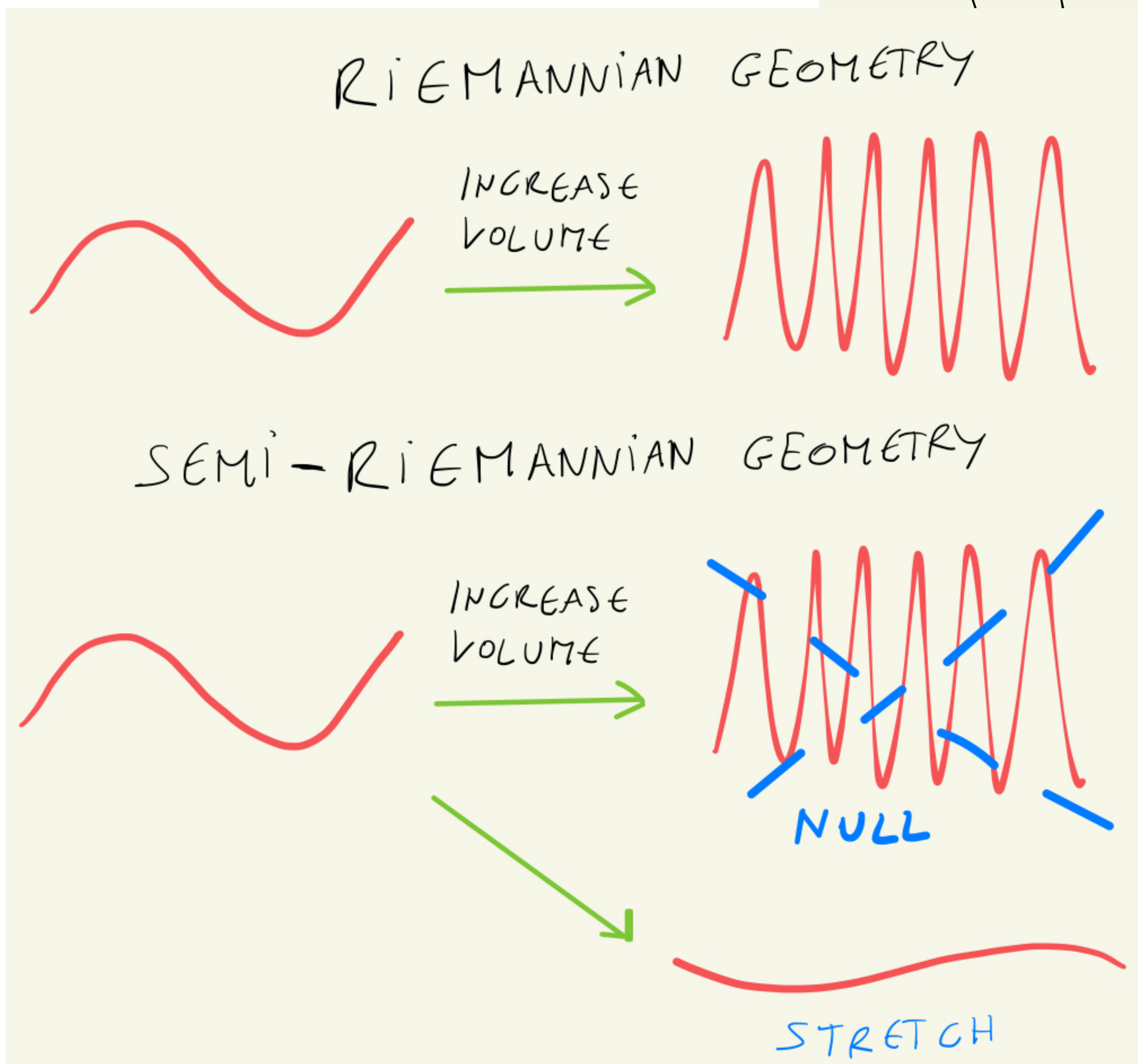
Mean Curvature Flow

– Important facts:

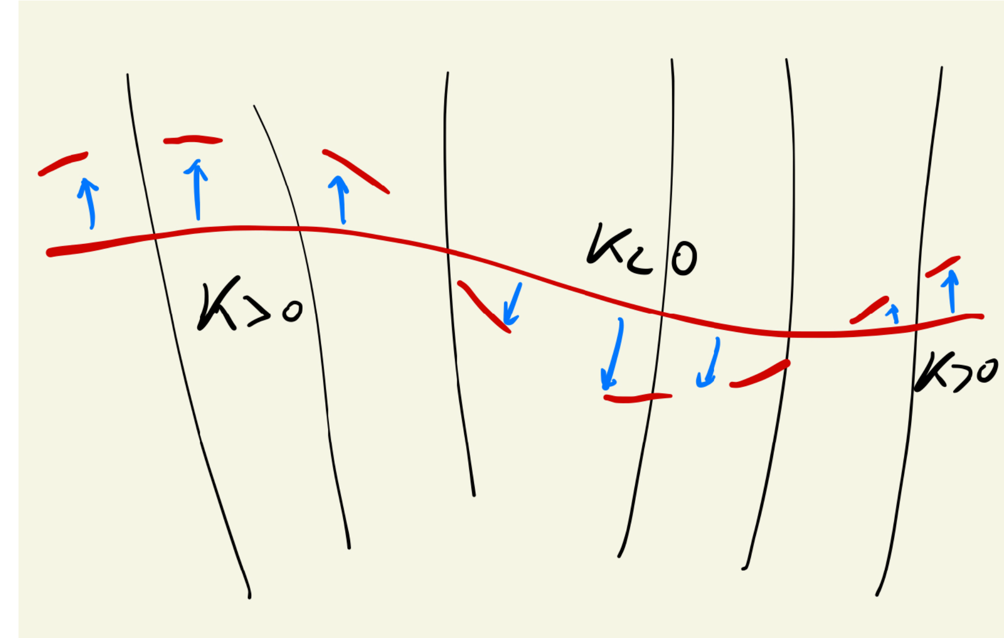
with Creminelli, Vasy, **Comm Math Phys 2020**
evolution of Ecker Huisken **1989**

– The flow stays regular, and so exists, at all times





Mean Curvature Flow

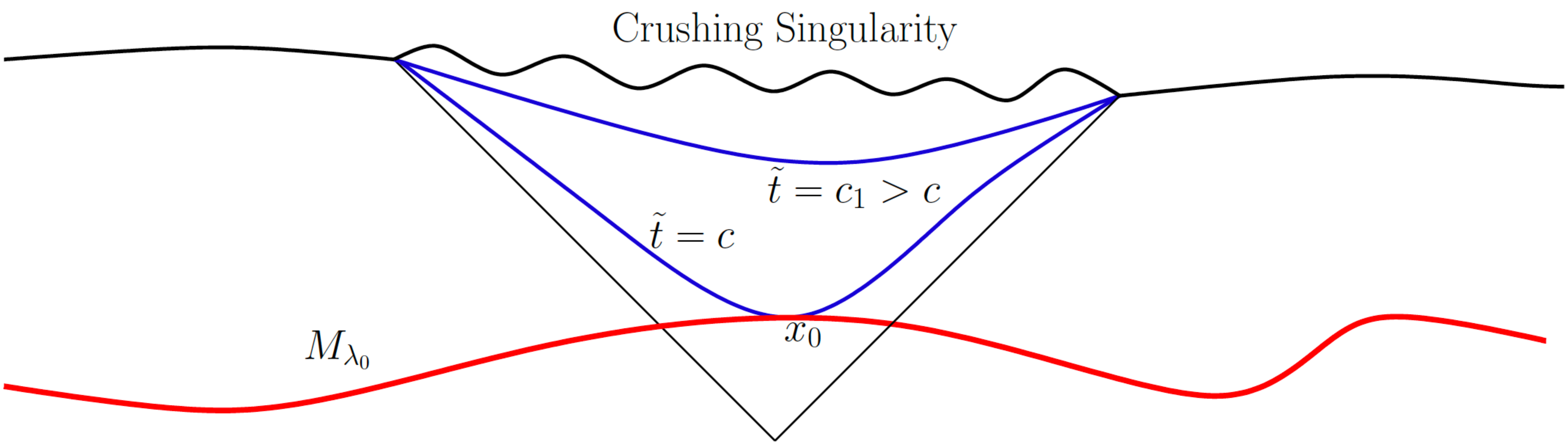


– Important facts:

with Creminelli, Vasy, **Comm Math Phys 2020**

– Stays away from singularities

– as there spatial volume decreases



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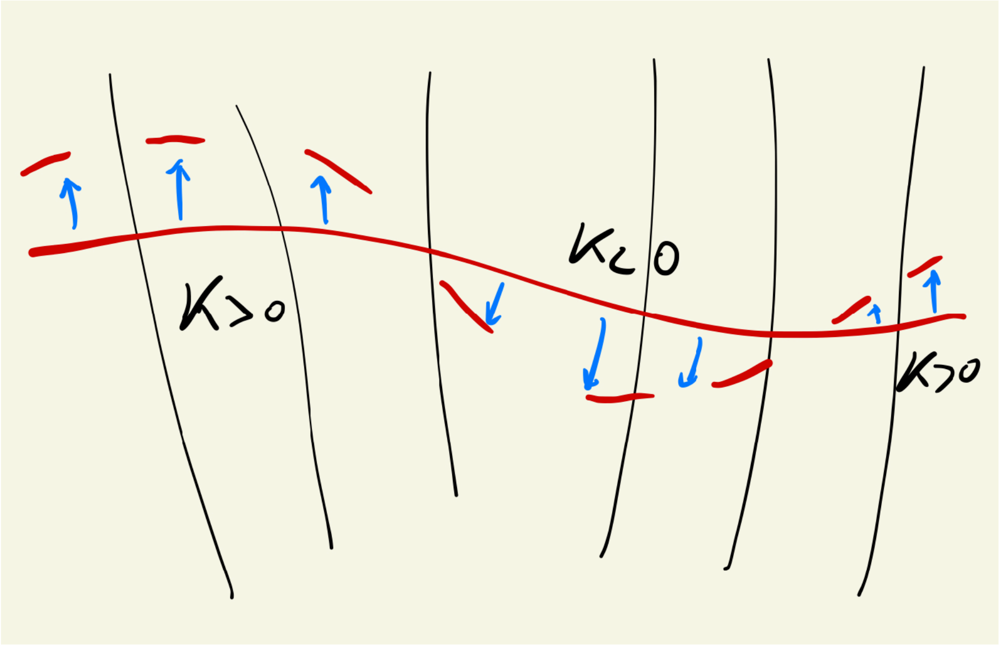


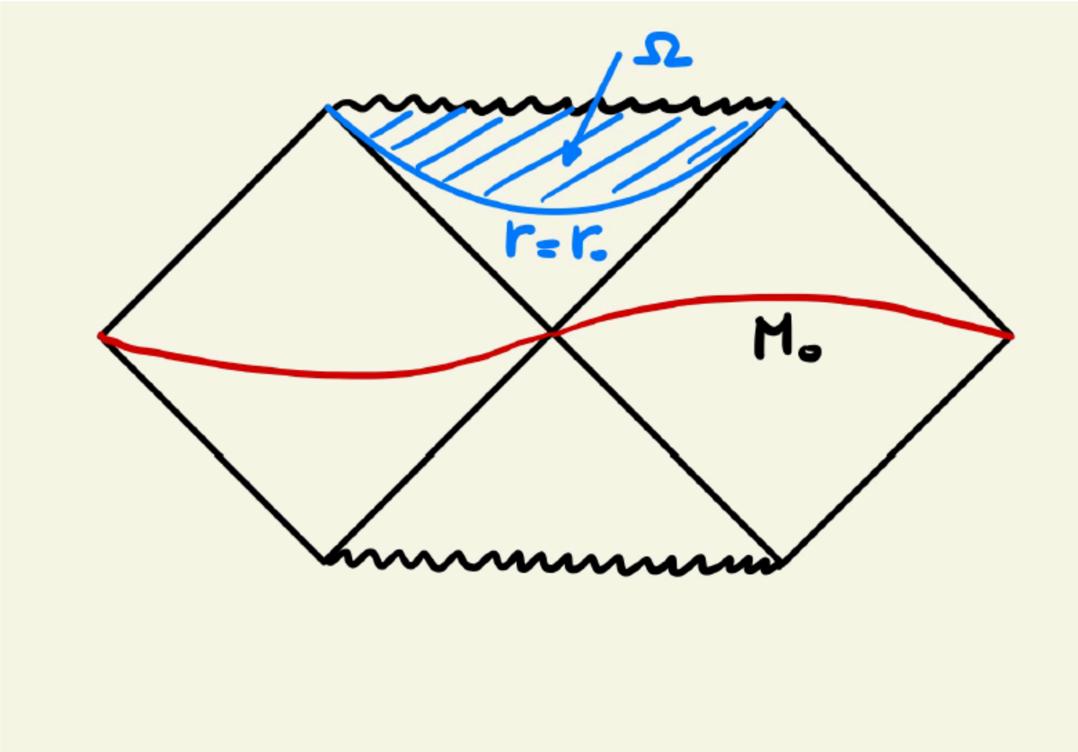
– Important facts:

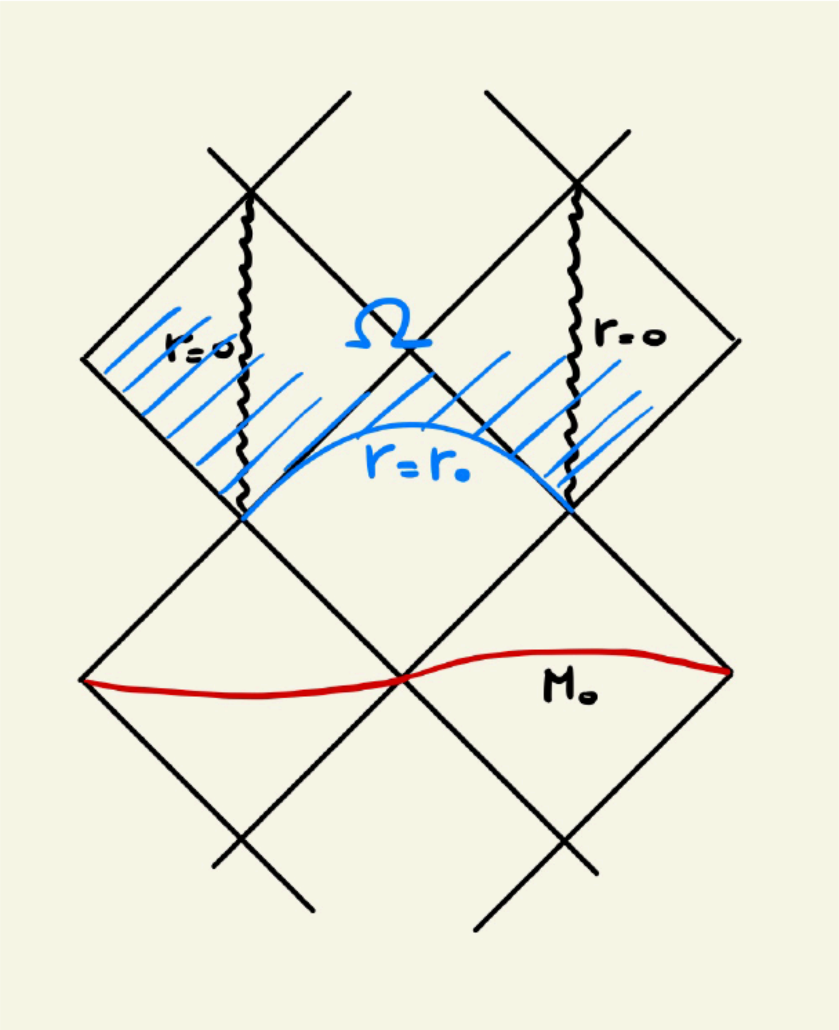
with Creminelli, Vasy, **Comm Math Phys 2020**

