The Euclidean vacuum state for linearized gravity on de Sitter spacetime

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CCR$*$-algebras

Let $(\mathcal{Y}, \mathbf{q})$ be a Hermitian space.

- One can introduce the abstract CCR $*$-algebra $\mathrm{CCR}(\mathcal{Y}, \mathbf{q})$ generated by the symbols $\psi(y), \psi^*(y)$ for $y \in \mathcal{Y}$ with relations:

$$\begin{array}{ll} 1) & \mathcal{Y} \ni y \mapsto \psi^*(y) \text{ resp. } \psi(y) \text{ linear resp. anti-linear,} \\ 2) & [\psi(y_1), \psi^*(y_2)] = \overline{y}_1 \cdot \mathbf{q} y_2 \mathbb{1}, \ y_1, y_2 \in \mathcal{Y}, \\ 3) & [\psi(y_1), \psi(y_2)] = [\psi^*(y_1), \psi^*(y_2)] = 0, \ y_1, y_2 \in \mathcal{Y}, \\ 4) & \psi(y)^* = \psi^*(y), \ y \in \mathcal{Y}. \end{array}$$

Quasi-free states

- A quasi-free state ω on $\mathrm{CCR}(\mathcal{Y}, \mathbf{q})$ is determined by a pair of Hermitian forms $\lambda^{\pm}$ on $\mathcal{Y}$ (called the covariances) by

$$\omega(\psi(y_1)\psi^*(y_2)) = \overline{y}_1 \cdot \lambda^+ y_2,$$
$$\omega(\psi^*(y_2)\psi(y_1)) = \overline{y}_1 \cdot \lambda^- y_2,$$
$$\omega(\psi(y_1)\psi(y_2)) = \omega(\psi^*(y_1)\psi^*(y_2)) = 0.$$

- Necessary and sufficient conditions for $\lambda^{\pm}$ to be covariances are

$$1) \quad \lambda^+ - \lambda^- = \mathbf{q} \text{ (CCR)},$$
$$2) \quad \lambda^{\pm} \geq 0 \text{ (positivity)}.$$

- Useful to introduce $c^{\pm} =: \pm \mathbf{q}^{-1} \circ \lambda^{\pm}$. Then $c^+ + c^- = \mathbb{1}$ and ω is pure iff $c^{\pm}$ are projections.

Quasi-free states for matter fields

- Let $(M, \mathbf{g})$ a globally hyperbolic spacetime, $V \xrightarrow{\pi} M$ a finite rank Hermitian bundle.
- Let D a second order differential operator acting on $C^\infty(M; V)$ such that $D = D^*$ with principal symbol $\xi{\cdot}\mathbf{g}^{-1}\xi \mathbb{1}_V$.
- standard example is the Klein-Gordon operator $D = -\Box$, acting on scalar functions.
- D has unique advanced/retarded inverses $G_{\rm ret/adv}$, $G := G_{\rm ret} - G_{\rm adv}$ is the commutator function.

The various Hermitian spaces

- 'off shell' Hermitian space is $\dfrac{C_0^\infty(M;V)}{DC_0^\infty(M;V)}$ with

$$\overline{[u]}\cdot\mathbf{Q}[u] = \mathrm{i}(u|Gu)_V.$$

- 'on shell' Hermitian space is $\mathrm{Ker}_{\mathrm{sc}} D$ (space of solutions) with

$$\overline{u}\cdot\mathbf{q}u = (u|[D, \mathrm{i}\mathbb{1}_{J^+(\Sigma)}]u)_V, \ \Sigma \text{ space-like Cauchy surface}$$

- 'Cauchy surface' Hermitian space is $C_0^\infty(\Sigma; V\otimes\mathbb{C}^2)$ with

$$\overline{f}\cdot\mathbf{q}_\Sigma f = \int_\Sigma (f_1|f_0)_V + (f_0|f_1)_V dVol_{\mathrm{h}}, \ f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

