Secular growths and their relation to Equilibrium states in perturbative Quantum Field Theories

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Recent results based on a joint work:
with S. Galanda and L. Sangaletti [arXiv: 2312.00556]

Plan of the talk

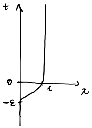
1. Secular effects as artefacts of perturbation theory in a simple toy model
2. Stability and return to equilibrium in quantum statistical mechanics
3. The case of interacting quantum field theories
4. Stability for generic states
5. A condition which avoids secular effects for generic states

References

- N. Drago, F. Faldino, np [arXiv:1609.01124 in CMP]
- N. Drago, J. Braga Vasconcellos, np, [arXiv:1906.04098 in AHP]
- S. Galanda, np, L. Sangaletti, [arXiv: 2312.00556]

Simple toy model

- Consider a scalar field with a mass which varies in time $m^2 + \delta m^2 \chi(t)$.



$$(\Box - m^2 - \chi\, \delta m^2)\phi = 0.$$

  where $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 0$ for $t < -\epsilon$ and $\chi(t) = 1$ for $t > 0$.

- A **pure**, **translation invariant**, **Gaussian state** of the corresponding quantised theory has the two-point function of the form:

$$\omega_2(x, y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{p}\, \overline{\xi_p(t_x)} \xi_p(t_y) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}.$$

  given in terms of the **modes** $\xi_p(t)$ which are solution of

$$\ddot{\xi}_p(t) + (|\mathbf{p}|^2 + m^2 + \delta m^2 \chi(t))\xi_p(t) = 0.$$

- If the state was prepared in the vacuum

$$\xi_p(t) = \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0}}, \quad t < -\epsilon \qquad \xi_p(t) = \alpha_p \frac{e^{i\omega_1 t}}{\sqrt{2\omega_1}} + \beta_p \frac{e^{-i\omega_1 t}}{\sqrt{2\omega_1}} \quad t > 0,$$

where $\omega_0 = \sqrt{|\mathbf{p}|^2 + m^2}$, $\omega_1 = \sqrt{|\mathbf{p}|^2 + m^2 + \delta m^2}$ and α_p, β_p are complex functions and β_p decays rapidly at large $\mathbf{p}$.

- At late time

$$\omega_2(x,y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{2\omega_1}\left(\overline{\alpha_p}\beta_p e^{-i\omega_1(t_x+t_y)} + \alpha_p\overline{\beta_p}e^{+i\omega_1(t_x+t_y)} + \dots\right)e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})},$$

  where we see the contribution which is **not invariant** under time translation $(t_x, t_y) \to (t_x + a, t_y + a)$.
- $\overline{\alpha_p}\beta_p$ decays rapidly for large p. The contribution which depends on $t_x + t_y$ remains bounded. **No secular** growths are visible at late time **in the exact solution**.
- Expanding in **powers** of δm^2 and truncating the power series at order n, there are contributions which grows as

$$O_n = C(t_x + t_y)^{n-5/2}(\delta m^2)^n,$$

- The presence of these **secular growths** is an **artefact of perturbation theory**. When they are present, perturbation theory is **not reliable**.
  - Examples: Dirac fields in external potential; perturbations over equilibrium states; interacting fields over Schwarzschild spacetimes

Question

Can we avoid these kind of problems?

Yes, if the state after the mass change is invariant under time translation (example **equilibrium states**)

Basic settings of quantum statistical mechanics

- Let $\mathcal{A}$ be the C^*-algebra describing the **observables** of the theory.

- **Time evolution** (also called **dynamics**) is described by a one-parameter group of $*-$automorphisms $t \mapsto \tau_t$, $\tau_t : \mathcal{A} \to \mathcal{A}$.

- A C^*-algebra $\mathcal{A}$ equipped with a continuous time evolution τ_t forms a C^*-**dynamical system**

- A **state** ω over $\mathcal{A}$ is a linear functional which is positive and normalized $\omega(1) = 1$.

C^*-dynamical systems and equilibrium states

Equilibrium states are characterized by the **Kubo Martin Schwinger (KMS)** condition

Definition (KMS states)

A state ω for $\mathcal{A}$, is a $(\beta, \tau_t)-$KMS state if $\forall A, B \in \mathcal{A}$ the map

$$t \mapsto \omega(A\tau_t(B))$$

can be extended to an analytic function in the strip $\Im(t) \in (0, \beta)$ and if

$$\omega(A\tau_{i\beta}(B)) = \omega(BA).$$

β is the inverse temperature.