– Stays away from singularities

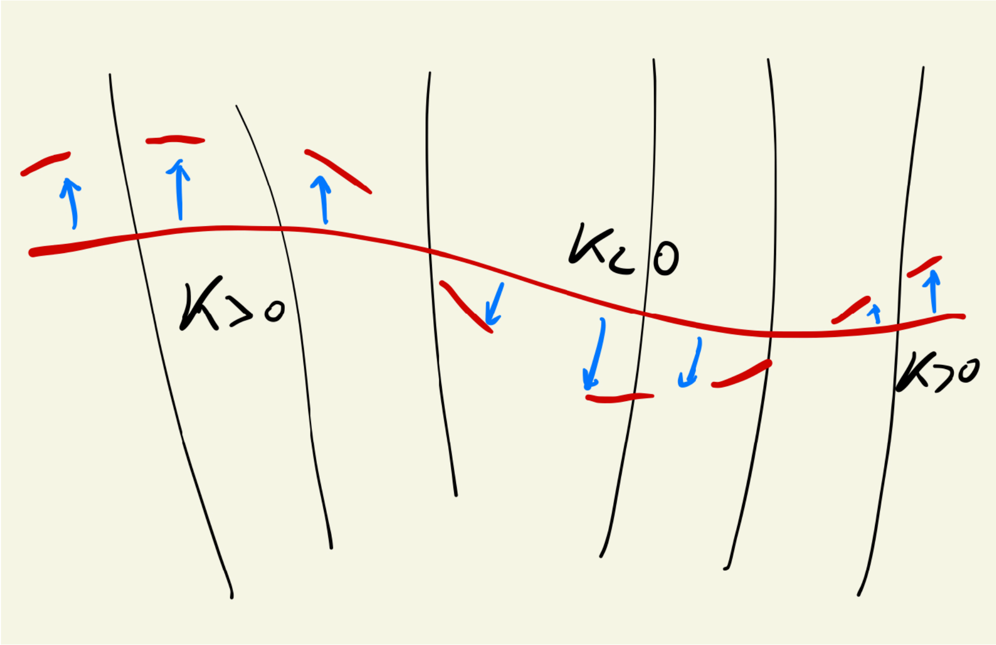
– as there spatial volume decreases

– e.g. barriers in Swartzschild and Kerr





Mean Curvature Flow



– Important facts:

with Creminelli, Vasy, **Comm Math Phys 2020**

– The flow stays regular, and so exists, at all times

– The maximum of the extrinsic curvature on a slice, K_m , decays exponentially towards K_Λ: $K_m(\lambda_1) \leq K_\Lambda + e^{-\frac{2}{3}K_\Lambda^2\lambda_1}(K_m(0) - K_\Lambda)$

$$\frac{dK}{d\lambda} - \Delta K + \frac{1}{n}K\left(K^2 - K_\Lambda^2\right) + \sigma^2 K + R^{(m)}_{\mu\nu}n^\mu n^\nu K = 0 \ ,$$

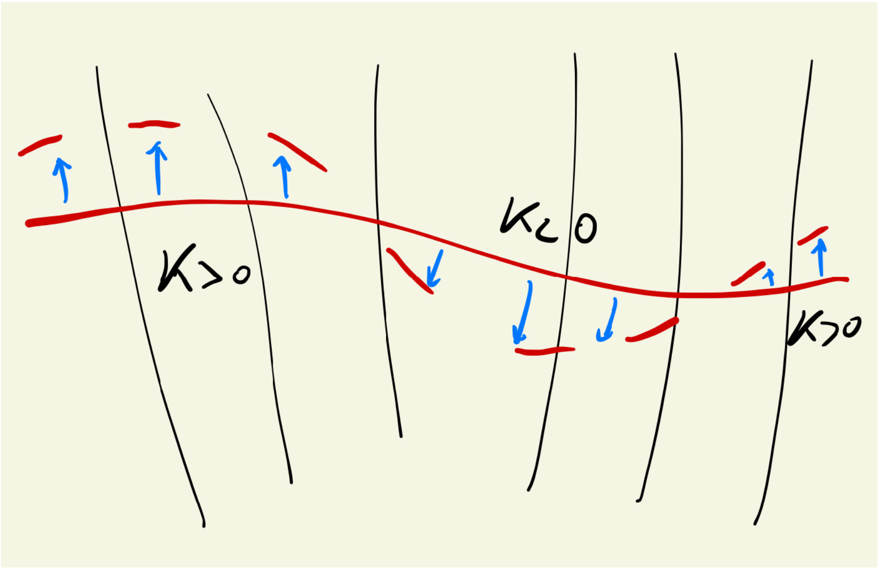
$$\Rightarrow \quad \frac{dK}{d\lambda} - \Delta K + \frac{1}{n}K\left(K^2 - K_\Lambda^2\right) + \sigma^2 K \leq 0 \ ,$$

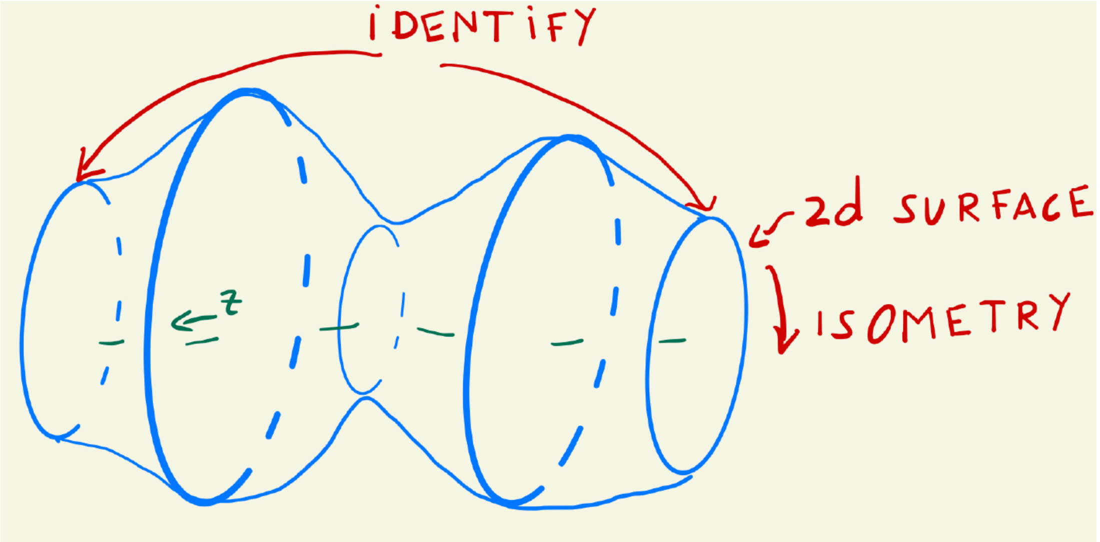
– $K \geq 0$ at all times Ecker, Ecker Huiskin

– Simple application of maximum principle

Steps of Proof

Steps of proof: (1)





– Growth of geometric quantities:

– from some flow-time on:

$$(1+\delta)^{-1} \le \frac{S_{\min}(\lambda_2)}{S_{\min}(\lambda_1)e^{\frac{2}{3}K_\Lambda^2(\lambda_2-\lambda_1)}} \le 1+\delta \ ,$$

$$(1+\delta)^{-1} \le \frac{L(\lambda)}{L(\lambda_{0,2})e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_{0,2})}} \le 1+\delta \ .$$

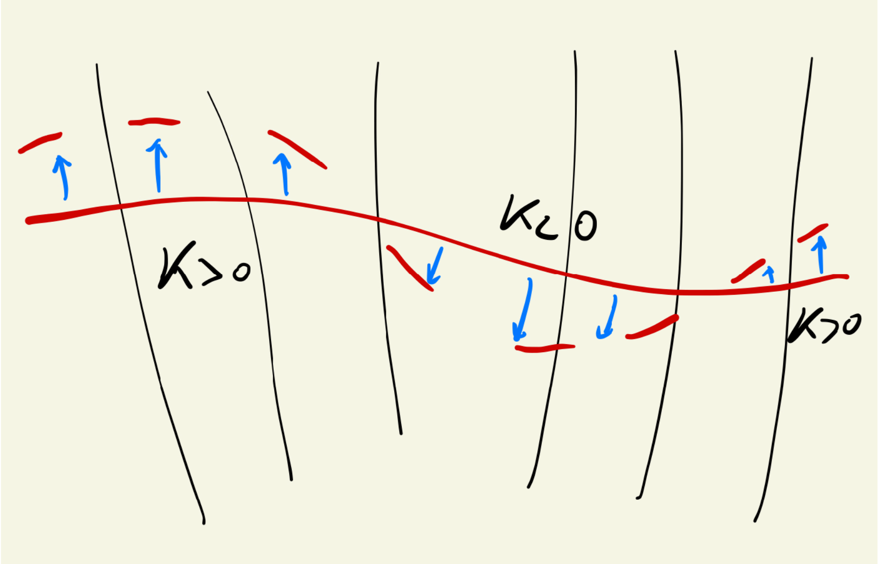
$$(1+\delta)^{-1} \le \frac{V(\lambda)}{V(\lambda_{0,2})e^{K_\Lambda^2(\lambda-\lambda_{0,2})}} \le 1+\delta$$

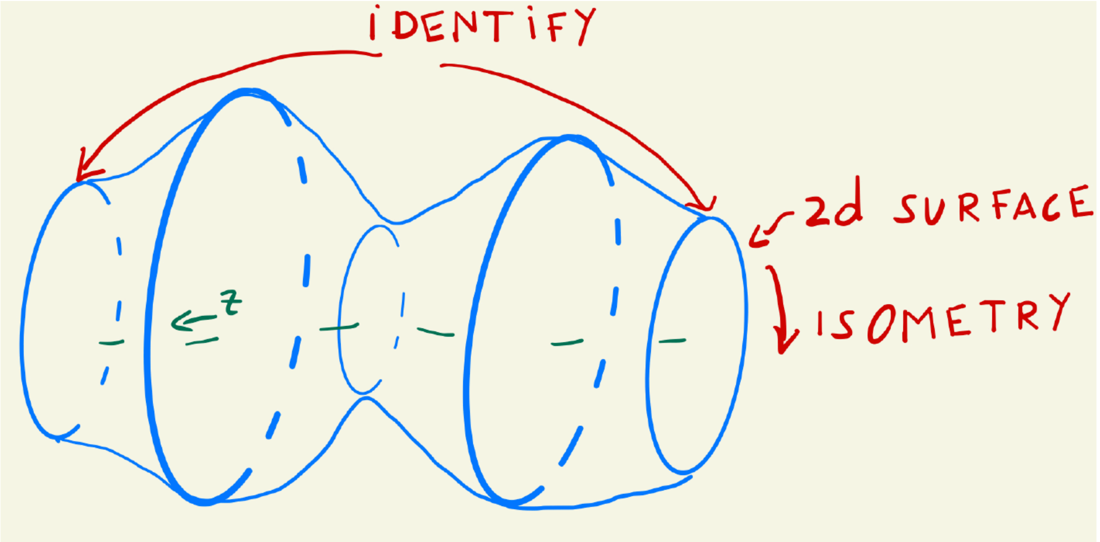
– uniformity of expansion rate close to dS:

$$\int_{\mathcal{M}_\lambda} dV \left(|K_\Lambda^2 - K^2| + \sigma^2\right) \le \frac{1}{K_\Lambda} C_7 \, e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_{0,2})}$$

(for the surface, see also Mirbabayi **2018**)

Steps of proof: (2)





– Closeness to exponentially expanding spatial slices:

– Define

$$\mathbf{g} := g(\lambda_0)e^{\frac{2}{3}K_\Lambda^2(\lambda-\lambda_0)}$$

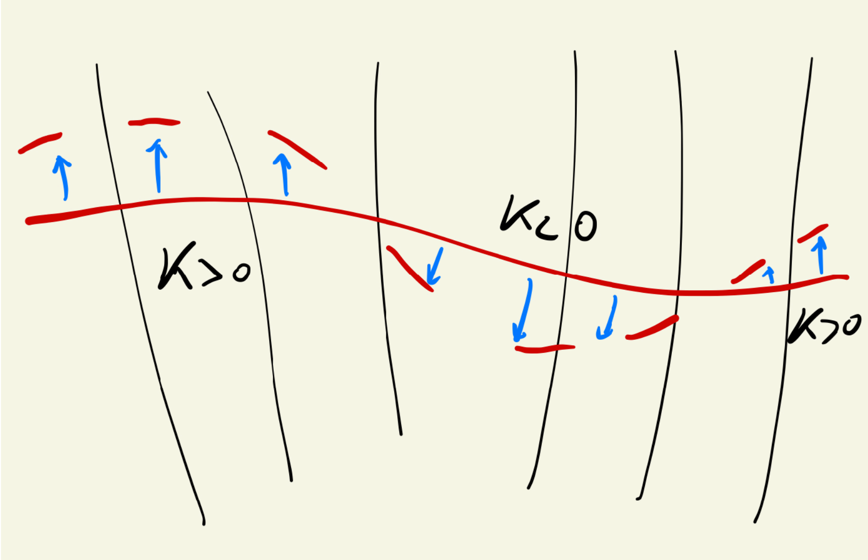
• i.e.: the spatial metric at some time, then let it grow as in dS

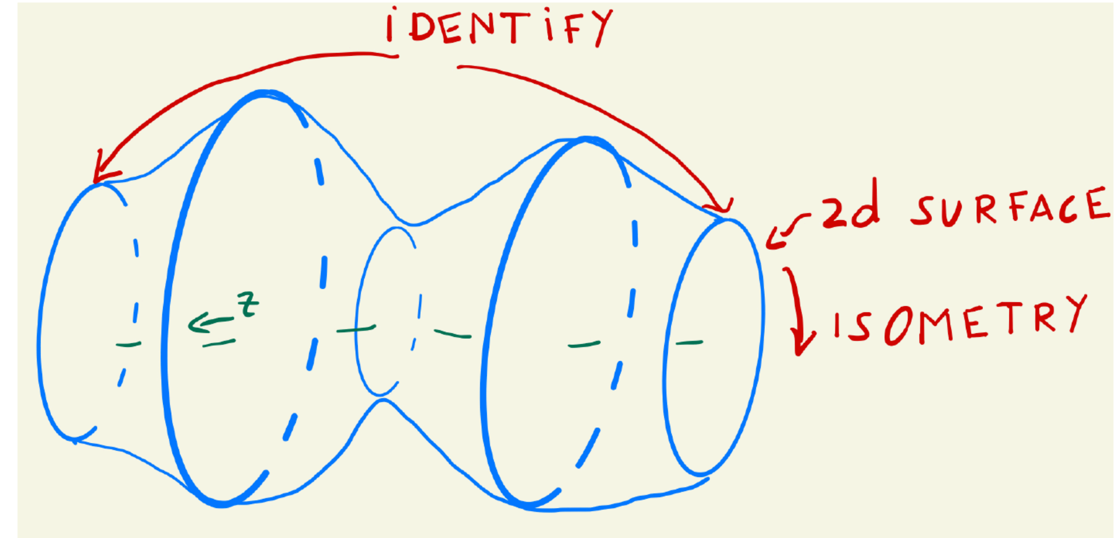
– from some flow-time on:

$$||g(\lambda) - \mathbf{g}(\lambda)||_{\mathbf{g}} \leq C_{11}e^{-\frac{1}{6}K_\Lambda^2\lambda_0} \ ,$$

• pointwise

Steps of proof: (3)





–Closeness to dS spatial slices over exponentially expanding Balls:

–Define the spatial slices of dS:

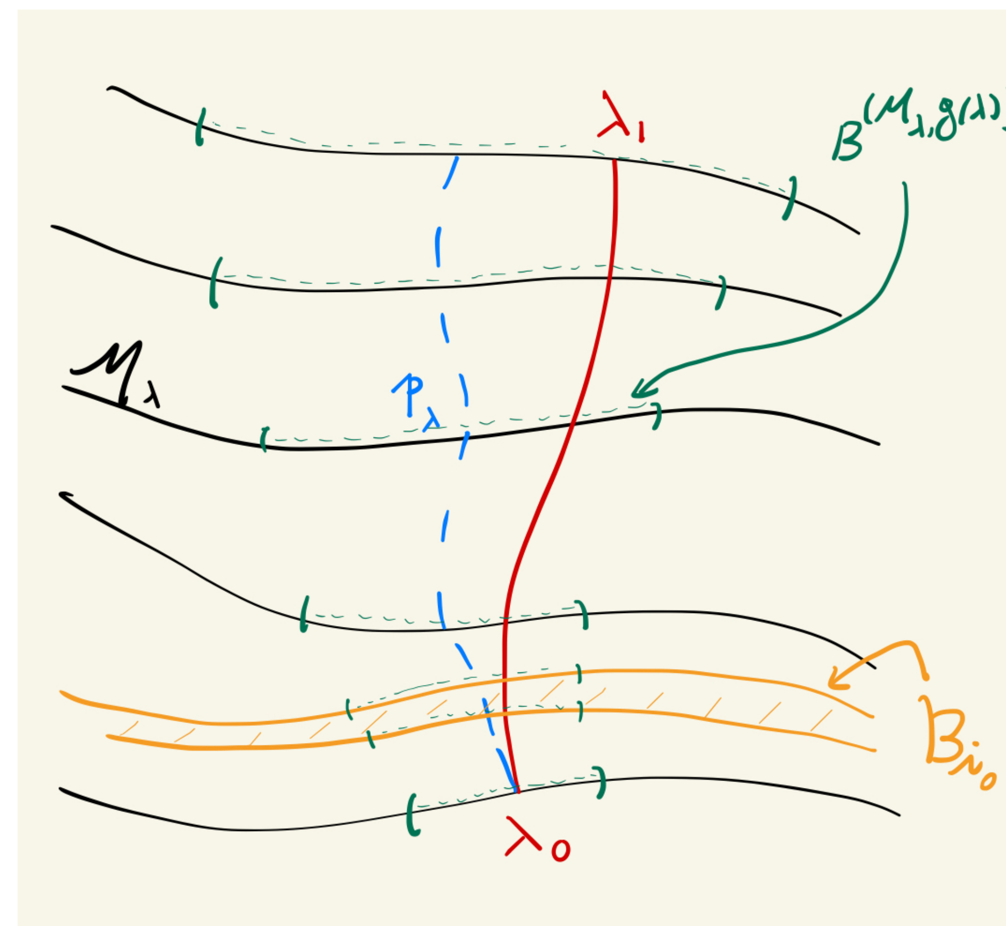
$$\mathbf{g}_{\mathrm{dS}}(\lambda) := e^{\frac{2}{3}K_\Lambda^2(\lambda-\lambda_0)} g_{\mathrm{Euc}} \ ,$$

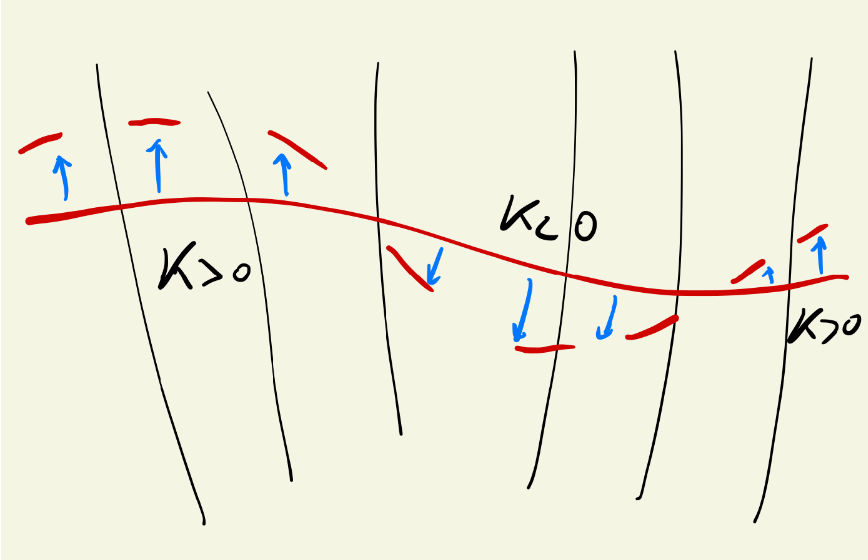
–from some flow-time on:

$$||g(\lambda) - \mathbf{g}_{\mathrm{dS}}(\lambda)||_{g(\lambda)} < e^{-\frac{1}{12}K_\Lambda^2\lambda_0} \ ,$$

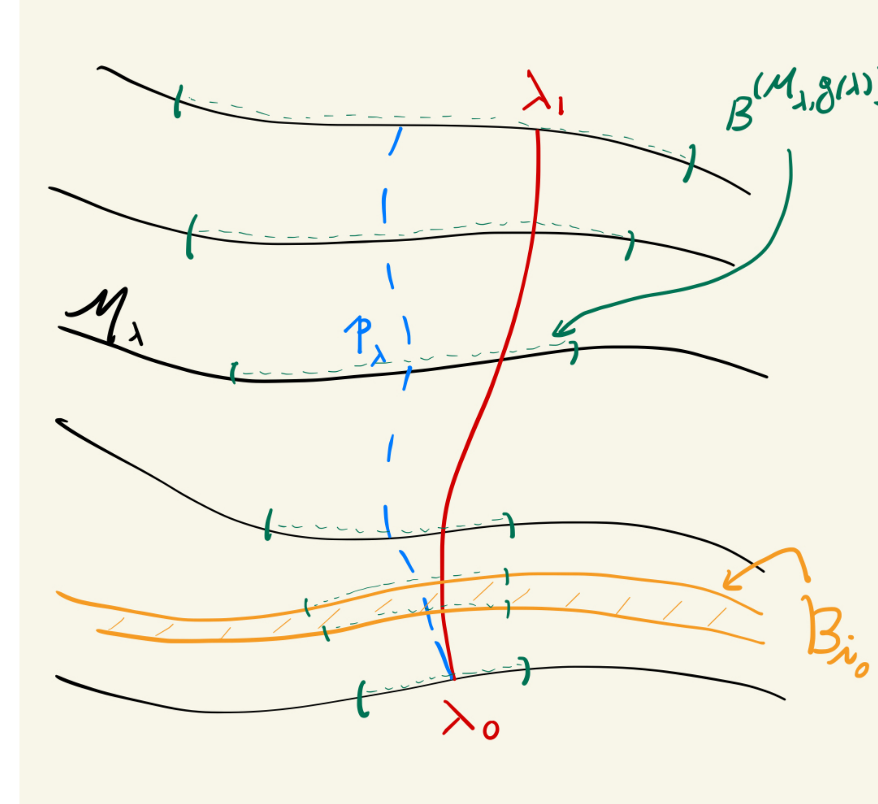
pointwise on $B^{(\mathcal{M}_\lambda, g(\lambda))}\left(p_\lambda, \frac{1}{K_\Lambda} e^{\frac{1}{12}K_\Lambda^2\lambda_0} \cdot e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_0)}\right)$.

Here p_λ *results from following* p *along the flow.*





Steps of proof: (4)



– Spacetime Closeness to dS over exp. growing Balls:

– The flow defines a natural 4-metric

$$ds_4^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu = -K^2 d\lambda^2 + g_{ij} dx^i dx^j$$

- foliates the whole spacetimes (so there are no singularities, geodesic completeness)

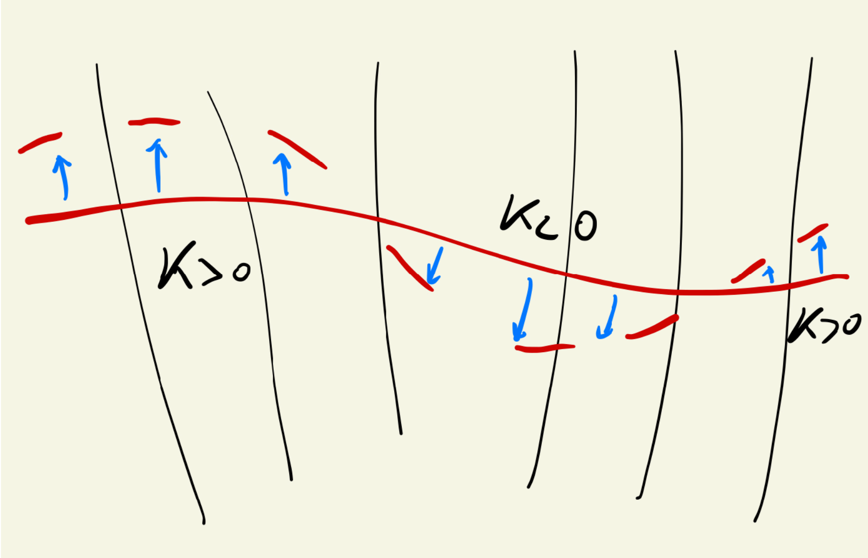
– Define the dS metric $ds^2_{\rm dS} := \mathbf{g}^{(4)}_{\rm dS} := -K_\Lambda^2 d\lambda^2 + (\mathbf{g}_{\rm dS})_{ij} dx^i dx^j.$

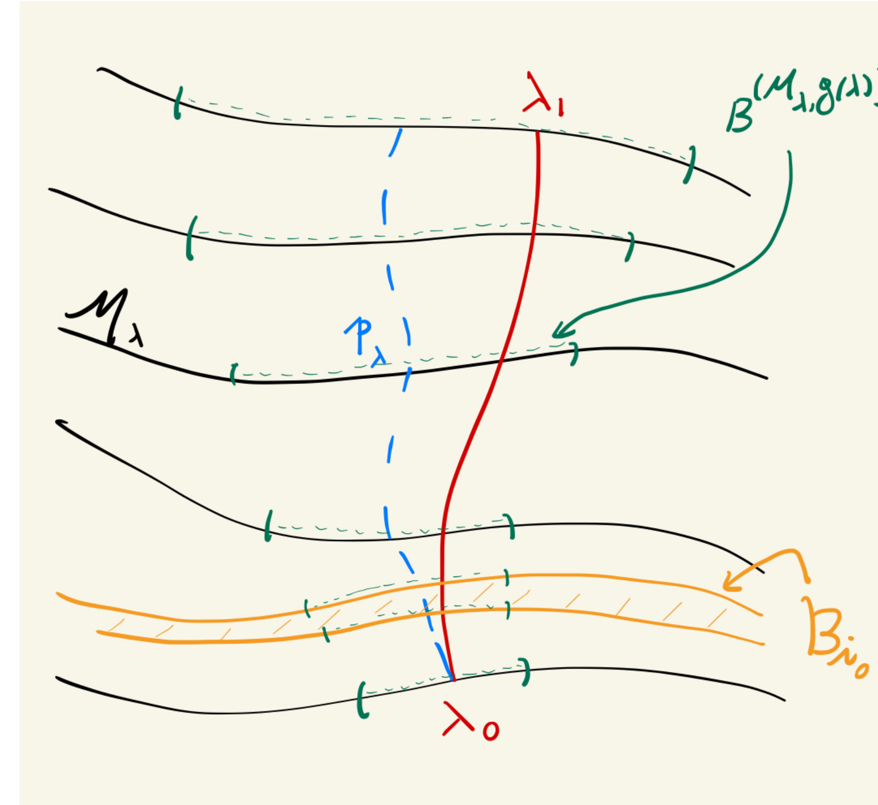
– Take any future oriented timelike or null curve

– from some flow-time on:

$$\left| L^{ds_4^2}[\gamma] - L^{ds^2_{\rm dS}}[\gamma] \right| \le \frac{C_{13}}{K_\Lambda} e^{-\frac{1}{18} K_\Lambda^2 \lambda_0} + 8 K_\Lambda e^{-\frac{1}{24} K_\Lambda^2 \lambda_0} (\lambda_1 - \lambda_0) .$$

Steps of proof: (5)



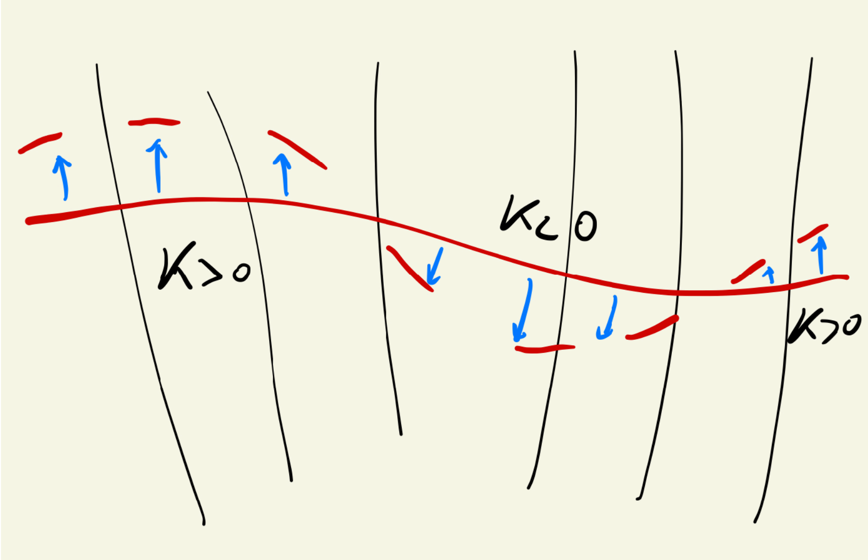


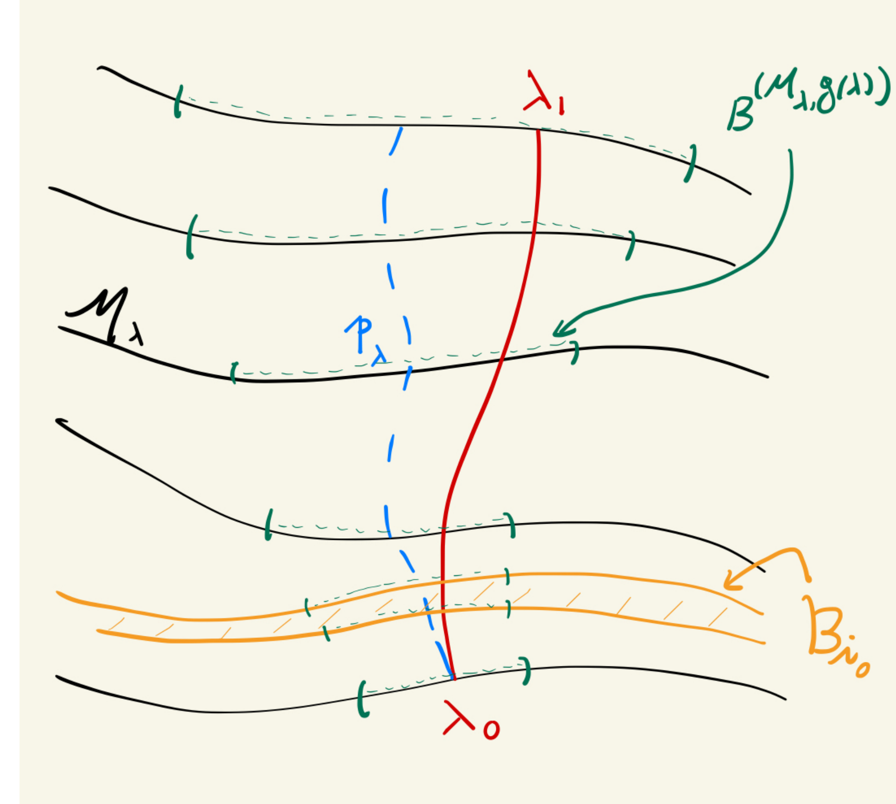
– Dilution of matter:

  – from some flow-time on

$$16\pi G_N \int dz \; \left| T_{\mu\nu} e^{\mu a} e^{\nu b} \right| \leq 16\pi G_N \int dz \; T_{\mu\nu} n^{\mu} n^{\nu} \leq C_{14} K_{\Lambda} e^{-\frac{1}{3} K_{\Lambda}^2 \lambda}$$

Steps of proof: (6)