The various covariances

- All three Hermitian spaces are isomorphic. One can use any of the three to construct $\mathrm{CCR}(\mathcal{Y}, \mathbf{q})$.
- 'off shell' covariances: a pair $\Lambda^{\pm}: C_0^\infty(M; V) \to \mathcal{D}'(M; V)$ such that

$$(1)\quad D\circ\Lambda^{\pm} = \Lambda^{\pm}\circ D = 0 \text{ (field equation)},$$

$$(2)\quad \Lambda^{+} - \Lambda^{-} = \mathrm{i}G, \text{ (CCR)},$$

$$(3)\quad (u|\Lambda^{\pm}u)_V \geq 0,\ u\in C_0^\infty(M; V), \text{ (positivity)}.$$

- 'Cauchy surface' covariances: a pair $\lambda_\Sigma^{\pm}: C_0^\infty(\Sigma; V\otimes\mathbb{C}^2) \to \mathcal{D}'(\Sigma; V\otimes\mathbb{C}^2)$ such that:

$$(1)\quad \lambda_\Sigma^{+} - \lambda_\Sigma^{-} = \mathbf{q}_\Sigma, \text{ (CCR)},$$

$$(2)\quad (f|\lambda_\Sigma^{\pm}f)_{V\otimes\mathbb{C}^2} \geq 0, f\in C_0^\infty(\Sigma; V\otimes\mathbb{C}^2), \text{ (positivity)}.$$

The various covariances

- The two types of covariances are related by

$$\lambda_\Sigma^{\pm} = (\varrho^* \mathbf{q}_\Sigma)^* \Lambda^{\pm} (\varrho^* \mathbf{q}_\Sigma),$$
$$\Lambda^{\pm} = (\varrho G)^* \lambda_\Sigma^{\pm} (\varrho G),$$

where $\varrho u = \begin{pmatrix} u\restriction_\Sigma \\ \mathrm{i}^{-1}\nabla_n u\restriction_\Sigma \end{pmatrix}$ is the trace of u on Σ.

The Hadamard condition

- The Hadamard condition on $\Lambda^{\pm}$ singles out the physically meaningful states:
- $$\mathrm{WF}(\Lambda^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm},$$
  where:
- $$\mathcal{N} = \{(x,\xi) \in T^*M \setminus o : \chi \cdot \mathbf{g}^{-1}(x)\xi = 0\},$$
  characteristic manifold aka lightcone,
  $$\mathcal{N}^{\pm} = \text{ positive/negative energy components of } \mathcal{N},$$
- $\mathrm{WF}(\Lambda^{\pm})' \subset T^*(M \times M) \setminus o$ is the wavefront set of $\Lambda^{\pm} \in \mathcal{D}'(M \times M; V \boxtimes V)$ (distributional kernel of $\Lambda^{\pm}$).

Wick rotation

- Assume that $M = I_t \times \Sigma$, $\mathbf{g} = -dt^2 + \mathrm{h}_t(x)dx^2$ and h_t real analytic in t near $t = 0$.
- Wick rotation amounts to set $t =: \mathrm{i}s$ ($dt = \mathrm{i}ds$ etc). We obtain $\tilde{M} = \tilde{I}_s \times \Sigma$ with a metric $\tilde{\mathbf{g}} = ds^2 + \mathrm{h}_{\mathrm{i}s}(x)dx^2$.
- Note that $\tilde{\mathbf{g}}$ is in general not Riemannian.
- The operator D becomes $\tilde{D}$, which is elliptic, at least near $s = 0$.

Calderón projectors

- Let $\Omega^{\pm} = \tilde{M} \cap \{\pm s > 0\}$. For $u \in \overline{C^{\infty}}(\Omega^{\pm})$ we set

$$\tilde{\varrho} u = \begin{pmatrix} u\restriction_{\Sigma} \\ -\partial_s u\restriction_{\Sigma} \end{pmatrix}.$$

- Key fact: the spaces

$$E^{\pm} = \{\tilde{\varrho} u : u \in \overline{C^{\infty}}(\Omega^{\pm}),\ \tilde{D} u = 0 \text{ in } \Omega^{\pm}\}$$

are not equal to $C^{\infty}(\Sigma; \mathbb{C}^2)$: one cannot solve the Cauchy problem for an elliptic equation !

