Null Infinity is a Weakly Isolated Horizon!

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Organization

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0. Preamble

- Null infinity, $\mathscr{I}$, of asymptotically flat space-times, and horizons, Δ, of BHs in equilibrium, are null 3-manifolds but have very different physical connotations. Typically Δ lies in the strong field region and there is no radiation flux across it, while $\mathscr{I}$ lies in the asymptotic, weak field regime with possibly large fluxes of radiation across it! Yet, surprisingly, they share a large number of geometric properties, making them both Weakly Isolated Horizons (WIHs) $\mathfrak{h}$. Consequently symmetry groups of Δ and $\mathscr{I}$ are almost the same. The origin of the drastic differences in their physics is remarkably simple: Einstein's equations hold on Δ, while conformal Einstein's equations hold on $\mathscr{I}$! These considerations apply also for cosmological horizons.

- General WIHs $\mathfrak{h}$, are null surfaces $\Rightarrow$ can extract (constraint-free) DOF and introduce a convenient Hamiltonian framework to obtain fluxes (and charges) associated with symmetries of WIHs $\mathfrak{h}$. It reproduces the standard results at $\mathscr{I}$ without having to extend symmetry vector fields to the space-time interior, or to find preferred symplectic potentials, even though $\mathscr{I}$ is a 'leaky' boundary. At Δ, on the other hand, it yields zero fluxes, just as one would physically expect.

- This unified framework paves a way to explore the relation between horizon dynamics in the strong field region and waveforms at infinity. Should also be useful in the analysis of black hole evaporation in quantum gravity.

1. Geometrical WIHs

• A *Non Expanding Horizon (NEH)* in 4-d Space-time $(\bar{M}, \bar{g}_{ab})$ is a null submanifold $\mathfrak{h}$, topologically $\mathbb{S}^2 \times \mathbb{R}$, such that:

(i) Every null normal $\bar{k}^a$ to $\mathfrak{h}$ is expansion-free, $\theta_{\bar{k}} = 0$; and,

(ii) On $\mathfrak{h}$ the Ricci tensor satisfies $\bar{R}_a{}^b \bar{k}^a = \alpha \bar{k}^b$ for some function α.

• Raychaudhuri Eq. implies that shear of $\bar{k}^a$ also vanishes $\Rightarrow$ the intrinsic (degenerate) metric $\bar{q}_{ab}$ satisfies $\mathcal{L}_{\bar{k}} \bar{q}_{ab} = 0$; it is 'time independent'. As a result, by pull-back, the space-time derivative operator ∇ induces a canonical intrinsic derivative D on $\mathfrak{h}$: $\underleftarrow{\bar{\nabla}} = \bar{D}$. It satisfies: $\bar{D}_a \bar{q}_{ab} = 0$, and, $\bar{D}_a \bar{k}^b = \bar{\omega}_a \, \bar{k}^b$ for some 1-form $\bar{\omega}_a$.

• We can always restrict ourselves to geodesic null normals $\bar{k}^a \bar{D}_a \bar{k}^b = 0$. Then NEH conditions $\Rightarrow \mathcal{L}_{\bar{k}} \bar{\omega}_a = 0$; $\bar{\omega}_a$ is also 'time-independent'. Furthermore, we can now severely restrict the rescaling freedom in $\bar{k}^a$ by demanding that $\bar{\omega}_a$ be divergence-free, i.e, $\bar{q}^{ab} \bar{D}_a \bar{\omega}_b = 0$. Only remaining freedom: $\bar{k}^a \rightarrow c\bar{k}^a$ where c is a positive constant.

• Thus, every NEH can be naturally equipped with a (small) equivalence class of null normals $[\bar{k}^a]$ (where $\bar{k}'^{\,a} \approx \bar{k}^a$ iff $\bar{k}'^{\,a} = c\bar{k}^a$) such that $\mathcal{L}_{\bar{k}} \bar{\omega}_a = 0$. An NEH equipped with an equivalence class $[\bar{k}^a]$ of null normals with this property is called a *Weakly Isolated Horizon (WIH)*. On a WIH we also have $\mathcal{L}_{\bar{k}} \bar{q}_{ab} = 0$ automatically. The triplet $([\bar{k}^a], \bar{q}_{ab}, \bar{D})$ is said to constitute the geometry of the WIH $\mathfrak{h}$.