– Physical equivalence to dS over exp. growing Balls

  – Consider an observer

  – equivalence of lengths

    » $\Rightarrow$ same geodesics

    » $\Rightarrow$ same horizon as in dS

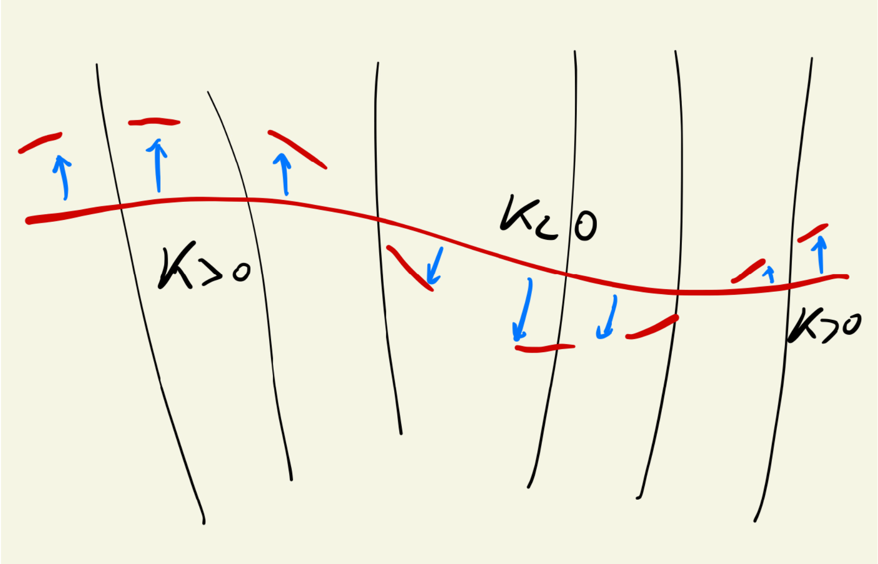
  – Observer has access to matter only from finite volume. All times, in this volume:

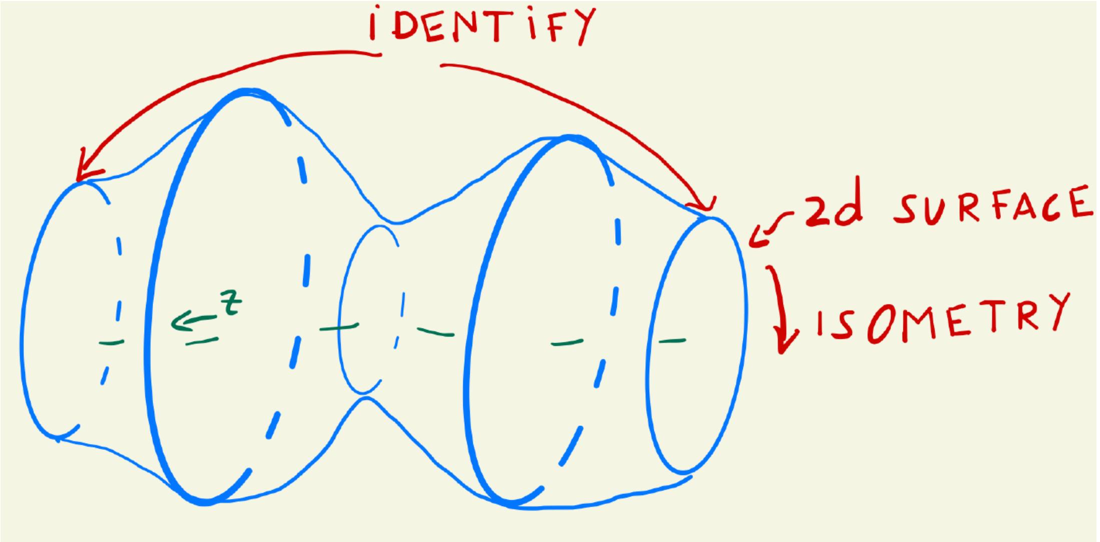
$$16\pi G_N \int_{\mathcal{M}_\lambda \cap y_\lambda(y_{\lambda_2}^{-1}(B_c(\lambda_2)))} \left|T_{\mu\nu} e^{\mu a} n^{\nu b}\right| \leq \frac{\pi (12)^2 C_{14}}{K_\Lambda} e^{-\frac{1}{3} K_\Lambda^2 \lambda_2} \ ,$$

  – available energy-momentum is below any threshold: it feels as dS

Proof

Steps of proof: (1)





– Growth of geometric quantities:

– from some flow-time on:

$$(1+\delta)^{-1} \leq \frac{S_{\min}(\lambda_2)}{S_{\min}(\lambda_1) e^{\frac{2}{3}K_\Lambda^2(\lambda_2-\lambda_1)}} \leq 1+\delta \ ,$$

$$(1+\delta)^{-1} \leq \frac{L(\lambda)}{L(\lambda_{0,2}) e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_{0,2})}} \leq 1+\delta \ .$$

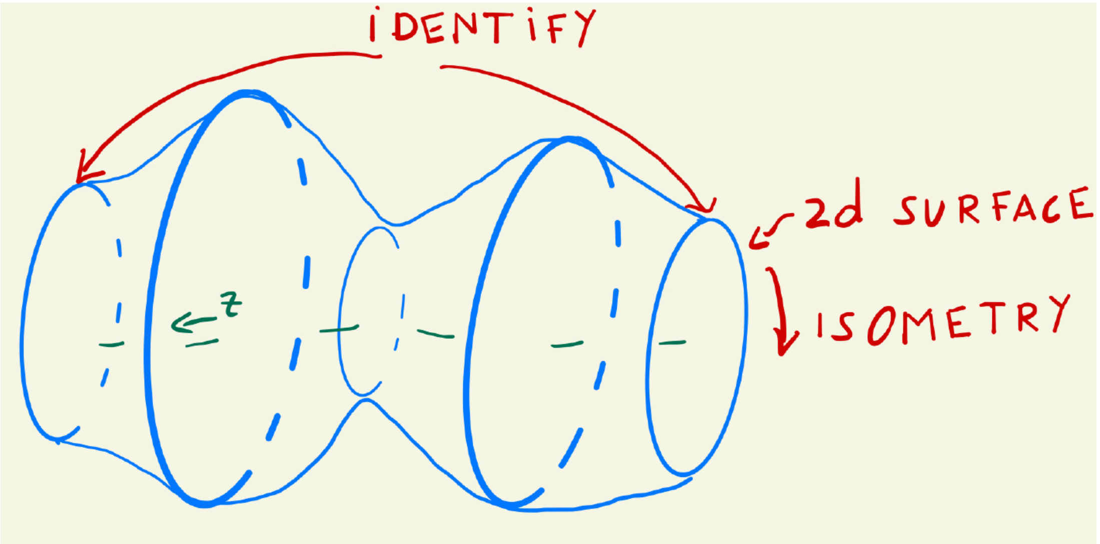
$$(1+\delta)^{-1} \leq \frac{V(\lambda)}{V(\lambda_{0,2}) e^{K_\Lambda^2(\lambda-\lambda_{0,2})}} \leq 1+\delta$$

– uniformity of expansion rate:

$$\int_{\mathcal{M}_\lambda} dV \left(|K_\Lambda^2 - K^2| + \sigma^2\right) \leq \frac{1}{K_\Lambda} C_7 \, e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_{0,2})}$$

(for the surface, see also Mirbabayi **2018**)

Steps of proof: (1)



– From Gauss-Codazzi:

$$G_{\mu\nu}n^\mu n^\nu = 8\pi G_N T_{\mu\nu}n^\mu n^\nu,$$

$$\Rightarrow \quad {}^{(3)}R + \frac{2}{3}K^2 - \sigma^2 = \frac{2}{3}K_\Lambda^2 + 16\pi G_N T_{\mu\nu}n^\mu n^\nu,$$

$$\Rightarrow \quad {}^{(3)}R + \frac{2}{3}K^2 - \sigma^2 \geq \frac{2}{3}K_\Lambda^2 \ ,$$

$$\Rightarrow \quad {}^{(3)}R \geq -C_2 K_\Lambda^2 e^{-\frac{2}{3}K_\Lambda^2\lambda} \ ,$$

– Take global coordinates $\quad g = dz^2 + h_z \ ,$

– a Combination of Riccati-Gauss-Codazzi:

$$H' + A_{\mu\nu}A^{\mu\nu} = -{}^{(3)}R_{zz} = \frac{{}^{(2)}R - {}^{(3)}R + A_{\mu\nu}A^{\mu\nu} - H^2}{2}$$

– On a minimal slice: $H = 0$ and $H' \geq 0$,

$$ {}^{(3)}R = {}^{(2)}R - A_{\mu\nu}A^{\mu\nu} - H^2 - 2H' \leq {}^{(2)}R$$

$$\Rightarrow \quad \frac{2}{3}K_\Lambda^2 - \frac{2}{3}K^2 + \sigma^2 \leq {}^{(2)}R \ ,$$

– But: ${}^{(2)}R(z,\lambda) \leq \dfrac{4\pi\chi_0}{S(z,\lambda)} \leq \dfrac{4\pi\chi_0}{S_{\rm min}(\lambda)}$

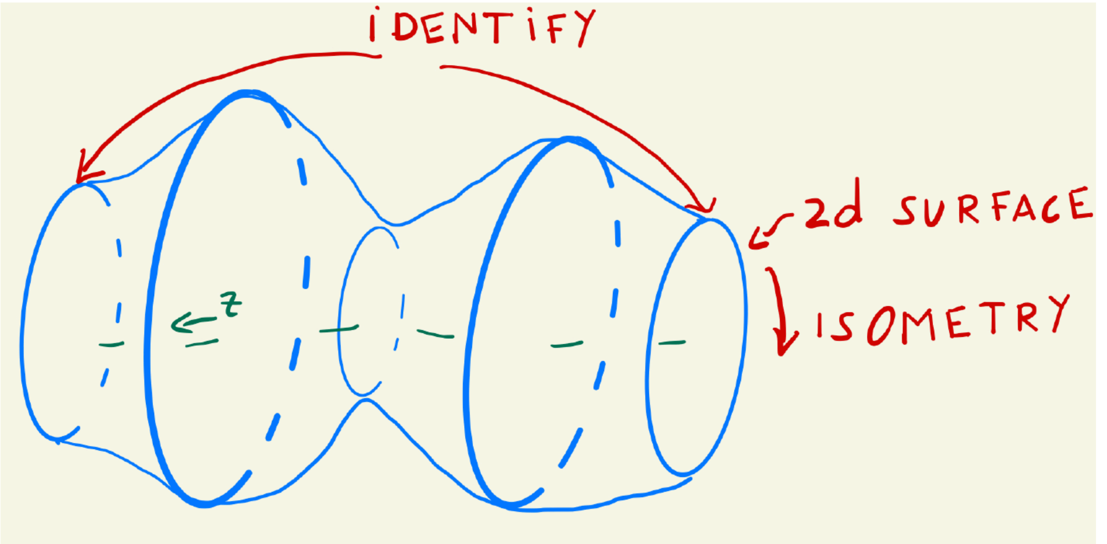
$$\Rightarrow \quad \frac{2}{3}\left|K_\Lambda^2 - K^2\right| + \sigma^2 \leq {}^{(2)}R + 2C_2K_\Lambda^2 e^{-\frac{2}{3}K_\Lambda^2\lambda} \leq \frac{4\pi\chi_0}{S_{\rm min}} + 2C_2K_\Lambda^2 e^{-\frac{2}{3}K_\Lambda^2\lambda} \ .$$

Steps of proof: (1)

–Evolution equation of metric:

$$\frac{dg_{ij}}{d\lambda} = 2KK_{ij} = \frac{2}{3}K^2 g_{ij} + 2K\sigma_{ij} = \frac{2}{3}K_\Lambda^2 g_{ij} + E_{S,ij} \ ,$$

$$E_{S,ij} = \tfrac{2}{3}(K^2 - K_\Lambda^2)g_{ij} + 2K\sigma_{ij}$$



–. $\Rightarrow \quad \frac{2}{3}\left|K_\Lambda^2 - K^2\right| + \sigma^2 \leq {}^{(2)}R + 2C_2 K_\Lambda^2 e^{-\frac{2}{3}K_\Lambda^2 \lambda} \leq \frac{4\pi\chi_0}{S_{\min}} + 2C_2 K_\Lambda^2 e^{-\frac{2}{3}K_\Lambda^2\lambda} \ .$

–. $\frac{d}{d\lambda}dS = \left(\frac{2}{3}K_\Lambda^2 + \frac{1}{2}E_S\right) dS \ .$

$$|E_S| \leq 2\left(\left[\frac{4\pi\chi_0}{S_{\min}} + 2C_2 K_\Lambda^2 e^{-\frac{2}{3}K_\Lambda^2\lambda}\right] + 2K_\Lambda(1+C_1)\left[\sqrt{\frac{4\pi\chi_0}{S_{\min}}} + \sqrt{2C_2}K_\Lambda e^{-\frac{1}{3}K_\Lambda^2\lambda}\right]\right)$$

– after proving that this derivative makes sense almost everytime.

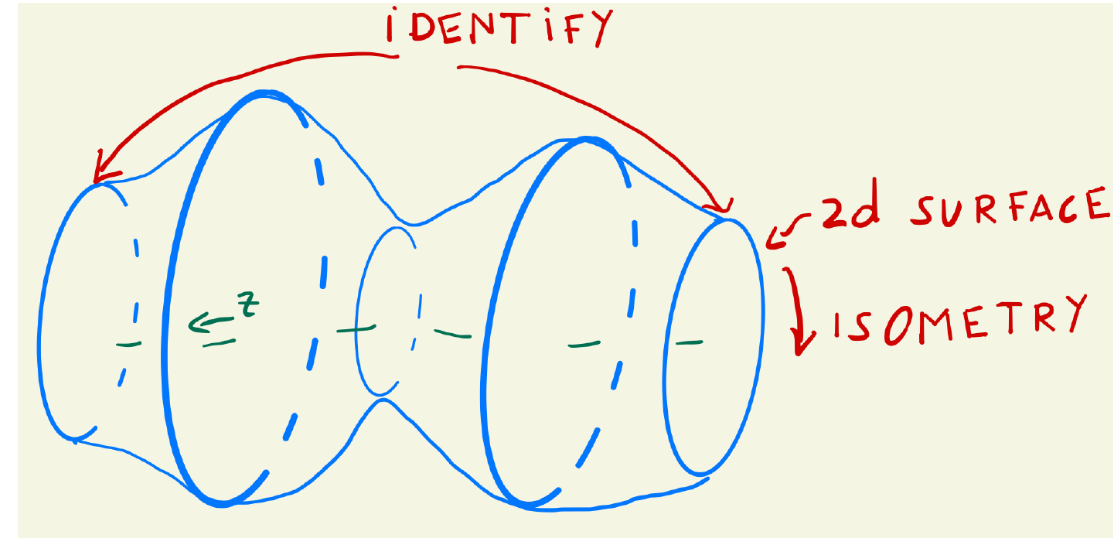
– for the sphere, ensure first E_S decays, then

$$\Rightarrow \quad \left|\log\left(\frac{S_{\min}(\lambda_2)}{S_{\min}(\lambda_1)}\right) - \frac{2}{3}K_\Lambda^2(\lambda_2 - \lambda_1)\right| \leq \log 2$$

Steps of proof: (1)

–Now with traverse length:

$$^{(3)}R = -A_{\mu\nu}A^{\mu\nu} - H^2 + {}^{(2)}R - 2H'$$



$$\Rightarrow \quad \int_0^{L(\lambda)} dz\ {}^{(3)}R \leq \int_0^{L(\lambda)} dz\ {}^{(2)}R \leq \int_0^{L(\lambda)} dz\ \frac{2\cdot 4\pi\chi_0}{S_{\rm min}(\lambda_{0,1})e^{\frac{2}{3}K_\Lambda^2(\lambda-\lambda_{0,1})}} = K_\Lambda^2 C_5 e^{-\frac{2}{3}K_\Lambda^2\lambda} L(\lambda)$$