- Suppose that in a representation π, τ_t is described by e^{itH} generated by H selfadjoint. If $e^{-\beta H}$ is trace class a **Gibbs state** is obtained averaging

$$\omega^\beta(A) = \mathrm{Tr}(\rho\pi(A)), \qquad \rho := e^{-\beta H}/\mathrm{Tr}(e^{-\beta H}) \qquad A \in \mathcal{A}$$

  where β is the inverse temperature.
- **Gibbs states** are KMS states
- KMS condition is meaningful for infinitely extended systems
- KMS states are stable under perturbation of the dynamics

Araki construction of perturbed KMS states

Consider a C^*-algebra $\mathcal{A}$ and $P = P^* \in \mathcal{A}$ the **perturbation Hamiltonian**.

Then the **perturbed dynamics** τ^P is such that

$$\tau_t^P(A) = U(t)\tau_t(A)U(t)^*, \qquad \text{where} \qquad -i\frac{d}{dt}U\Big|_{t=0} = P$$

where $U(t)$ is the cocycle generated by P, $U(t+s) = U(t)\tau_t U(s)$.

Theorem (Araki)

Let ω be an extremal $(\beta,\tau)-$KMS state and τ^P the perturbed dynamics. Consider

$$\omega^P(A) := \frac{\omega(AU(i\beta))}{\omega(U(i\beta))}$$

where $\omega(AU(i\beta))$ is the analytic continuation of $\omega(AU(t))$, then $\omega^P(A)$ is a $(\beta,\tau^P)-$KMS state.

Stability of KMS states for C^*-dynamical systems

If the following **clustering** condition holds for ω

$$\lim_{t\to\pm\infty} \omega(A\tau_t^P(B)) = \omega(A)\omega(B)$$

stability - return to equilibrium holds:

$$\lim_{t\to\infty} \omega(\tau_t^P(A)) = \omega^P(A)$$

[Haag Kastler Trych-Pohlmeyer, Bratteli Robinson, Bratteli Robinson Kishimoto]

- In *[Drago, Faldino, np]* it is proved that in the case of an scalar self interacting **quantum field theory** strong clustering **does not hold** under the adiabatic limit, and equilibrium states cannot be constructed with a limit $t \to \infty$.

- For this reason return to equilibrium does not hold in general.

- In *[Fredenhagen, Lindner]* the **Araki construction** is used to obtain equilibrium states for interacting quantum field theories constructed with perturbation theory

Real scalar quantum field

- The **Lagrangian density** of the theory on Minkowski space (sign. $(-,+,+,+)$)

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \lambda\phi^n$$

- **Observables** for the linear theory $\lambda = 0$, $(\Box - m^2)\phi = 0$
- They form a $*-$**algebra** (off-shell) $\mathcal{A}_q$ gen. by

$$\phi(f) = \int \phi(x)f(x)dx, \qquad f \in C_0^\infty(M).$$

(in a concrete representation $\phi(x)$ is a smooth field configuration)

- The **product** is given in terms of the causal propagator $\Delta = \Delta_R - \Delta_A$

$$e^{i\phi(f)} \star e^{i\phi(g)} = e^{-\frac{i}{2}\langle f,\Delta g\rangle} e^{i\phi(f+g)}$$

- $\mathcal{A}_q$ can be extended to $\mathcal{A}$ which contains normal ordered **local fields** like $\int \phi^n(x)f(x)dx$ in $\mathcal{F}_{loc}$. *[Hollands Wald, Brunetti Fredenhagen Köhler]* (Extensive use of Microlocal Analysis)

Algebra of interacting fields

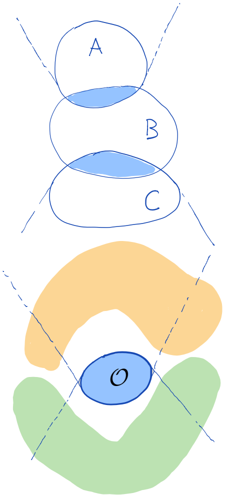
[Brunetti, Fredenhagen, Duetsch, Rejzner, Hollands, Wald]