Time Dependence on Geometric WIHs

- On any geometrical WIH $(\mathfrak{h}, [\bar{k}^a], \bar{q}_{ab}, \bar{D})$, fields $\bar{q}_{ab}$, $\bar{\omega}_a$ are time independent :

$$\dot{\bar{q}}_{ab} := \mathcal{L}_{\bar{k}}\, \bar{q}_{ab} = 0 \quad \text{and} \quad \dot{\bar{\omega}}_a := \mathcal{L}_{\bar{k}}\, \bar{\omega}_a = 0 \;\Leftrightarrow\; \dot{\bar{D}}_a \bar{k}^b = 0.$$

Furthermore, one can show that, given any horizontal 1-form h_a (i.e. $h_a \bar{k}^a = 0$),

$$\dot{\bar{D}}_a h_b := (\mathcal{L}_{\bar{k}} \bar{D}_a - \bar{D}_a \mathcal{L}_{\bar{k}}) h_b = 0.$$

- Thus, time dependence of $\bar{D}$ is completely determined by $\dot{D}_a \bar{j}_b$ for any 1-form $\bar{j}_b$ satisfying $\bar{j}_a \bar{k}^a = -1$. It is given by:

$$\dot{\bar{D}}_a \bar{j}_b = \big(\bar{D}_a \bar{\omega}_b + \bar{\omega}_a \bar{\omega}_b\big) + \big(\bar{k}^c\, \bar{C}_{\underleftarrow{cab}}{}^d\, \bar{j}_d + \tfrac{1}{2} \bar{S}_{\underleftarrow{ab}} + \alpha\, \bar{q}_{ab}\big) \qquad (1).$$

($\bar{C}_{abc}{}^d$ and $\bar{S}_{ab}$ are the 4-d Weyl and 4-d Schouten tensors and $\bar{R}_a{}^b \bar{k}^a = \alpha\, \bar{k}^b$.) None of the terms on the right hand side vanishes on $\mathfrak{h}$! Thus part of the geometry of a WIH is dynamical. This dynamics is driven by the pull-back to $\mathfrak{h}$ of the 4-d curvature tensor $\bar{R}_{abc}{}^d$, since $\bar{\omega}_a$ is part of $\bar{D}$.

- So far no field equations have been imposed; we only have a geometric condition on $\bar{R}_a{}^b$. So Eq. (1) holds both on BH (and cosmological) horizons Δ and null infinity $\mathscr{I}^+$. We will find that the diametrically opposite physics of Δ and $\mathscr{I}^+$ emerges from the fact that field equations imply that complementary terms on the right side of (1) vanish in the two cases.

2. BH (and Cosmological) WIHs Δ

• To discuss these horizons, let us now assume Einstein's vacuum field equations on them. Following literature, we will drop 'bars' over symbols $\bar{g}_{ab},\ \bar{R}_{abc}{}^{d},\ \bar{D}$ and use the notation: $\mathfrak{h} \to \Delta; \qquad \bar{k}^a \to \ell^a; \qquad \bar{j}_b \to n_a.$

• Now the Ricci tensor terms in equation (1) for $\dot{D}$ vanish. Furthermore, already on geometric WIHs $\mathfrak{h}$, the the Weyl term that enters the equation (1) for $\dot{D}$ is given by $\ell^c\, \bar{C}_{\underleftarrow{cab}}{}^{d}\, n_d = -(\frac{1}{4}\mathcal{R}\, q_{ab} + D_{[a}\omega_{b]})$ (with $\mathcal{R}$ the 2-d scalar curvature). Therefore,

$$\dot{D}_a n_b = D_{(a}\omega_{b)} + \omega_a\omega_b - \tfrac{1}{4}\mathcal{R}\, q_{ab} \qquad (2)$$

• Thus, even on BH WIHs Δ, the derivative operator D is time-dependent! But this dependence is highly constrained, because the right side of (2), is time independent. $(\mathcal{L}_\ell q_{ab} = 0, \quad \mathcal{L}_\ell \omega = 0,\ (\text{and } \omega_a \ell^a = 0).)$ Therefore, (q_{ab}, D) on Δ are completely determined by its values on a 2-sphere cross-section; they are 'corner data', representing 'Coulombic fields'. There are no 3-d degrees of freedom that are hallmarks of radiation. That is why physical quantities on Δ –mass, angular momentum, as well as all higher multipole moments– are absolutely conserved.