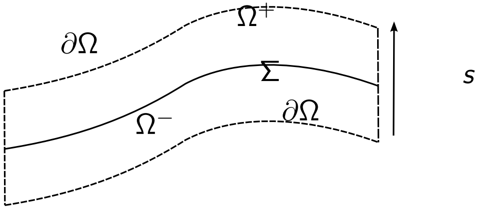
Calderón projectors

- The Calderón projectors $\tilde{c}^{\pm}$ are the projections on $E^{\pm}$ along $E^{\mp}$.
- This requires that

$$\begin{aligned} E^{+} \cap E^{-} &= \{0\}, \ (\tilde{D} \text{ injective}), \\ E^{+} + E^{-} &= C^{\infty}(\Sigma; \mathbb{C}^{2}), \ (\tilde{D} \text{ surjective}), \end{aligned}$$

- $\tilde{D}$ has to be defined as a linear operator, not only as a formal expression: boundary conditions on $\partial\Omega$!

Calderón projectors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Hadamard states from Calderón projectors

- For scalar fields one can put Dirichlet boundary conditions on $\partial\Omega$ to make $\tilde{D}$ invertible.

Theorem (GW)

Let

$$\lambda_{\Sigma}^{\pm} = \pm \mathbf{q}_{\Sigma} \circ \tilde{c}^{\pm}.$$

Then $\lambda_{\Sigma}^{\pm}$ are the Cauchy surface covariances of a Hadamard state.

- For ultrastatic spacetimes $\mathbf{g} = -dt^2 + \mathrm{h}$, $\tilde{\mathbf{g}} = ds^2 + \mathrm{h}$, the state obtained with Calderón projectors with no boundary conditions (ie $\tilde{I} = \mathbb{R}$) is the vacuum state.

Einstein's equations

- Einstein's equations:

$$\mathrm{Ric}(\mathbf{g}) = \Lambda \mathbf{g}, \ \Lambda \text{ cosmological constant.}$$

- fix a background metric $\mathbf{g}$ solution of Einstein equations

$$\mathrm{Ric}(\mathbf{g} + \epsilon u) - \Lambda(\mathbf{g} + \epsilon u) = \epsilon P u + O(\epsilon^2),$$

  for $u \in C^\infty(M; \otimes_{\mathrm{s}}^2 T^* M)$.

-
$$Pu = 0 \text{ linearized Einstein equations.}$$

- Similarly linearize a diffeomorphism χ around $\mathbb{1}$:

$$\chi^* = \mathbb{1} + \epsilon \mathcal{L}_v + O(\epsilon^2),$$

  $\mathcal{L}_v$ Lie derivative associated to the vector field v.

Some background

- Set $V_k = \mathbb{C} \otimes_{\rm s}^k T^*M$, $k = 0, 1, 2$.
  V_k equipped with canonical Hermitian form $(\cdot|\cdot)_{V_k}$
- *physical Hermitian form*: $(u|v)_{I,V_2} := (u|Iv)_{V_2}$,
- I *trace reversal*: orthogonal symmetry w.r.t. $\mathbb{C}\mathbf{g}$:

$$Iu_2 = u_2 - \frac{1}{4}\mathbf{g}(\mathbf{g}|u_2)_{V_2},$$

- one has

$$I^2 = \mathbb{1}, I\mathbf{g} = -\mathbf{g},\ I = I^* \text{ for } (\cdot|\cdot)_{V_2}.$$

Symmetric differential and co-differential

- symmetric differential: we set

$$d: \begin{array}{l} C^\infty(M; V_k) \to C^\infty(M; V_{k+1}) \\ (du)_{a_1 \dots, a_{k+1}} = \nabla_{(a_1} u_{a_2 \dots, a_{k+1})}, \end{array}$$

$u_{(a_1 \dots a_k)}$ is the symmetrization of $u_{a_1 \dots a_k}$,

- symmetric co-differential

$$\delta: \begin{array}{l} C^\infty(M; V_k) \to C^\infty(M; V_{k-1}) \\ (\delta u)_{a_1, \dots, a_{k-1}} = -k \nabla^a u_{a a_1 \dots a_{k-1}}. \end{array}$$

$d^* = \delta$ w.r.t. the Hermitian form $(\cdot|\cdot)_{V_k}$.