- Consider an interaction Lagrangian $\mathcal{L}_I(\phi)$ (like $-\phi^n$) and the corresponding local functional
$$V \doteq \lambda \int g\mathcal{L}_I(\phi)dx$$
with the **cutoff** $g \in C_0^\infty(M)$, inserted to have $V \in \mathcal{F}_{\text{loc}}$.
- **Time ordered exponential** as an element of $\mathcal{A}[[\lambda]]$ (S-matrix)
$$S(V) \doteq \exp_T(iV).$$
- Relative $S-$matrices are then defined as
$$S_V(F) \doteq S(V)^{-1}S(V+F), \qquad F \in \mathcal{F}_{\text{loc}}.$$
- **Bogoliubov map** (also called Møller map)
$$R_V(F) \doteq -i\,\frac{d}{d\mu}S_V(\mu F)\bigg|_{\mu=0} = S(V)^{-1}T(e^{iV}F).$$
- **Interacting observables** supported in $\mathcal{O}$ can now be represented in $\mathcal{A}[[\lambda]]$ as the smaller subalgebra containing $R_V(F)$ for every $F \in \mathcal{F}_{\text{loc}}(\mathcal{O})$
$$\mathcal{A}_I(\mathcal{O}) \doteq [\{S_V(F) | F \in \mathcal{F}_{\text{loc}}(\mathcal{O})\}] \subset \mathcal{A}[[\lambda]].$$

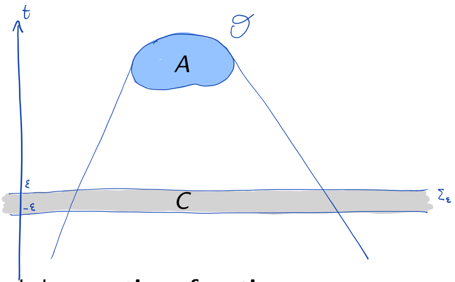
Adiabatic limit $g \to 1$

[Hollands Wald, Brunetti Fredenhagen]

- **Causal properties** of the S matrix. If $A \gtrsim C$
  $(J^+(\mathrm{supp} A) \cap J^-(\mathrm{supp} C) = \emptyset)$
  $$S(A+B+C) = S(A+B)\, S(B)^{-1}\, S(B+C).$$
  $$V = \lambda \int g \mathcal{L}_I dx$$
- If g, g' coincide on $J^+(\mathcal{O}) \cap J^-(\mathcal{O})$
  $$V' - V = W_+ + W_-$$
  with $\mathrm{supp} W_+ \cap J^-(\mathcal{O}) = \emptyset$ and $\mathrm{supp} W_- \cap J^+(\mathcal{O}) = \emptyset$.
- For interacting observables supported in $\mathcal{O}$, the map
  $$R_V(F) \mapsto R_{V'}(F) = S_V(W_-)^{-1}\, R_V(F)\, S_V(W_-)$$
  defined for $F \in \mathcal{F}_{\mathrm{loc}}(\mathcal{O})$,
  extends to an **isomorphism** $\mathcal{A}_I^g(\mathcal{O}) \to \mathcal{A}_I^{g'}(\mathcal{O})$.
- The limit $g \to 1$ can now be taken at algebraic level (direct limit).



Adiabatic limit $g \to 1$



- The $S-$matrix and the **equation of motion**:

$$S(V) \cdot_T \mathcal{L}^{(1)} = S(V) \star \mathcal{L}_0^{(1)}$$

- **Time slice axiom** permits to restrict observables on *[Chilian Fredenhagen]*

$$\Sigma_\epsilon \doteq \{p \in M \mid t(p) \in (-\epsilon, \epsilon)\} \ .$$

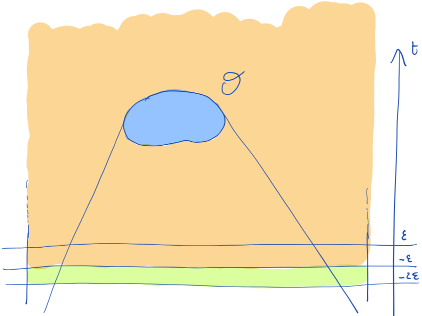
For every $A \in \mathcal{A}_I(\mathcal{O})$ it exists a $C \in \mathcal{A}_I(\Sigma_\epsilon \cap J(\mathcal{O}))$ such that

$$A = C + W$$

where $W \in \mathcal{A}_I$ vanishes on solutions hence

$$\omega(A) = \omega(C)$$

States in the adiabatic limit



- To construct a state for the int. alg. $\mathcal{A}_I(M)$ it suffices to know it on $\mathcal{A}_I(J^+(\Sigma_\epsilon))$.
- We choose a **cutoff function** g in the interaction Lagrangian of the form

$$g(t, \mathbf{x}) = \chi(t) h(\mathbf{x})$$

  - $\chi(t)$ is a smooth function which is equal to 1 for $t > -\epsilon$ and 0 for $t \leq -2\epsilon$.
  - h is a space cutoff which is compactly supported on Σ.