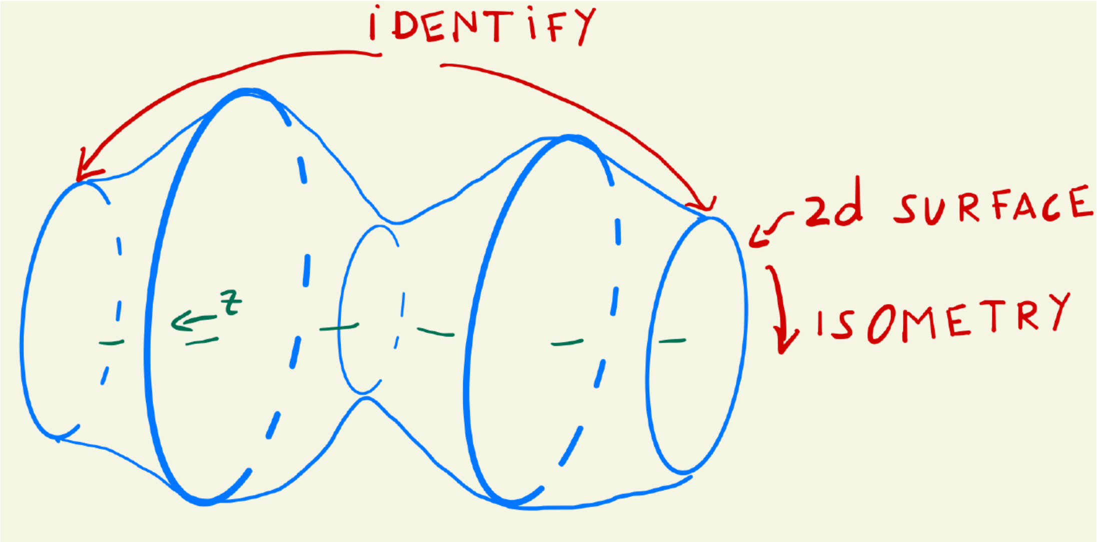
$$\Rightarrow \int_0^{L(\lambda)} dz\ \left(\frac{2}{3}\left(K_\Lambda^2 - K^2\right) + \sigma^2\right) \leq K_\Lambda^2 C_5\ e^{-\frac{2}{3}K_\Lambda^2\lambda} L(\lambda)\ .$$

$$\Rightarrow \int_0^{L(\lambda)} dz\ \sigma^2 \leq K_\Lambda^2 C_5 e^{-\frac{2}{3}K_\Lambda^2\lambda} L(\lambda) + \int_0^{L(\lambda)} dz\ \frac{2}{3}\left(K^2 - K_\Lambda^2\right) \leq K_\Lambda^2\left(C_5 + C_2\right) e^{-\frac{2}{3}K_\Lambda^2\lambda} L(\lambda)$$

$$\Rightarrow \int_0^{L(\lambda)} dz \left(\frac{2}{3}|K_\Lambda^2 - K^2| + \sigma^2\right) \leq K_\Lambda^2 C_6 e^{-\frac{2}{3}K_\Lambda^2\lambda} L(\lambda)\ ,$$

Steps of proof: (1)

– Now with traverse length:



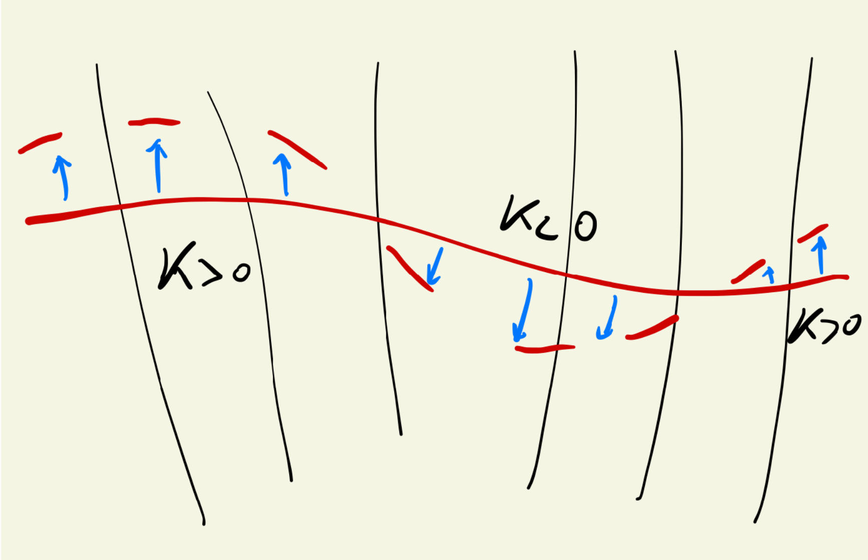
$$L'(\lambda) = \int_0^{L(\lambda)} dz\ KK_{zz} = \int_0^{L(\lambda)} dz\ \left(\frac{K^2}{3} + K\sigma_{zz}\right) =$$

$$= \frac{K_\Lambda^2}{3} L(\lambda) + \int_0^{L(\lambda)} dz\ \left(\frac{1}{3}(K^2 - K_\Lambda^2) + K\sigma_{zz}\right)\ ,$$

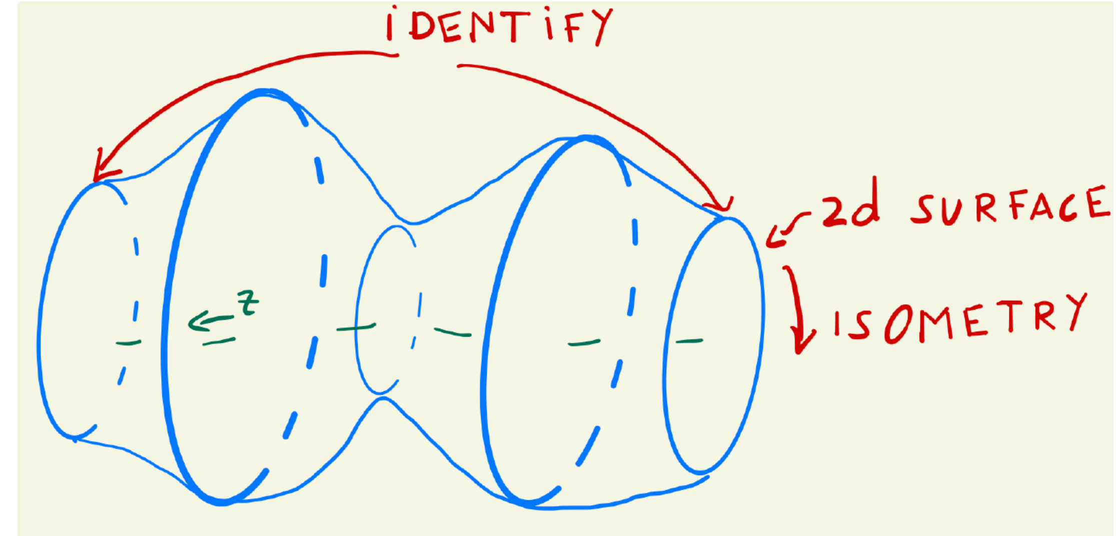
$$\Rightarrow \quad |L'(\lambda) - \frac{K_\Lambda^2}{3} L(\lambda)| \leq E_L(\lambda)$$

$$|E_L(\lambda)| \leq K_\Lambda^2 \frac{C_6}{2} e^{-\frac{2}{3}K_\Lambda^2\lambda} L(\lambda) + K_\Lambda^2 \left(1 + C_1 e^{-\frac{2}{3}K_\Lambda^2\lambda}\right)(C_5 + C_2)^{1/2} e^{-\frac{1}{3}K_\Lambda^2\lambda} L(\lambda)$$

– and one proceeds as before (and similarly for the volume)



Steps of proof: (1)



– Growth of geometric quantities:

– from some flow-time on:

$$(1+\delta)^{-1} \leq \frac{S_{\min}(\lambda_2)}{S_{\min}(\lambda_1) e^{\frac{2}{3}K_\Lambda^2(\lambda_2-\lambda_1)}} \leq 1+\delta\ ,$$

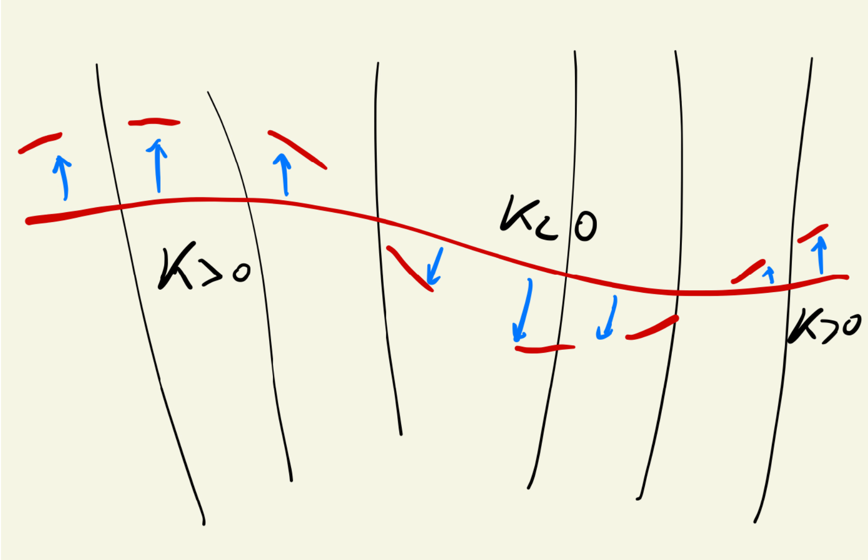
$$(1+\delta)^{-1} \leq \frac{L(\lambda)}{L(\lambda_{0,2}) e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_{0,2})}} \leq 1+\delta\ .$$

$$(1+\delta)^{-1} \leq \frac{V(\lambda)}{V(\lambda_{0,2}) e^{K_\Lambda^2(\lambda-\lambda_{0,2})}} \leq 1+\delta$$

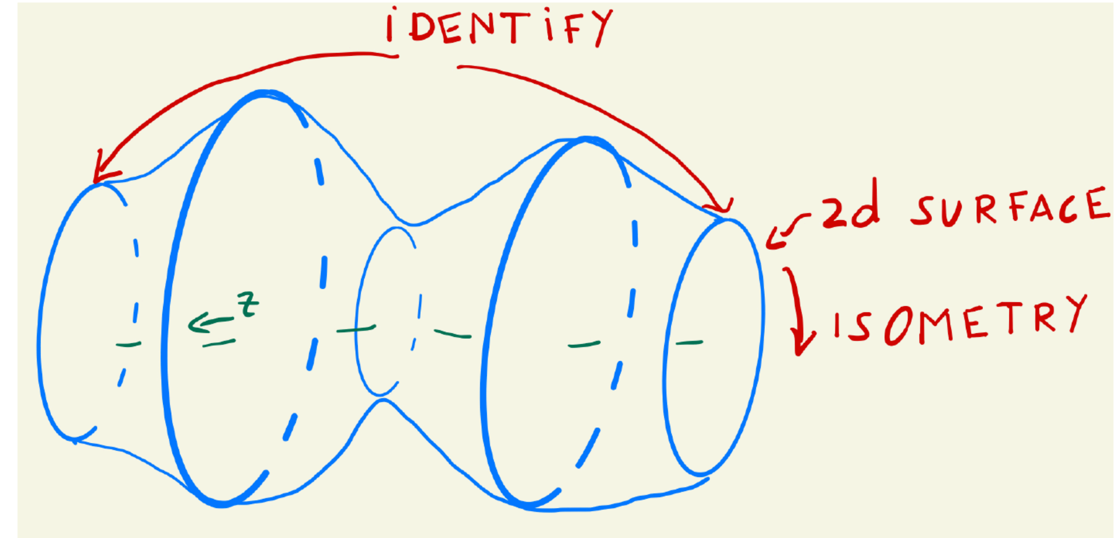
– uniformity of expansion rate:

$$\int_{\mathcal{M}_\lambda} dV\,\left(|K_\Lambda^2 - K^2| + \sigma^2\right) \leq \frac{1}{K_\Lambda} C_7\, e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_{0,2})}$$

(for the surface, see also Mirbabayi **2018**)



Steps of proof: (2)



– Closeness to exponentially expanding spatial slices:

– Define $$\mathbf{g} := g(\lambda_0) e^{\frac{2}{3} K_\Lambda^2 (\lambda - \lambda_0)}$$

- the spatial metric at some time, then let it grow as in dS

– from some flow-time on:

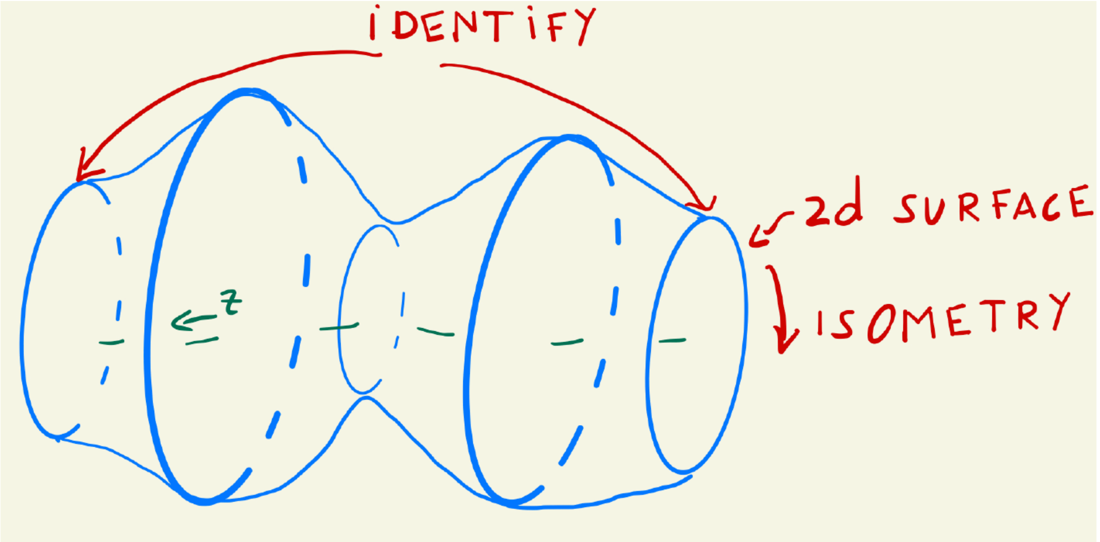
$$||g(\lambda) - \mathbf{g}(\lambda)||_{\mathbf{g}} \leq C_{11} e^{-\frac{1}{6} K_\Lambda^2 \lambda_0} ,$$

- pointwise

– The idea is to notice that, from some time on, the expansion rate is the one of dS.

- however, this proof is quite subtle.