Linearized gravity as a gauge theory

- replace u_2 by Iu_2.
- P becomes

$$P = -\Box - I \circ d \circ \delta + 2\mathbf{Riem},$$

- set

$$K := I \circ d.$$

- The gauge invariance of P is expressed by

$$P \circ K = 0,$$

- u_2 and $u_2 + Ku_1$ are equivalent solutions of $Pu_2 = 0$.
- $$\text{canonical Hermitian space: } \frac{\mathrm{Ker}_{\rm sc} P}{\mathrm{Ran}_{\rm sc} K},$$

  ie solutions of linearized Einstein modulo gauge equivalence.

Quantization of linearized gravity

- To quantize linearized gravity we need to equip $\dfrac{\mathrm{Ker}_{\mathrm{sc}} P}{\mathrm{Ran}_{\mathrm{sc}} K}$ with a Hermitian form $\mathbf{q}_P$.
- $\overline{u}\cdot\mathbf{q}_P u := (u|I[P, \mathrm{i}\mathbb{1}_{J^+(\Sigma)}]u)_{V_2}$, for Σ space-like Cauchy surface, $u \in \mathrm{Ker}\, P$.
- $\mathbf{q}_P$ is independent on the choice of Σ.
- $\mathbf{q}_P$ passes to quotient on $\dfrac{\mathrm{Ker}_{\mathrm{sc}} P}{\mathrm{Ran}_{\mathrm{sc}} K}$ (ie $\overline{u}\cdot\mathbf{q}_P K v = 0$ for all $u \in \mathrm{Ker}_{\mathrm{sc}} P$, $v \in C^\infty_{\mathrm{sc}}(M; V_2)$).

Harmonic gauge

- P not hyperbolic (admits compactly supported solutions).
- one adds the gauge condition $K^\star u = 0$ ie $\delta u = 0$ (harmonic gauge condition).
- here $A^\star$ is the adjoint w.r.t. the physical Hermitian form

$$(u|u)_{I,V_2} = (u|Iu)_{V_2}.$$

- for any u_2 with $Pu_2 = 0$ there exists u_1 such that $K^\star(u_2 + Ku_1) = 0$.
- u_1 is unique modulo a solution of $K^\star K v_1 = 0$ (residual gauge freedom).

- It follows that

$$\frac{\mathrm{Ker}_{\mathrm{sc}} P}{\mathrm{Ran}_{\mathrm{sc}} K} \sim \frac{\mathrm{Ker}_{\mathrm{sc}} D_2 \cap \mathrm{Ker}_{\mathrm{sc}} K^\star}{K\, \mathrm{Ker}_{\mathrm{sc}} D_1},$$

  where

- $$D_2 := P + K \circ K^\star = -\Box + 2\mathbf{Riem},$$

  $$D_1 := K^\star \circ K = -\Box + \Lambda,$$

- $D_i = D_{i,L} - 2\Lambda$ where $D_{i,L}$ are the Lichnerowicz d' Alembertians, D_i are hyperbolic operators.
- They admit advanced/retarded inverses.

Further gauge fixing

- It is possible to impose further gauge fixing conditions, for example the traceless gauge

$$K_0^\star u_2 = 0$$

for $K_0^\star u_2 = -\mathrm{tr}_{\mathbf{g}} u_2$, $K_0 u_0 = u_0 \mathbf{g}$.

- One obtains then the equivalent Hermitian space

$$\frac{\mathrm{Ker}_{\mathrm{sc}}\, D_2 \cap \mathrm{Ker}_{\mathrm{sc}}\, K^\star \cap \mathrm{Ker}_{\mathrm{sc}}\, K_0^\star}{K\, \mathrm{Ker}_{\mathrm{sc}}\, D_1 \cap \mathrm{Ker}_{\mathrm{sc}}\, K_0^\star},$$

- It is also possible to change the gauge fixing condition.

$$\delta u_2 + \epsilon d \mathrm{tr}_{\mathbf{g}} u_2 = 0,$$

for $\epsilon \in \mathbb{R}$ has been used in the Euclidean framework.

- Leads to different operators D_i (leading term no more scalar).

- $\dfrac{\mathrm{Ker_{sc}}\, P}{\mathrm{Ran_{sc}} K}$ represents the 'on shell' Hermitian space.
- the corresponding ' off shell' Hermitian space is

$$\mathcal{V} = \frac{\mathrm{Ker_c}\, K^\star}{\mathrm{Ran_c} P}.$$

- One equips it with the Hermitian form

$$\overline{[u]} \cdot \mathbf{Q}[u] = \mathrm{i}(u|G_2 u)_{I,V_2},$$

for $G_2 = G_{2\mathrm{ret}} - G_{2\mathrm{adv}}$.