- To obtain a state in the **adiabatic limit**, we consider the limit where h **tends to** 1 keeping fixed the time cutoff χ.

Equilibrium states for the free theory

- A state is characterized by its $n-$point functions

$$\omega_n(f_1, \dots, f_n) = \omega(\phi(f_1)\dots\phi(f_n)), \qquad f_i \in C_0^\infty(M)$$

  a state is quasi-free (Gaussian) if its $n-$point functions can be given in terms of the two-point function only.

- Fix the spacetime to be **Minkowski**. The free time evolution is given in terms of time translations

$$\tau_t(\phi(f)) \doteq \phi(f_t)\ , \qquad f_t(s, \mathbf{x}) \doteq f(s-t, \mathbf{x})\ .$$

Proposition

It exists an unique quasifree KMS state ω^β at inverse temperature β wrt τ_t $(m > 0)$.

$$\widehat{\omega_2^\beta}(p) = \frac{1}{2\pi}\frac{1}{1-e^{-\beta p_0}}\delta(p^2+m^2)sign(p_0)$$

Interacting time evolution

- Time evolution $\tau_t F(\phi) \doteq F_t(\phi) \doteq F(\phi_t)$, $\qquad \phi_t(x) = \phi(x + te_0)$.
- The **interacting time evolution** τ_t^V in $\mathcal{A}_I(\mathcal{O})$ is such that

$$\tau_t^V(R_V(F)) \doteq R_V(F_t) \qquad F \in \mathcal{F}_{\mathrm{loc}}.$$

- The **causal factorisation property** implies that

$$\tau_t^V(R_V(F)) = S_V(V_t - V)\,\tau_t(R_V(F))\,S_V(V_t - V)^{-1}, \qquad F \in \mathcal{F}_{loc}(J^+\Sigma_\epsilon), t \geq 0,$$

where $U(t) \doteq S_V(V_t - V)$ are unitary elements which intertwines the free and interacting time evolutions. $U(t)$ satisfies the cocycle condition

$$U(t+s) = U(t)\tau_t U(s), \qquad H_I \doteq -i\,\frac{d}{dt}U(t)\Big|_{t=0},$$

where, H_I is the **interaction Hamiltonian** which is given in terms of the interaction Hamiltonian density $\mathcal{H}_I$ by

$$H_I = \int h(\mathbf{x})\mathcal{H}_I(\mathbf{x})d^3\mathbf{x}, \qquad \mathcal{H}_I(\mathbf{x}) \doteq \int \dot{\chi}(t)R_V(-\mathcal{L}_I(t,\mathbf{x}))dt.$$

Equilibrium state for the interacting theory

For every $A \in \mathcal{A}_I(J^+\Sigma_\epsilon)$ with fixed h

$$t \mapsto \omega^\beta(AU(t))$$

can be analytically continued to $\mathrm{Im}\, t \in (0, \beta)$. Hence,

$$\omega_h^{\beta,V}(A) \doteq \frac{\omega^\beta(AU(i\beta))}{\omega^\beta(U(i\beta))}, \qquad A \in \mathcal{A}_I(J^+\Sigma_\epsilon)$$

defines a $\beta-$KMS state with respect to τ_t^V.

- The state **does not** depend on χ
- If $m > 0$ the limit $h \to 1$ (**Adiabatic Limit**) can be taken thanks to suitable **clustering properties** of the truncated $n-$point functions for large spatial separations *[Fredenhagen Lindner]*

Expectation values in the state $\omega^{\beta,V}$ can be computed by the following formula

$$\omega_h^{\beta,V}(A) = \sum_n \int_{0\leq u_1 \leq \ldots u_n \leq \beta} du_1 \ldots du_n \int_{\mathbb{R}^{3n}} d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n h(\mathbf{x}_1)\ldots h(\mathbf{x}_n)$$
$$\omega_T^\beta(A; \tau_{iu_1}(\mathcal{H}_I(\mathbf{x}_1)); \ldots ; \tau_{iu_n}(\mathcal{H}_I(\mathbf{x}_n)))\,.$$

Here ω_T^β denotes the truncated functional associated to ω^β.