Steps of proof: (2)



- First we show that the metric becomes less and less dependent on z.
- In fact, notice that $\mathcal{L}_{\partial_z} g_{\mu\nu} = 2A_{\mu\nu}$
- .

$$H' + \frac{H^2 + A_{\mu\nu}A^{\mu\nu}}{2} \leq C_8 K_\Lambda^2 \, e^{-\frac{2}{3}K_\Lambda^2 \lambda} \,, \quad \Rightarrow \quad |H| \leq \frac{2}{\sqrt{3}} \sqrt{C_8} K_\Lambda e^{-\frac{1}{3}K_\Lambda^2 \lambda} := \varepsilon_\lambda$$

- Integrate above

$$\int_0^z A_{\mu\nu}A^{\mu\nu} \leq \frac{3}{2}\varepsilon_\lambda^2 z + 4\varepsilon_\lambda \,, \Rightarrow \quad \int_0^z |A| \leq \sqrt{\frac{3}{2}\varepsilon_\lambda^2 z^2 + 4\varepsilon_\lambda z} \,.$$

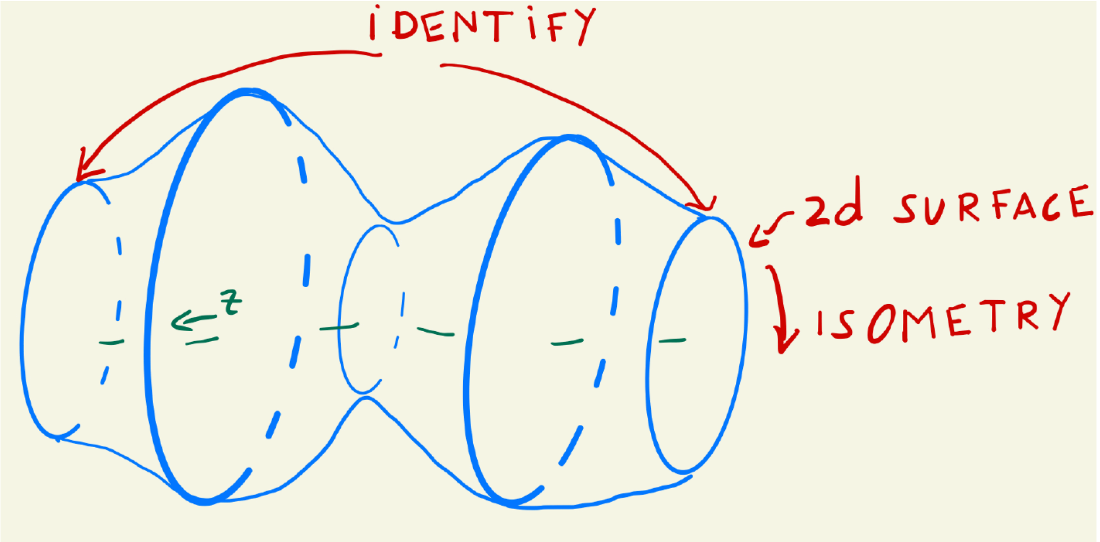
- So we see that the extrinsic curvature is small. Therefore, with $g^{\text{prod}} = dz^2 + g^{z_1}$

$$e^{-4\sqrt{2\varepsilon_\lambda (z_2 - z_1)}} \leq \frac{g(W,W)}{g^{\text{prod}}(W,W)} \leq e^{4\sqrt{2\varepsilon_\lambda (z_2 - z_1)}}$$

- Therefore there is a long distance over which the metric changes by less than $(1+\delta)$

$$d_\lambda^\delta := \frac{\log^2(1+\delta)}{32\varepsilon_\lambda} = \frac{1}{K_\Lambda}\frac{1}{64}\sqrt{\frac{3}{C_8}} \log^2(1+\delta) e^{\frac{1}{3}K_\Lambda^2 \lambda}$$

Steps of proof: (2)

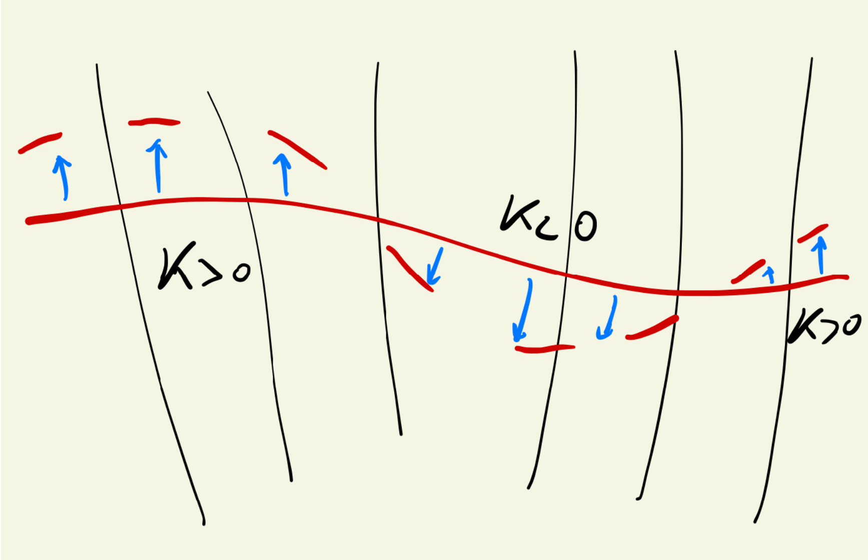


- New we show that the expansion rate converges in some average to the dS one.
- Recollect $\mathbf{g} := g(\lambda_0)e^{\frac{2}{3}K_\Lambda^2(\lambda-\lambda_0)}$
- Define $E(\lambda) := \int_{\mathcal{M}_\lambda} ||g(\lambda) - \mathbf{g}(\lambda)||^2_{\mathbf{g}(\lambda)} dV_{g(\lambda)}$
- Notice $E(\lambda_0) = 0$
- Evolution:

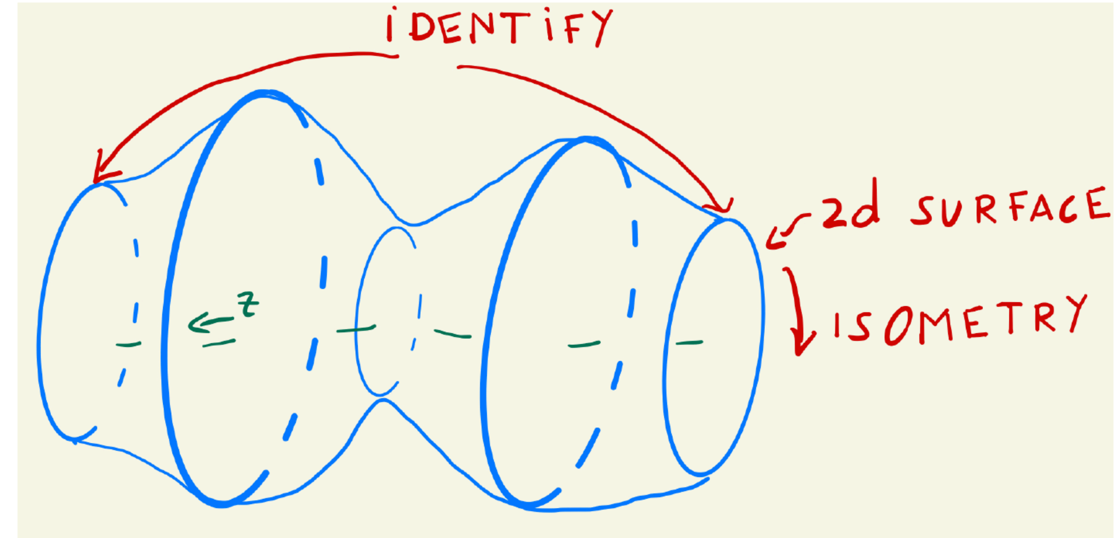
$$E'(\lambda) = \int_{\mathcal{M}_\lambda} K^2 ||g - \mathbf{g}||^2_{\mathbf{g}} dV_g + 2\int_{\mathcal{M}_\lambda} \langle g - \mathbf{g}, 2KK_{ij} - \frac{2}{3}K_\Lambda^2 \mathbf{g}\rangle_{\mathbf{g}} dV_g - \frac{4}{3}K_\Lambda^2 E(\lambda)$$

- Estimate all terms: $\int_{\mathcal{M}_\lambda} dV_g \, K^2 ||g - \mathbf{g}||^2_{\mathbf{g}} \leq K_\Lambda^2 \left(1 + \frac{3}{2}C_2 e^{-\frac{2}{3}K_\Lambda^2 \lambda}\right) E(\lambda)$
- .... $\Rightarrow \quad E \leq \dfrac{C_{10}}{K_\Lambda^3} e^{-\frac{2}{3}K_\Lambda^2\lambda_0} e^{K_\Lambda^2 \lambda}$, as good as could be.
  - –some of these terms require conditional proximity to the reference metric. $||g(\lambda') - \mathbf{g}(\lambda')||_{\mathbf{g}(\lambda')} < \gamma$
- Putting this with the fact that the metric is quasi constant in z, leads unconditionally:

$$\Rightarrow \quad ||g(\lambda) - \mathbf{g}(\lambda)||_{\mathbf{g}} \leq C_{11} e^{-\frac{1}{6}K_\Lambda^2 \lambda_0} \ ,$$



Steps of proof: (3)



– Closeness to dS spatial slices over exponentially expanding Balls:

– Define the spatial slices of dS:

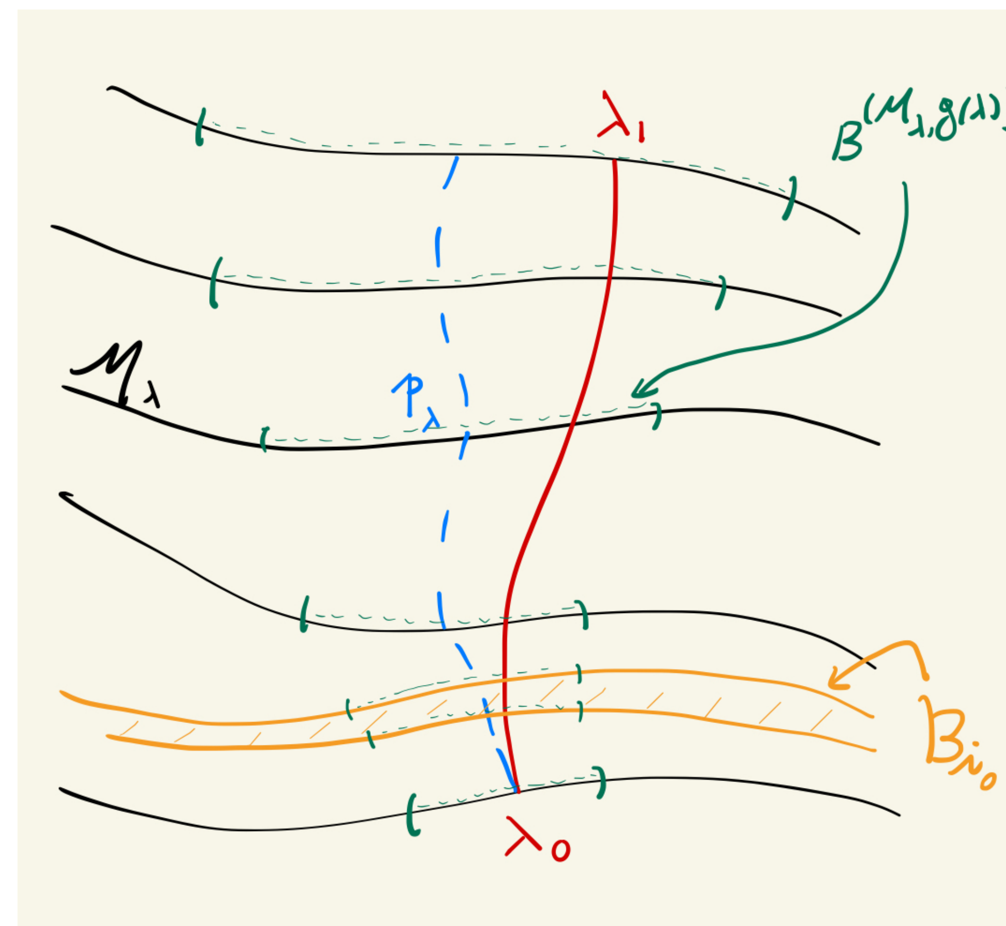
$$\mathbf{g}_{\mathrm{dS}}(\lambda) := e^{\frac{2}{3}K_\Lambda^2(\lambda-\lambda_0)} g_{\mathrm{Euc}} \ ,$$

– from some flow-time on:

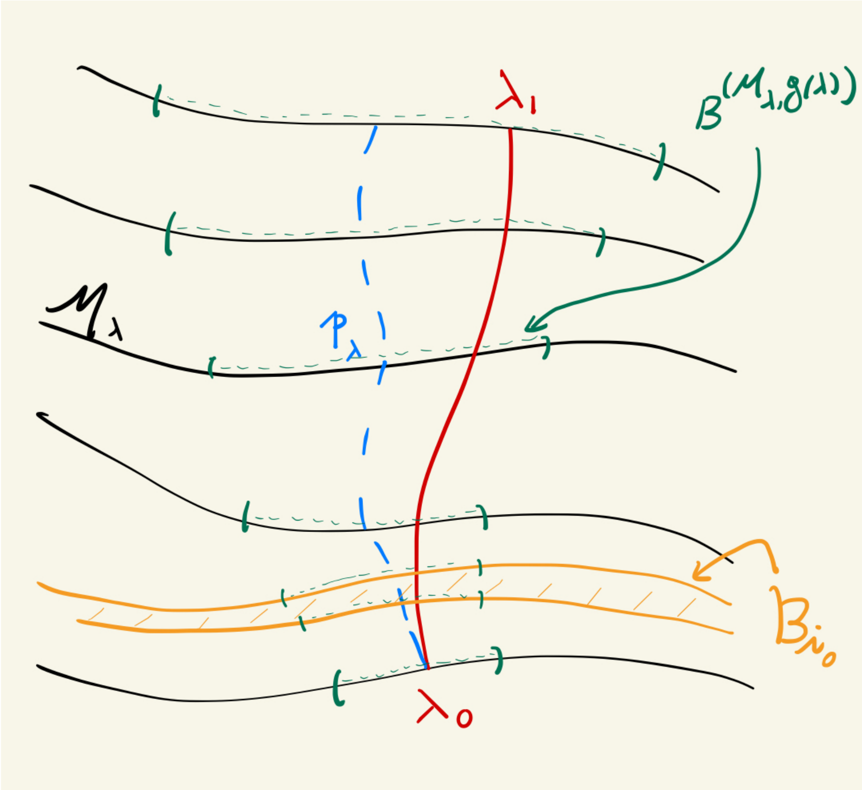
$$||g(\lambda) - \mathbf{g}_{\mathrm{dS}}(\lambda)||_{g(\lambda)} < e^{-\frac{1}{12}K_\Lambda^2\lambda_0} \ ,$$

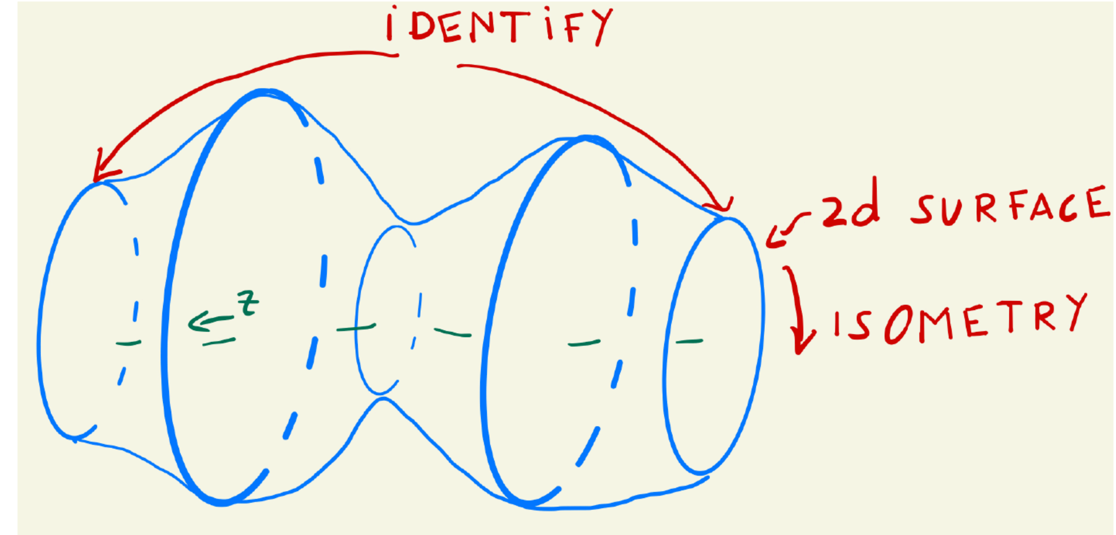
pointwise on $B^{(\mathcal{M}_\lambda, g(\lambda))}\left(p_\lambda, \frac{1}{K_\Lambda} e^{\frac{1}{12}K_\Lambda^2\lambda_0} \cdot e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_0)}\right)$.

Here p_λ *results from following* p *along the flow.*



Steps of proof: (3)





– Intuitive: the metric takes the form: $g = dz^2 + h_z$,

– Each 2-slice is expanding, so it gets flatter. Then also it is slowly varying in z .

– Furthermore, the growth is pointwise the same (by the former theorem), so, once they are close, they remain close.