Cauchy surface Hermitian space

- We have $D_2 \circ K = K \circ D_1$ and (taking adjoints) $K^\star \circ D_2 = D_1 \circ K^\star$.
- therefore

$$K : \mathrm{Ker}_{\mathrm{sc}}\, D_1 \to \mathrm{Ker}_{\mathrm{sc}}\, D_2,$$

$$K^\star : \mathrm{Ker}_{\mathrm{sc}}\, D_2 \to \mathrm{Ker}_{\mathrm{sc}}\, D_1$$

- We denote by K_Σ, $K_\Sigma^\dagger$ the 'Cauchy data' versions of K, K^*.
- For example if $D_1 u_1 = 0$, $f_1 = \varrho_{1\Sigma} u_1$, then

$$K_\Sigma f_1 = \varrho_{2\Sigma} K u_1.$$

- Since $D_1 = K^\star \circ K$ we have

$$K_\Sigma^\dagger \circ K_\Sigma = 0.$$

Cauchy surface Hermitian space

- We have $I \circ D_2 = D_2 \circ I$, so $I : \mathrm{Ker}_{\rm sc}\, D_2 \to \mathrm{Ker}_{\rm sc}\, D_2$. ($I$ = trace reversal).
- We denote by I_Σ the Cauchy data version of I.
- The Cauchy surface Hermitian space is

$$\frac{\mathrm{Ker}_{\rm c}\, K_\Sigma^\dagger}{\mathrm{Ran}_{\rm c} K_\Sigma},$$

equipped with the Hermitian form

$$\overline{[f]}\cdot \mathbf{q}_{2,I}[f] = (f|\mathbf{q}_{2\Sigma} \circ I_\Sigma f)_{V_2 \otimes \mathbb{C}^2}, \ \ f \in \mathrm{Ker}_{\rm c}\, K_\Sigma^\dagger.$$

Off-shell covariances

Let $\Lambda_2^{\pm} \in L(C_0^\infty(M; V_2); C_0^\infty(M; V_2)^*)$.

- assume that

$$(1) \quad D_2 \circ \Lambda_2^{\pm} = \Lambda_2^{\pm} \circ D_2 = 0 \text{ (field equation)},$$

$$(2) \quad \Lambda_2^{+} - \Lambda_2^{-} = \mathrm{i} G_2 \text{ on } \operatorname{Ker}_{\mathrm{c}} K^\star \text{ (CCR)},$$

$$(3) \quad \Lambda_2^{\pm} = 0 \text{ on } \operatorname{Ker}_{\mathrm{c}} K^\star \to \mathrm{Ran}_{\mathrm{c}} KK^\star \text{ (gauge invariance)},$$

$$(4) \quad (u|I\Lambda_2^{\pm} u)_{V_2} \geq 0, \ \forall u \in \operatorname{Ker}_{\mathrm{c}} K^\star \text{ (positivity)}.$$

- Then

$$\overline{[u]} \cdot \Lambda_2^{\pm}[u] := \overline{u} \cdot \Lambda_2^{\pm} u$$

are the covariances of a quasi-free state on $\left(\frac{\operatorname{Ker}_{\mathrm{c}} K^\star}{\mathrm{Ran}_{\mathrm{c}} P}, \mathbf{Q}\right)$.

Cauchy surface covariances

- Let $\lambda_{2\Sigma}^{\pm} : C_0^{\infty}(\Sigma; V_2 \otimes \mathbb{C}^2) \to \mathcal{D}'(\Sigma; V_2 \otimes \mathbb{C}^2)$ a pair of Cauchy surface covariances. We set $\lambda_{2\Sigma}^{\pm} =: \pm \mathbf{q}_{2\Sigma} \circ c_2^{\pm}$.
- Assume that

$$(1) \quad c_2^{+} + c_2^{-} = \mathbb{1} \text{ on } \operatorname{Ker}_{\rm c} K_{\Sigma}^{\dagger} \text{ (CCR)},$$

$$(2) \quad c_2^{\pm} : \mathrm{Ran}_{\rm c} K_{\Sigma} \to \mathrm{Ran} K_{\Sigma} \text{ (strong gauge invariance)},$$

$$(3) \quad \pm (f | I_{\Sigma} \mathbf{q}_{2\Sigma} c_2^{\pm} f)_{V_2 \otimes \mathbb{C}^2} \geq 0, \ \forall f \in \operatorname{Ker}_{\rm c} K_{\Sigma}^{\star} \text{ (positivity)}.$$