Thermal states in perturbation theory

- In this way one obtains the **KMS state** for the **interacting theory** in the **adiabatic limit**. *[Fredenhagen Lindner]*

- The case $\mathcal{L}_I = -\phi^4$, $m = 0$ can be treated with the use of the **thermal mass**. *[Drago, Hack, np]*.

$$:\phi^4:_\infty =:\phi^4:_\beta + M_\beta^2 :\phi^2:_\beta$$

- Limit $t \to \infty$ can be easily taken because $\omega^{\beta,V}$ is invariant under time translations

$$\omega^{\beta,V}(R_V(A)) = \omega^{\beta,V}(\tau_t^V R_V(A)) = \omega^{\beta,V}(R_V(A_t)) = \omega^{\beta,V_{-t}}(R_{V_{-t}}(A))$$

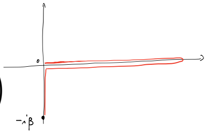
- In some cases, the obtained correlation functions differ from predictions in the traditional **Real Time formalisms**. (where equilibrium states are obtained by means of stability) *[Braga Vasconcellos, Drago,np]*.

Comparison with the physical literature

$$\omega^{\beta,V}(S_V(F)) = \frac{\omega^\beta(S_V U(i\beta))}{\omega^\beta(U(i\beta))} = \frac{\omega^\beta(S^{-1}S(V+F)U(i\beta))}{\omega^\beta(U(i\beta))}$$

A direct comparison requires a bit of work. Notice in particular that formally

$$S_V(F)U_V(t) = S_V\left(F - \int_0^t \tau_s \dot{V} ds\right) = \tilde{S}\left(F + \int_C \tau_s \dot{V} ds\right)$$



where C is the known **Keldysh contour** and $\tilde{S}$ is the time ordered exponential wrt C.

In the literature: two methods to study interacting field theory at finite temp:

- **Matsubara or imaginary time method**: Suited to compute correction to global thermodynamical quantities. Example

$$F = -\frac{1}{\beta}\log(\omega(U(i\beta)))$$

  It is not possible to compute the correlation functions of localized field in space.
- The **real time formalisms**: assuming stability the state is essentially constructed as

$$\lim_{t\to\infty}\omega^\beta(\tau_t^V R_V(F))$$

  however this fails sometime as we have seen above.

- $\omega^{\beta,V}$ contains corrections to the correlation functions already at lower orders.

Return to equilibrium and KMS condition

We start with an h of compact spatial support.

Theorem (Return to equilibrium)

If $V = V_{\chi h}$ *is a* **spatially compact** *interaction Lagrangian*

$$\lim_{t\to\infty} \omega^{\beta,V}(\tau_t(A)) = \lim_{t\to\infty} \frac{\omega^{\beta}(\tau_t(A)U(i\beta))}{\omega^{\beta}(U(i\beta))} = \omega^{\beta}(A)$$

where A *is an element of* $\mathcal{A}_I(\Sigma_\epsilon)$.

The limits are taken in the sense of perturbation theory.

Idea of the Proof:
Decay of ω_2^β implies $\omega_2^\beta(x, y + te_0) \leq \frac{C}{t^{3/2}}$ for $t >> 1$ *[Buchholz Bros]*. Hence we have clustering of ω^β.

$$\lim_{\tau\to\infty} \omega^{\beta}(A\tau_t(B)) = \omega^{\beta}(A)\omega^{\beta}(B)$$

from which we obtain the thesis.

Stability and KMS condition

Theorem (Stability)

If $V = V_{\chi,h}$ is a spatially compact interaction Lagrangian

$$\lim_{t\to\infty} \omega^{\beta}(\tau_t^V(A)) = \omega^{\beta,V}(A)$$

where A is an element of $\mathcal{A}_I(\Sigma_\epsilon)$.

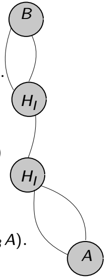
The limits are taken in the sense of perturbation theories.