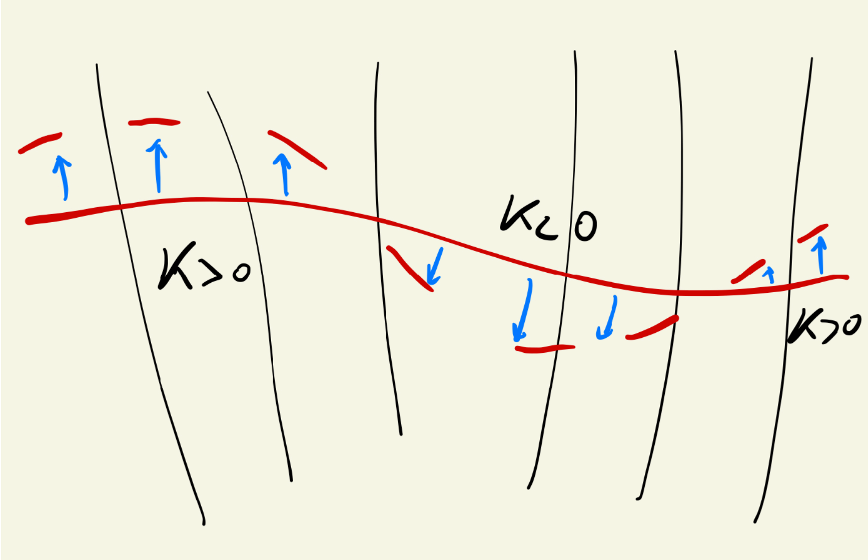
– $\Rightarrow$ Closeness to dS spatial slices over exponentially expanding Balls:

– Define the spatial slices of dS

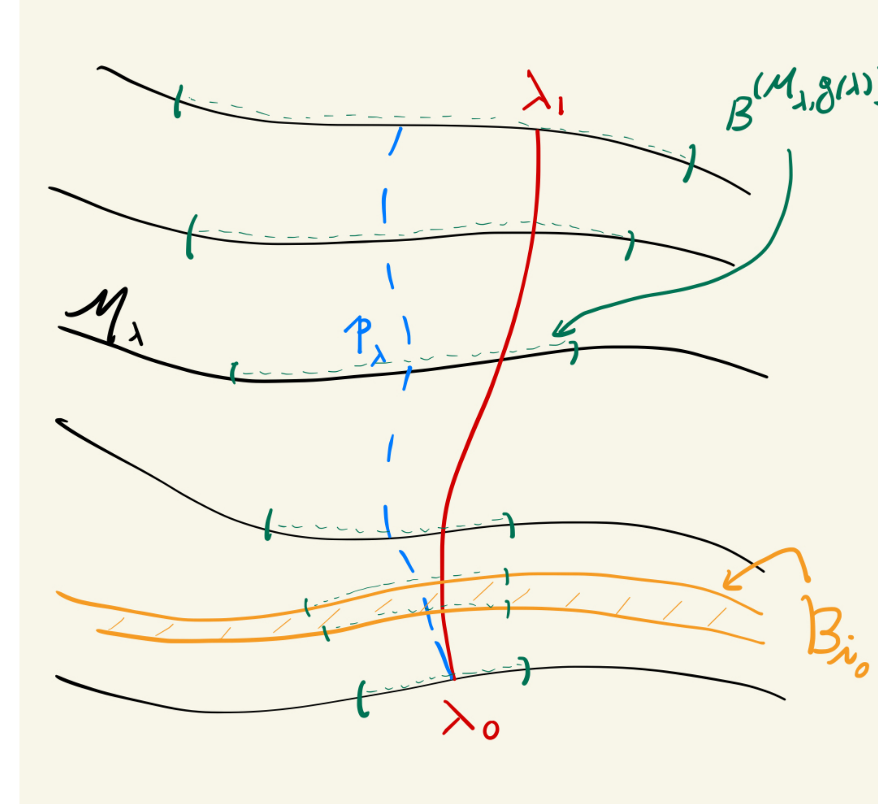
$$\mathbf{g}_{\mathrm{dS}}(\lambda) := e^{\frac{2}{3}K_\Lambda^2(\lambda-\lambda_0)} g_{\mathrm{Euc}}$$

$$||g(\lambda) - \mathbf{g}_{\mathrm{dS}}(\lambda)||_{g(\lambda)} < e^{-\frac{1}{12}K_\Lambda^2\lambda_0} ,$$

pointwise on $B^{(\mathcal{M}_\lambda, g(\lambda))}\left(p_\lambda, \frac{1}{K_\Lambda} e^{\frac{1}{12}K_\Lambda^2\lambda_0} \cdot e^{\frac{1}{3}K_\Lambda^2(\lambda-\lambda_0)}\right)$.
Here p_λ *results from following* p *along the flow.*



Steps of proof: (4)



– Spacetime Closeness to dS over exp. growing Balls:

– The flow defines a natural 4-metric

$$ds_4^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu = -K^2 d\lambda^2 + g_{ij} dx^i dx^j$$

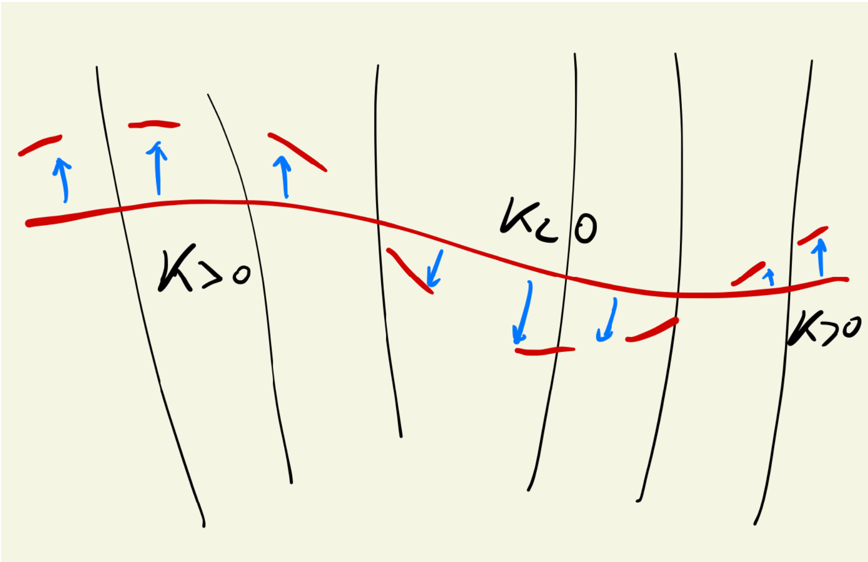
- foliates the whole spacetimes (so there are no singularities, geodesic completeness)

– Define the dS metric $ds^2_{\mathrm{dS}} := \mathbf{g}^{(4)}_{\mathrm{dS}} := -K_\Lambda^2 d\lambda^2 + (\mathbf{g}_{\mathrm{dS}})_{ij} dx^i dx^j.$

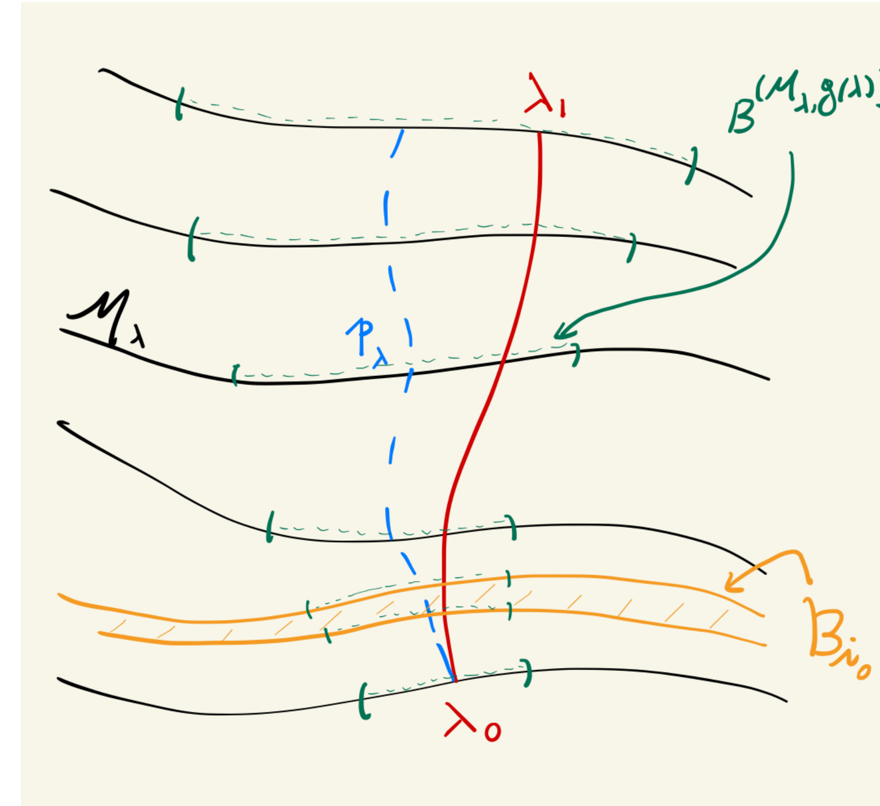
– Take any future oriented timelike or null curve

– from some flow-time on:

$$\left| L^{ds_4^2}[\gamma] - L^{ds^2_{\mathrm{dS}}}[\gamma] \right| \leq \frac{C_{13}}{K_\Lambda} e^{-\frac{1}{18} K_\Lambda^2 \lambda_0} + 8 K_\Lambda e^{-\frac{1}{24} K_\Lambda^2 \lambda_0} \left(\lambda_1 - \lambda_0 \right).$$



Steps of proof: (4)



- Given $ds_4^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu = -K^2 d\lambda^2 + g_{ij} dx^i dx^j$
- we need more control on K :
  - K is almost-always z independent:

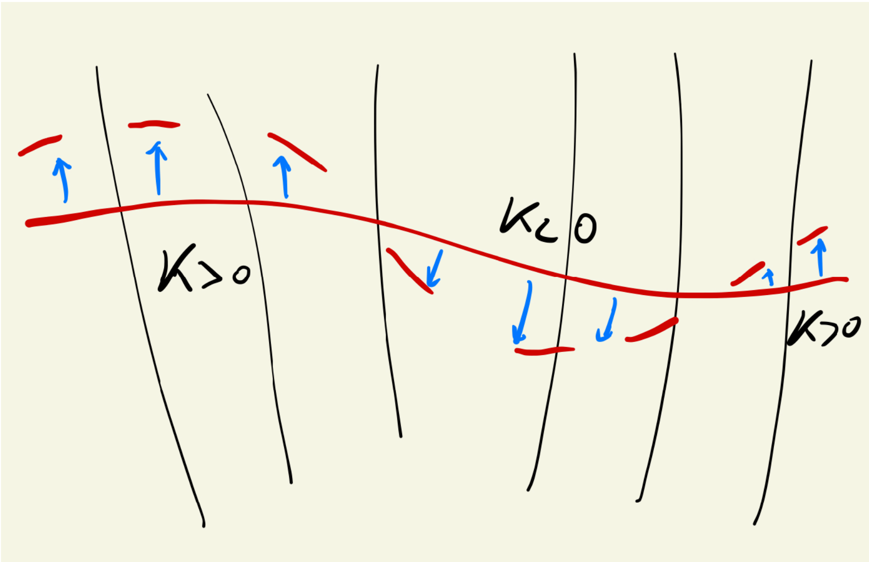
$$\frac{dK}{d\lambda} - \Delta K + \frac{1}{3} K \left(K^2 - K_\Lambda^2\right) + \sigma^2 K \le 0 \quad \Rightarrow \quad \int_\lambda^{\lambda + \frac{1}{K_\Lambda^2}} d\lambda' \int_{\mathcal{M}_{\lambda'}} dV \, |\nabla K|^2 \le \frac{C'_{12}}{K_\Lambda} \, e^{\frac{1}{3} K_\Lambda^2 \lambda}$$

- We can define Bad gradient times (with very small measure):

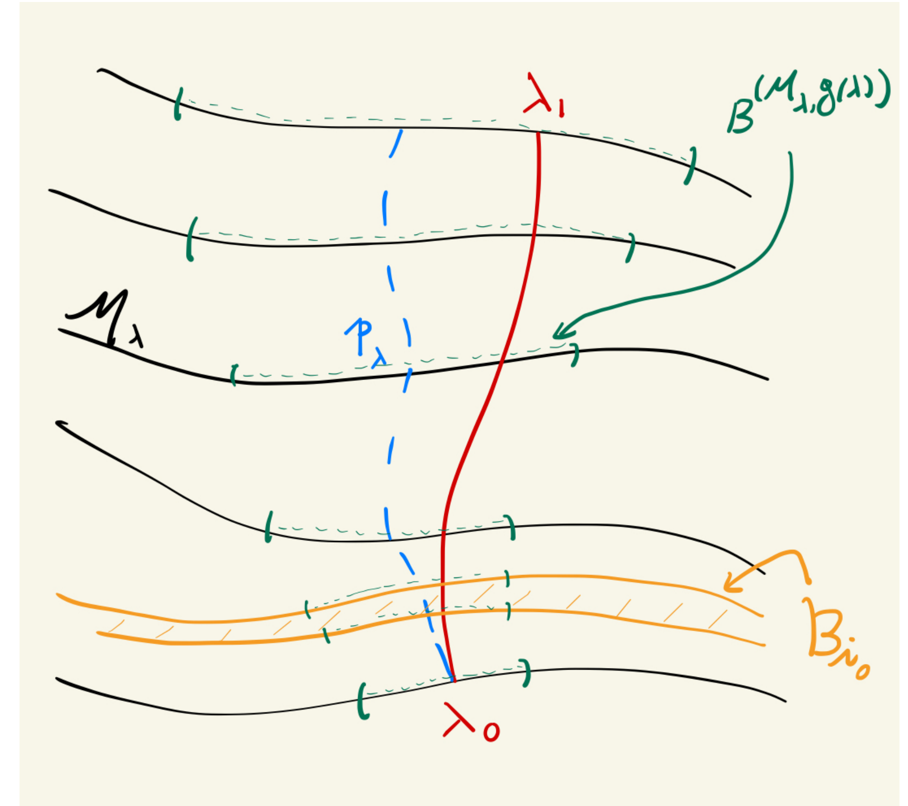
$$B_i := \{K_\Lambda^2 \lambda \in [i, i+1) \mid \int dz \, |\nabla K(\lambda, z)|^2 \ge C_{12} \, K_\Lambda^3 e^{-\frac{2}{9} i}\} \quad \Rightarrow \quad |\mathcal{B}_{i_0}| \le \frac{1}{K_\Lambda^2} \frac{e^{1/9}}{e^{1/9} - 1} e^{-\frac{i_0}{9}}.$$

- In the complementary (good gradient times): pointwise

$$|K - K_\Lambda| \le 2 K_\Lambda \sqrt{C_{12}} \, e^{-\frac{1}{9} i},$$



Steps of proof: (4)



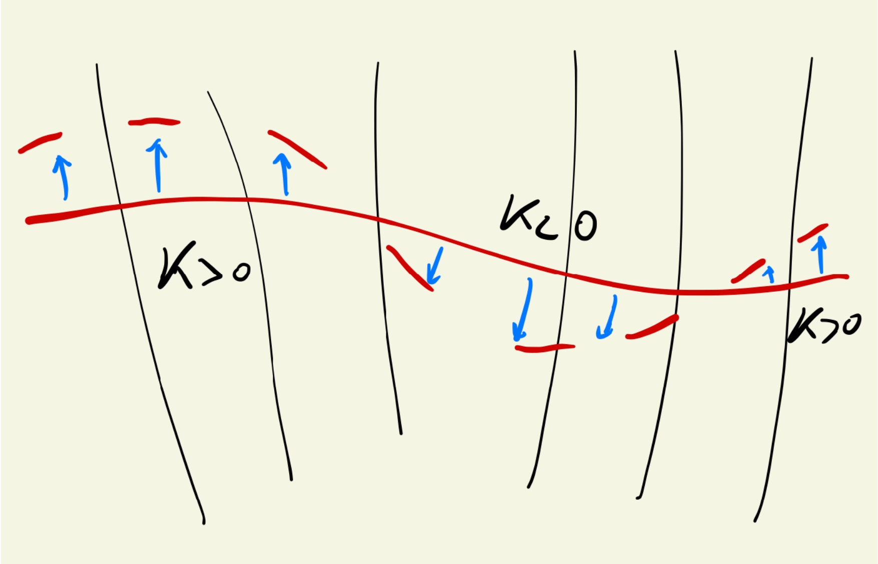
- Given $ds_4^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu = -K^2 d\lambda^2 + g_{ij} dx^i dx^j$
- Foliation of spacetime by this metric:

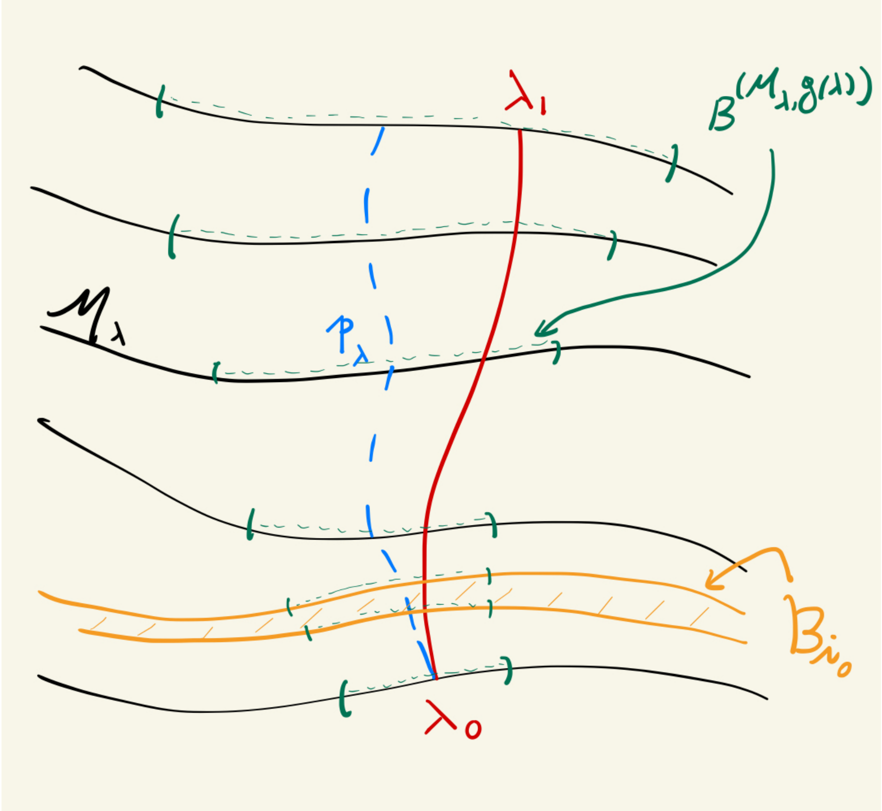
$$\frac{d}{d\lambda} t_{\rm min}(\lambda) = \frac{\partial t}{\partial \lambda}(x_\lambda, \lambda) = K(x_\lambda, \lambda) \quad \Rightarrow \quad t_{\rm min}(\lambda) = t_{\rm min}(\lambda_{0,4}) + \int_{[\lambda_{0,4},\lambda]} d\lambda'\; K(x_{\lambda'}, \lambda') \geq$$

$$\geq t_{\rm min}(\lambda_{0,4}) + \int_{[\lambda_{0,4},\lambda]\cap\mathcal{G}} d\lambda'\; K(x_{\lambda'}, \lambda') \geq t_{\rm min}(\lambda_{0,4}) + \frac{K_\Lambda}{2}\left(\lambda - \lambda_{0,4} - \frac{1}{K_\Lambda^2}\right) .$$

- so, no singularities, geodesic completeness.
- Since $K = K_\Lambda$ up to small quantities, *or* up to exponentially small time intervals (during which $K < 2K_\Lambda$), and since the spatial metric is always exponentially close to dS on exp. growing balls $\Longrightarrow$ upon integration on any curve, we get the same result.
  - –In particular, null curves stay in the ball (intuitive)

Steps of proof: (5)





–Dilution of matter:

–from some flow-time on

$$16\pi G_N \int dz\ |T_{\mu\nu} n^\mu n^\nu| = 16\pi G_N \int dz\ T_{\mu\nu} n^\mu n^\nu =$$

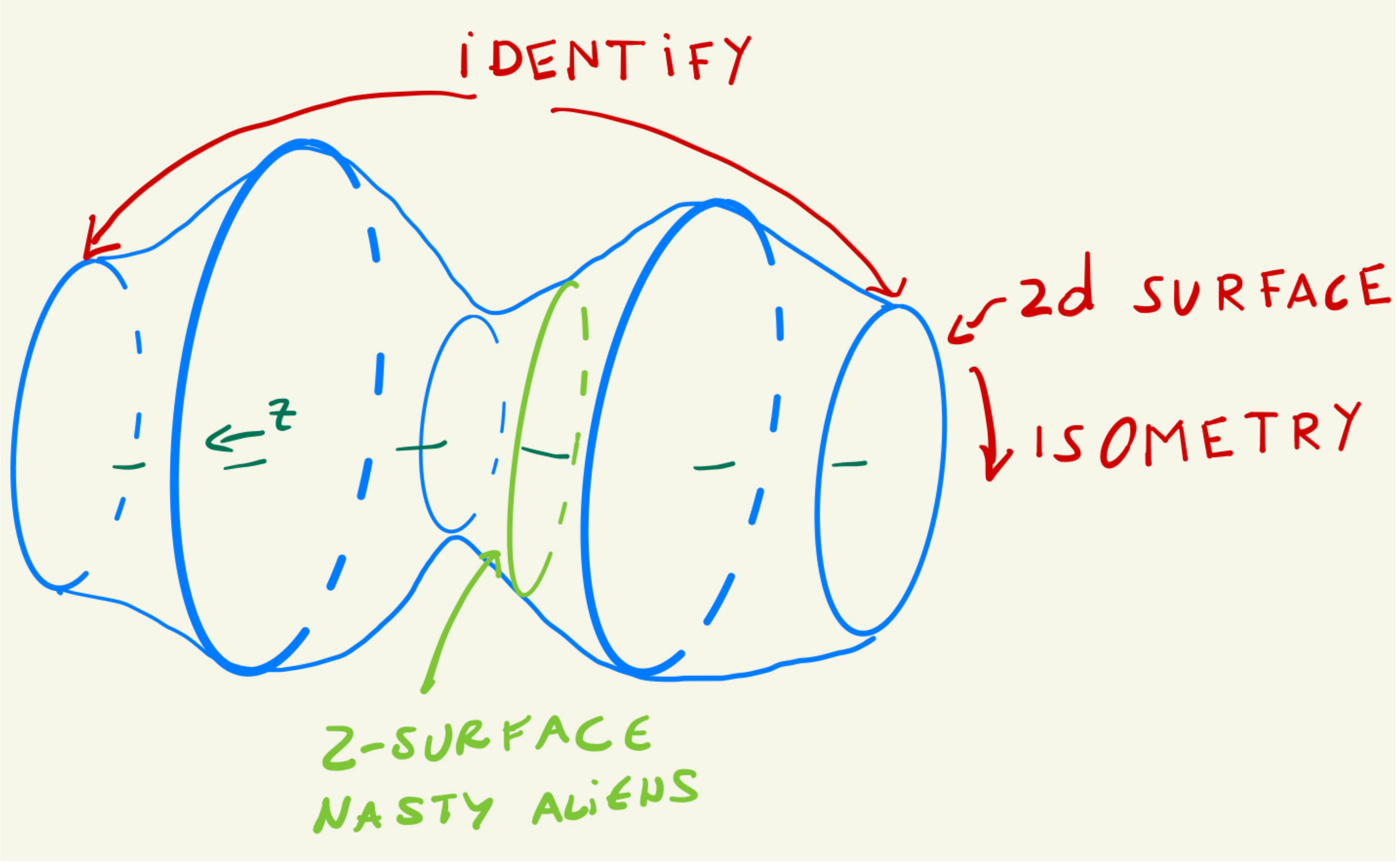
$$= \int dz\ \left({}^{(3)}R + \frac{2}{3}\left(K^2 - K_\Lambda^2\right) - \sigma^2 \right) \leq C_{14} K_\Lambda e^{-\frac{1}{3}K_\Lambda^2 \lambda}\ ,$$

$$16\pi G_N \int dz\ \left|T_{\mu\nu} e^{\mu a} e^{\nu b}\right| \leq 16\pi G_N \int dz\ T_{\mu\nu} n^\mu n^\nu \leq C_{14} K_\Lambda e^{-\frac{1}{3}K_\Lambda^2 \lambda}$$

–compatible with Israel junction condition for domain walls

Steps of proof: (5)

–How come we get pointwise converg.?



–Domain wall of aliens:

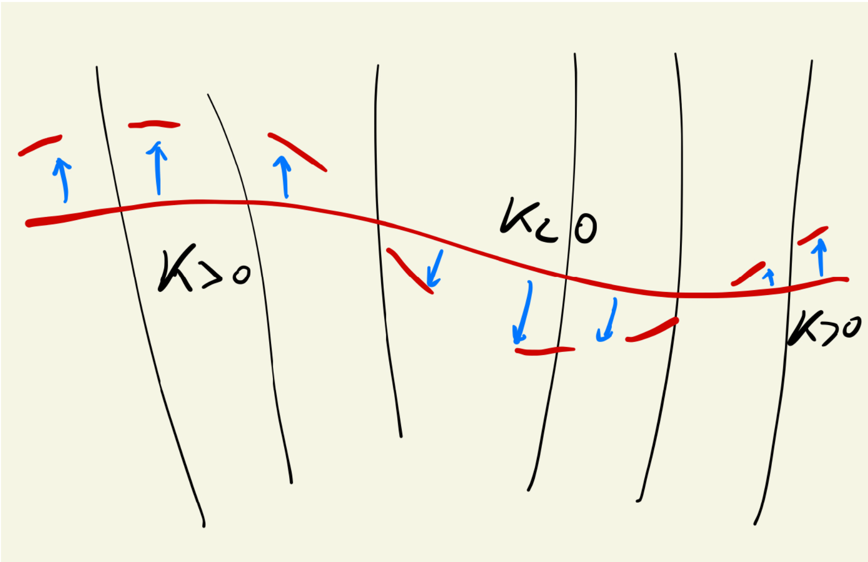
$$16\pi G_N \int dz \, \left|T_{\mu\nu} e^{\mu a} e^{\nu b}\right| \leq 16\pi G_N \int dz \, T_{\mu\nu} n^\mu n^\nu \leq C_{14} K_\Lambda e^{-\frac{1}{3}K_\Lambda^2 \lambda}$$

–Israel junction conditions: metric continues

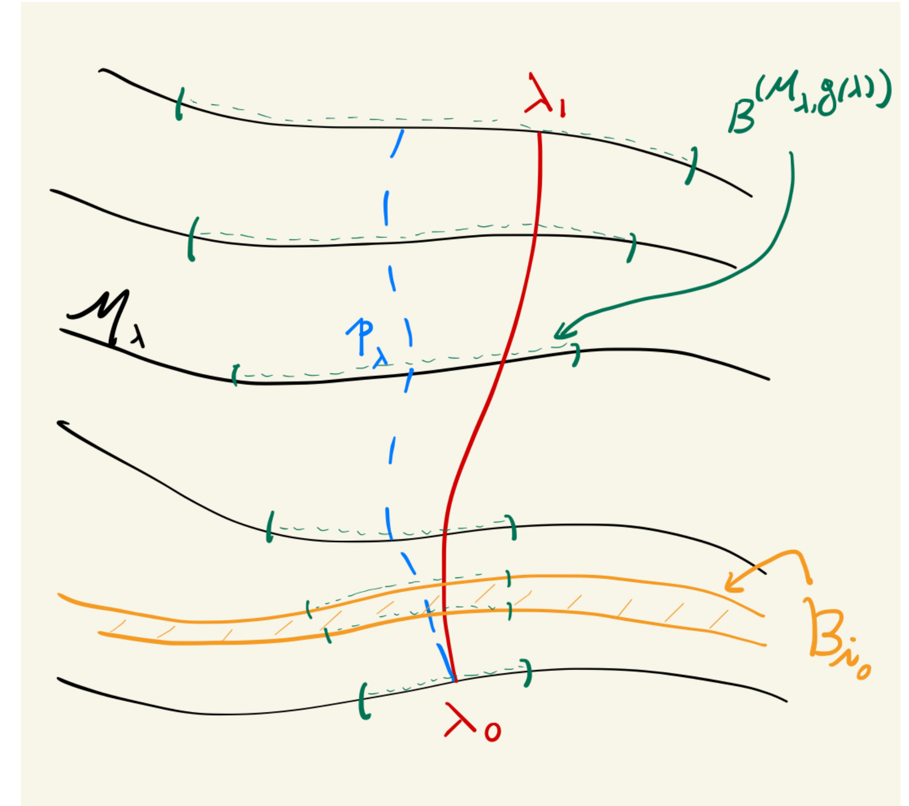
$$K^+_{\alpha\beta} - K^-_{\alpha\beta} = 8\pi G_N \left(S_{\alpha\beta} - \frac{g_{\alpha\beta}}{2} g^{\gamma\delta} S_{\gamma\delta} \right)$$

–Because of DES, $S \sim e^{-\frac{1}{3}K_\Lambda^2 \lambda}$

–In agreement with pointwise convergence of metric and with $|H| \leq \frac{2}{\sqrt{3}}\sqrt{C_8} K_\Lambda e^{-\frac{1}{3}K_\Lambda^2 \lambda}$



Steps of proof: (6)



- Physical equivalence to dS over exp. growing Balls
  - Consider an observer
  - equivalence of lengths
    - » $\Longrightarrow$ same geodesics
    - » $\Longrightarrow$ same horizon as in dS
  - Observer has access to matter only from finite volume. At all times, in this volume:

$$16\pi G_N \int_{\mathcal{M}_\lambda \cap y_\lambda(y_{\lambda_2}^{-1}(B_c(\lambda_2)))} |T_{\mu\nu} e^{\mu a} n^{\nu b}| \leq \frac{\pi (12)^2 C_{14}}{K_\Lambda} e^{-\frac{1}{3} K_\Lambda^2 \lambda_2} \ ,$$

  - available energy-momentum is below any threshold

Theorem

If the 2-surfaces have non-positive Euler characteristic (or in the case of 2-spheres, if the initial 2-spheres are large enough) and also if the initial spatial slice is expanding everywhere, then the asymptotically the spacetime becomes physically indistinguishable from de Sitter space on arbitrarily large regions of spacetime. This holds true notwithstanding the presence of initial arbitrarily-large density fluctuations and potential singularities.

with Creminelli, Hershkovits, Vasy **Advances in Math. 2023**

Mean Curvature Flow in de Sitter

with Hershkovits **2023**

Is Mean Curvature Flow good?

- The usage of MCF revealed itself useful for the 2-isometry case. But that was a symmetric case, with no singularities.
- In general, we expect singularities to form, so, we need to be able to describe some local regions of $\mathcal{M}_\lambda$
- We achieved this in with Hershkovits **2023**
- Consider a tube of de Sitter space:

$$\mathcal{N}_R = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} \mid |x| \leq R\}$$

$$g = e^{2t}(dx_1^2 + dx_2^2 + dx_3^2) - dt^2$$

Locally converging to FRW slicing

- **Theorem 1.3** (Main theorem - geometric version). *There exists a universal $R < \infty$ with the following significance: Let $(M_s)_{s\in[0,\infty)}$ be a graphical mean curvature flow in $\mathcal{N}_R \cap \{t \geq 0\}$ with bounded, non negative mean curvature, and with graphical function $u(x,s)$. Then*

(10) $$\lim_{\lambda\to\infty} u(0,\lambda) = \infty,$$

and setting $M_s^\lambda := O_{u(0,\lambda)}(M_{\lambda+s})$, we have

(11) $$M_s^\lambda \xrightarrow{C^\infty_{\text{loc}}(\mathbb{R}^3\times\mathbb{R})} \bar{M}_s$$

as $\lambda \to \infty$.

- where $\bar{M}_s = t^{-1}(3s)$ is te MCF given by the flat slices of the FRW slicing of dS.

Locally converging to FRW slicing

- Proof:
  - several ideas
  - perhaps most innovative/crucial:
    - study the evolution of $v^2\mu(r)$,
    - with $\mu(r) = (R - r)^p$

$$r(x_1, x_2, x_3, t) = e^{\alpha t}|x|^2$$

    - so effectively focus the study on the tube:

$$D_{\alpha,R} := \{(x,t) \mid e^{\alpha t}|x|^2 \leq R\}$$

    - eg:

$$\sup_{z \in M_s \cap D_{\alpha,R/2}} v \leq C$$