- Then $\overline{[f]} \cdot \lambda_{2\Sigma}^{\pm}[f] := \overline{f} \cdot \lambda_{2\Sigma}^{\pm} f$ are the Cauchy surface covariances of a quasi-free state on $\mathrm{CCR}(\frac{\operatorname{Ker}_{\rm c} K_{\Sigma}^{\dagger}}{\mathrm{Ran}_{\rm c} K_{\Sigma}}, \mathbf{q}_{2,I})$.

Cauchy surface covariances

- One can ask when $\Lambda_2^{\pm} = (\varrho_2 G_2)^* \lambda_{2\Sigma}^{\pm}(\varrho_2 G_2)$ generate a quasi-free state on $(\frac{\mathrm{Ker}_c\, K^\star}{\mathrm{Ran}_c P}, \mathbf{Q})$.
- leads to the weaker condition:

$$c_2^{\pm} : \mathrm{Ran}_c K_\Sigma \to \mathrm{Ran} K_\Sigma K_\Sigma^\dagger \text{ (weak gauge invariance)}$$

Hadamard condition

- In addition to the above conditions, we require the Hadamard condition ie
- $$\mathrm{WF}(\Lambda_2^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm},$$

  or equivalently

  $$\mathrm{WF}(U_{2\Sigma} \circ c_2^{\pm})' \subset \mathcal{N}^{\pm} \times T^*\Sigma,$$

Existence of Hadamard states

- Theorem (G 2023)

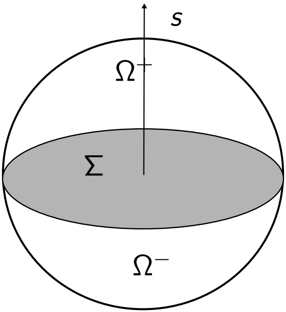
  Let $(M, \mathbf{g})$ be any Einstein manifold with compact Cauchy surfaces. Then there exist gauge invariant Hadamard states for linearized gravity on $(M, \mathbf{g})$.

  - The proof relies on pdo calculus and uses full gauge fixing:
  - this amounts to find a convenient supplementary space to $\mathrm{Ran} K_{\Sigma}$ inside Ker $K_{\Sigma}^{\dagger}$. The delicate gauge invariance property can now be forgotten.

de Sitter spacetime

- the de Sitter spacetime dS^4 is $\mathbb{R}_t \times \mathbb{S}^3$, equipped with the metric
$$\mathbf{g} = -dt^2 + \cosh^2(t)\mathrm{h},$$
h canonical metric on $\mathbb{S}^3 = \Sigma$.
- By Wick rotation $t \mapsto is$ we obtain the metric $\tilde{\mathbf{g}} = ds^2 + \cos^2(s)\mathrm{h}$, $s \in]-\pi/2, \pi/2[$,
- ie the sphere $\mathbb{S}^4$ by setting
$$x_0 = \sin s, \ (x_1, \dots, x_4) = \cos s \ \omega, \ \omega \in \mathbb{S}^3.$$
- The Wick rotations of D_i are denoted by $\tilde{D}_i$.
- They are selfadjoint for the natural scalar products on $\mathbb{S}^4$.

Wick rotated de Sitter spacetime



Euclidean dS^4

Wick rotated de Sitter spacetime

- The Wick rotation of dS^4 is compact: no need for boundary conditions to define $\tilde{D}_1$, $\tilde{D}_2$!
- $\tilde{D}_2$ is invertible,
- $\tilde{D}_2 \geq 2$ on $\mathrm{Ker}(\tilde{\mathbf{g}}|$ (traceless symmetric 2-tensors on $\mathbb{S}^4$).
- $\tilde{D}_1$ is not invertible,
- $\mathrm{Ker}\,\tilde{D}_1 = \mathrm{Ker}\,d = \mathrm{Ker}\,d \cap \mathrm{Ker}\,\delta =$ space of Killing 1-forms on $\mathbb{S}^4$.