Idea of the Proof: The following **clustering condition** holds

$$\lim_{t\to+\infty} \left[\omega^{\beta}(A\tau_t^V(B)) - \omega^{\beta}(A)\omega^{\beta}(\tau_t^V(B))\right] = 0,$$

for A and B in $\mathcal{A}_I(\mathcal{O})$,

$$\tau_t^V(B) = \tau_t(B) + \sum_{n\geq 1}(-i)^n \int_{0<t_1<\cdots<t_n<t} \left[\tau_{t_1}(H_I), \ldots, \left[\tau_{t_n}(H_I), \tau_t(B)\right]\right] dt_1 \ldots dt_n.$$



Now

$$\omega^{\beta}(\tau_t^V(A)) = \omega^{\beta}(\tau_{-t}\tau_t^V(A)) = \omega^{\beta}(U(-t)^{-1}AU(-t)) = \omega^{\beta}(U(-t)\tau_{i\beta}U(-t)^{-1}\tau_{i\beta}A)$$

where in the last equality we have used the **KMS condition**. The **co-cycle condition** for $U(t)$ implies that $\tau_{-t}(U(t))^{-1} = U(-t)$ and that $U(s)\tau_s(U(t)) = U(t+s) = U(t)\tau_t(U(s))$

$$\omega^{\beta}(\tau_t^V(A)) = \omega^{\beta}(U(-t)\tau_{-t}(U(i\beta)^{-1})U(-t)^{-1}U(i\beta)\tau_{i\beta}A) = \omega^{\beta}(\tau_{-t}^V(U(i\beta)^{-1})U(i\beta)\tau_{i\beta}A).$$

The clustering for τ_t^V

$$\lim_{t\to\infty} \omega^{\beta}(\tau_t^V(A)) = \omega^{\beta}(U(i\beta)\tau_{i\beta}A) \lim_{t\to\infty}(\omega^{\beta}(\tau_{-t}^V(U(i\beta)^{-1}))) = \omega^{\beta}(AU(i\beta))\omega^{\beta}(U(i\beta))^{-1}$$

Instabilities in the adiabatic limit - secular effects

- Under the adiabatic limit, the clustering condition fails at first order

$$\lim_{t\to\infty}\lim_{h\to 1}\left(\omega^{\beta}(A\tau_t(H_I)) - \omega^{\beta}(A)\omega^{\beta}(H_I)\right) \neq 0,$$

- **no return to equilibrium** is expected to hold.
- Counterexamples can be found:

  Consider the **ergodic mean** of $\omega^{\beta}\circ\tau_t^V$ to smoothen oscillations

$$\omega_T^{V,+}(A) \doteq \lim_{h\to 1}\frac{1}{T}\int_0^T \omega^{\beta}(\tau_t^V(A))dt$$

  the limit $T\to\infty$ produces a NESS.

Secular growth

Theorem

Let ω^β be an equilibrium state with respect to the free dynamics τ_t. If **return to equilibrium holds** *at all order in perturbation theory:*

$$\lim_{T\to\infty} \omega^\beta(\tau_T^V(A)) = \omega^{\beta,V}(A)\,, \quad \forall A \in \mathcal{A}$$

then **secular effects are absent***.*

Proof.

The proof is a consequence of invariance under time translations of the KMS states. □

- If $V = V_{\chi h}$ is spatially compact we have seen that return to equilibrium holds and hence **no secular growths** are present in that case.
- Once the large time limit is taken, also the adiabatic limit can be performed without introducing secular instabilities.
- If the limit $t \to \infty$ is taken after the limit $h \to 1$ we expect secular growth.

Generalization

Absence of secular growths for general states if the correlation functions satisfy certain conditions.

Theorem

Consider an interaction V **outside the adiabatic limit**. *Let ω be a state on $\mathcal{A}$ such that for every $\{A_i\}_{i\in\{1,\dots,n\}} \in \mathcal{A}$, $n \in \mathbb{N}$ the function:*

$$f_{A_1\dots A_n}(t_1,\dots,t_n) := \omega^{\mathcal{T}}(\tau_{t_1}(A_1)\otimes\cdots\otimes\tau_{t_n}(A_n))$$

is **secularly bounded**, *for $\omega^{\mathcal{T}}$ the truncated or connected functions of ω. Then, for every $A \in \mathcal{A}$ the following uniform bound holds in the sense of perturbation theory:*

$$|\omega(\tau_t^V(A))| \leq C.$$

In particular, **no secular effects** *are present.*

Proof: Write $\omega(\tau_t^V(A))$ in terms of sum of connected functions.

Secularily bounded

Definition

A function $f(t_1, \ldots, t_n)$ of real variables $(t_1, \ldots, t_n) \in \mathbb{R}^n$ is called **secularily bounded** *if it satisfies both the following two conditions:*

i) *f is* **absolutely integrable** *on $\mathbb{R}^{n-1}$ in the variables $t_{p_1}, \ldots, t_{p_{n-1}}$ for any choice of $\{p_1, \ldots, p_{n-1}\} \subset \{1, \ldots, n\}$:*

$$\int_{\mathbb{R}^{n-1}} |f(t_1, \ldots, t_n)| dt_{p_1} \cdots dt_{p_{n-1}} < \infty.$$

ii) *The function:*

$$g(t_{p_n}) := \int_{\mathbb{R}^{n-1}} |f(t_1, \ldots, t_n)| dt_{p_1} \cdots dt_{p_{n-1}}$$

satisfies a **bound** *$|g(t_{p_n})| \leq C_1$ for $C_1 \in \mathbb{R}^+$, for every $p_n \in \{1, \ldots, n\}$.*

Observation, this holds in particular if the **initial state** is quasi free, invariant under translations and

$$\left\langle \left| \partial_x^{(\alpha)} \partial_y^{(\beta)} \omega_2(x; y_0 + t, \mathbf{y}) \right| \right\rangle \leq \frac{C}{t^{1+\epsilon}}, \quad t > 1.$$

Summary

- Secular effects are artefacts of perturbation theory.
- They can be avoided if the final state is invariant under time translations.
- For equilibrium states this holds as a consequence of return to equilibrium.
- The condition can be generalized.

Thanks a lot for your attention

Graphical expansion of the correlation functions

Expectation values in the state $\omega^{\beta,V}$ can be computed by the following formula

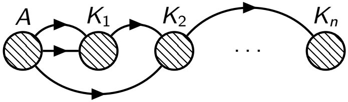
$$\omega_h^{\beta,V}(A) = \sum_n \int_{0\leq u_1 \leq \ldots u_n \leq \beta} du_1 \ldots du_n \int_{\mathbb{R}^{3n}} d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_n h(\mathbf{x}_1) \ldots h(\mathbf{x}_n)$$
$$\omega_T^{\beta}(A; \tau_{iu_1}(\mathcal{H}_I(\mathbf{x}_1)); \ldots ; \tau_{iu_n}(\mathcal{H}_I(\mathbf{x}_n)))$$

Here ω_T^{β} denotes the truncated functional associated to ω^{β}.

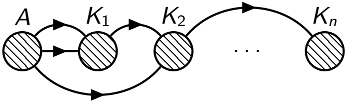
Let $\mathcal{G}_n$ be the set of connected graphs with n vertices.

$$\omega_T^{\beta}(A \otimes K_1 \otimes \cdots \otimes K_n) = \sum_{G \in \mathcal{G}^c_{n+1}} \frac{1}{\mathrm{Sym}(G)} \cdot$$
$$\cdot \left[\prod_{l \in E(G)} \int dx_l dy_l \, \Delta^{\beta}(x_l - y_l) \frac{\delta^2}{\delta\varphi_{s(l)}(x_l)\delta\varphi_{r(l)}(y_l)} \right] A \otimes K \otimes \cdots \otimes K \Big|_{\varphi_i = 0}$$

where the thermal propagator $\Delta^{\beta}(x-y) = \omega_2^{\beta}(x,y)$ is analytically continued



Comparison with the physical literature



In **physical literature**, there are two methods to analyze interacting field theory at finite temperature:

- **Matsubara or imaginary time method**: Suited to compute correction to global thermodynamical quantities. It is not possible to compute the correlation functions of localized field. ($\lim \chi \to \theta$)
- The **real time formalisms**: assuming stability the state is essentially constructed as

  $$\lim_{t\to\infty} \omega^{\beta}(\tau_t^V R_V(F))$$

  however this fails sometime.
- In *[Drago, Faldino, np]* it is proved that **clustering does not hold** in the adiabatic limit, hence the real time formalisms cannot be used in this case

- $\omega^{\beta,V}$ for a ϕ^4 theory contains corrections to the correlation functions already at lower orders. *[Braga Vasconcellos, Drago,np]*. back