Calderón projectors

- By Wick rotating the identity $D_2 \circ K = K \circ D_1$ we obtain

$$\tilde{D}_2 \circ \tilde{K} = \tilde{K} \circ \tilde{D}_1, \ (\tilde{K} = \tilde{I} \circ \tilde{d}).$$

- If Calderón projectors $\tilde{c}_i^{\pm}$ exist for $\tilde{D}_i$ then one would have

$$\tilde{c}_2^{\pm} \circ \tilde{K}_{\Sigma} = \tilde{K}_{\Sigma} \circ \tilde{c}_1^{\pm}$$

hence $\tilde{c}_2^{\pm}$ preserves $\mathrm{Ran}\tilde{K}_{\Sigma}$: one would get strong gauge invariance !

Calderón projectors

- $\tilde{c}_2^{\pm}$ exist since $\tilde{D}_2$ is invertible.
- $\tilde{c}_1^{\pm}$ do not exist on the whole space $C^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$, since $\tilde{D}_1$ is not invertible.
- this problem is due to the existence of Killing one-forms !
- it is still present with any of the alternative gauge fixing conditions explained above.

The Euclidean vacuum state

- Let us define Cauchy surface covariances

$$\overline{f}_2 \cdot \lambda^{\pm}_{2\Sigma} f_2 = \pm (f_2 | q_{2\Sigma} I_\Sigma \tilde{c}^{\pm}_2 f_2)_{V_2 \otimes \mathbb{C}^2}.$$

- We call the associated (pseudo) state $\omega_{\rm vac}$ the Euclidean (pseudo) vacuum.
- (pseudo reflects the fact that $\omega_{\rm vac}$ is not positive).
- use the 'mode expansion method' as a way to label states: $\omega_{\rm vac}$ is a Bunch Davies state.
- We will see that $\omega_{\rm vac}$ has nevertheless some problems.

Study of $\mathcal{E}_{\rm TT}$

- We denote by $\mathcal{E}_{\rm TT}$ the space of Cauchy data of transverse-traceless solutions $\mathrm{Ker}\, D_2 \cap \mathrm{Ker}\, K^\star \cap \mathrm{Ker}\, K_0^\star$, equipped with the Hermitian form $\mathbf{q}_{2,I}$.
- $\mathbf{q}_{2,I}$ is degenerate on $\mathcal{E}_{\rm TT}$, with an explicit 10-dimensional kernel $\mathcal{E}_{\rm TT,sing}$.
- We set $\mathcal{E}_{\rm TT,reg} = \mathcal{E}_{\rm TT,sing}^{\perp}$, orthogonal for the Riemannian scalar product.
- $\mathcal{E}_{\rm TT,sing}$ is included in $\mathrm{Ran} K_\Sigma$ ie consists of pure gauge solutions. Therefore $\mathbf{q}_{I,2}$ is non degenerate on the quotient space $\dfrac{\mathcal{E}_{\rm TT}}{\mathrm{Ran} K_\Sigma \cap \mathcal{E}_{\rm TT}}$.

Action of de Sitter symmetries on $\mathcal{E}_{\rm TT}$

- One can check that $\mathcal{E}_{\rm TT}$ is invariant under the full $O(1,4)$ symmetry group of dS^4.
- The only precaution is to implement time reversal $\tau : (t, \omega) \mapsto (-t, \omega)$ antilinearly ie by $u \mapsto \overline{\tau^* u}$ (Wigner time reversal).
- The same invariance is true of $\mathcal{E}_{\rm TT} \cap {\rm Ran} K_{\Sigma}$.

The Euclidean vacuum on $\mathcal{E}_{\rm TT}$

We now examine the properties of $\omega_{\rm vac}$ on $\mathcal{E}_{\rm TT}$. We start by the gauge invariance.

- Let $\mathcal{K} \subset \text{Ker}\, D_1$ the 10 dimensional space of Killing 1-forms on dS^4 and $\mathcal{K}_\Sigma = \varrho_1 \mathcal{K}$.
- $\lambda^{\pm}_{2\Sigma}$ are strongly gauge invariant only under gauge transformations in $\mathcal{K}^{\mathbf{q}_1}_\Sigma$, the orthogonal for $\mathbf{q}_1$ of $\mathcal{K}_\Sigma$.
- $\mathcal{K}^{\mathbf{q}_1}_\Sigma$ is the subspace on which the Calderón projectors $\tilde{c}^{\pm}_1$ for $\tilde{D}_1$ are well defined.
- But: $\lambda^{\pm}_{2\Sigma}$ are weakly gauge invariant !
- the space-time covariances $\Lambda^{\pm}_2$ are hence gauge invariant.

The Euclidean vacuum on $\mathcal{E}_{\rm TT}$

We now examine the positivity and Hadamard property of $\omega_{\rm vac}$.

- The Hadamard property is rather easy: $\omega_{\rm vac}$ satisfies the Hadamard condition, as does any state constructed with Calderón projectors.
- $\lambda_{2\Sigma}^{\pm}$ are not positive on the whole of $\mathcal{E}_{\rm TT}$. Their inertia indices are $(6, +\infty)$. They are negative definite on an explicit 6 dimensional subspace, included in $\mathcal{E}_{\rm TT, sing}$.
- This is the most delicate part of the analysis. It relies on the partial gauge invariance of $\lambda_{2\Sigma}^{\pm}$.
- as a consequence $\omega_{\rm vac}$ is not positive on the whole phase space.

The Euclidean vacuum on $\mathcal{E}_{\mathrm{TT}}$

Finally we examine the invariance of ω_{vac} under $O(1,4)$

- ω_{vac} is invariant under the full symmetry group $O(1,4)$.
- of course time reversal has to be implemented antilinearly.

Construction of α-vacua

α-vacua were discovered in the 80's for scalar fields on de Sitter. Their construction is made obscure by the use of mode expansions. It is actually very simple:

- Let $S : u \mapsto \tau^* u$ be the Racah time reversal and S_Σ its Cauchy surface version. S_Σ is now linear.
- Let $U_\alpha = \mathrm{e}^{\alpha S_\Sigma}$. U_α is a 1-parameter group of Bogoliubov transformations.
- The α-vacua ω_α are defined by the covariances

$$\lambda^{\pm}_{2,\alpha} = U^*_\alpha \lambda^{\pm}_{2\Sigma} U_\alpha.$$

- They have the same properties as $\omega_{\rm vac}$ except the Hadamard condition.

Modified Euclidean vacuum

It is possible to repair $\omega_{\rm vac}$ by additional gauge fixing. This amounts to add to the TT gauge condition an extra condition formulated in terms of Cauchy data on Σ.

- Let π the orthogonal projection on $\mathcal{E}_{\rm TT,reg} = \mathcal{E}_{\rm TT,sing}^{\perp}$.
- The physical phase space $\dfrac{\mathcal{E}_{\rm TT}}{{\rm Ran} K_{\Sigma}}$ is isomorphic to

$$\frac{\mathcal{E}_{\rm TT,reg}}{\mathcal{K}_{\Sigma}^{\mathbf{q}_1}}.$$

  ('correct' space in the denominator).
- We replace $\lambda_{2\Sigma}^{\pm}$ by

$$\lambda_{2\Sigma{\rm mod}}^{\pm} = \pi^{*}\lambda_{2\Sigma}^{\pm}\pi.$$

- Theorem (GW)

  The modified covariances $\lambda^{\pm}_{2\Sigma\mathrm{mod}}$ satisfy:

  (1) *the Hadamard condition,*
  (2) *the CCR on $\mathcal{E}_{\mathrm{TT}}$,*
  (3) *the strong gauge invariance on $\mathcal{E}_{\mathrm{TT}}$.*
  (4) *the positivity on $\mathcal{E}_{\mathrm{TT}}$,*
  (5) *the invariance under de Sitter isometries preserving Σ.*

  - We denote by ω_{mod} the associated state on $\mathcal{E}_{\mathrm{TT}}$. It is a true, gauge invariant, quasi-free state.
  - Its only defect is that it is not invariant under the full $O(1,4)$ group, only under its subgroup $O(4)$.
  - Thank you for your attention !