3. Asymptotic WIH $\mathscr{I}^+$

- A physical space-time $(M,\, g_{ab})$ is said to be asymptotically flat at future at future infinity if it admits a conformal completion $(\hat{M},\, \hat{g}_{ab})$, where $\hat{M} = M \cup \mathscr{I}^+$ is a manifold with a boundary $\mathscr{I}^+$, topologically $\mathbb{S}^2 \times \mathbb{R}$, and $\hat{g}_{ab} = \Omega^2 g_{ab}$ on M s.t.
  (i) At $\mathscr{I}^+$, we have $\Omega \hat{=} 0$ and $\hat{\nabla}_a \Omega \,\hat{\neq}\, 0$; and,
  (ii) g_{ab} satisfies Einstein's equations $G_{ab} = 8\pi G\, T_{ab}$, with $\Omega^{-2}\, T_{ab}$ admitting a smooth limit to $\mathscr{I}^+$.

These conditions imply: (a) $\mathscr{I}^+$ is null with null normal $\hat{n}^a := \hat{\nabla}^a \Omega$; and, (b) we can always choose Ω such that $\hat{\nabla}_a \hat{n}^a = 0$. As is standard, let us work with these divergence-free conformal frames. (If in addition the conformal factor is such that the (degenerate) metric on $\mathscr{I}$ is a unit 2-sphere metric, we are in a Bondi conformal frame: $g_{ab} \mathrm{d}x^a \mathrm{d}x^b \hat{=} 2\mathrm{d}u\mathrm{d}r + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$).

- Conformal Einstein's equations at $\mathscr{I}^+$ imply: (1) $\hat{R}_a{}^b \hat{n}^b \propto \hat{n}^a$ at $\mathscr{I}^+$, and, (2) $\hat{\nabla}_a \hat{n}^b = 0$ at $\mathscr{I}^+$. Thus $\mathscr{I}^+$ is a null 3-manifold, for which the expansion $\theta_{\hat{n}}$ of the null normal vanishes, and the Ricci tensor is such that it is an NEH. Also, condition (2) $\Rightarrow$ $\hat{\nabla}_a \hat{n}^b \equiv \hat{\omega}_a \hat{n}^b = 0$, whence $\hat{\omega}_a = 0$, and $(\mathscr{I}^+,\, \hat{n}^a)$ is a WIH.

Thus $(\mathscr{I}^+,\, \hat{n}^a,\, \hat{q}_{ab},\, \hat{D})$ is a WIH in the conformally completed space-time $(\hat{M}, \hat{g}_{ab})$.

Time dependence of the WIH geometry of $\mathscr{I}^+$

• Since $\mathscr{I}^+$ is a WIH, the metric $\hat{q}_{ab}$ is time independent. Recall that on any WIH, the time dependence of $\bar{D}$ is given by

$$\dot{\bar{D}}_a \bar{j}_b = \bar{D}_a \bar{\omega}_b + \bar{\omega}_a \bar{\omega}_b + \bar{k}^c\, \bar{C}_{c\underleftarrow{ab}}{}^d\, \bar{j}_d + \tfrac{1}{2} \bar{S}_{\underleftarrow{ab}} + \alpha\, \bar{q}_{ab} \qquad (1).$$

In the notation used at $\mathscr{I}^+$: $\quad \bar{k}^a \to \hat{n}^a,\ \bar{j}_b \to \hat{\ell}_b,\ \bar{D} \to \hat{D},\ \bar{\omega}_a \to\ \hat{\omega}_a.$

Interestingly, since $\hat{\omega}_a = 0,\ \text{and}\ \hat{C}_{abc}{}^d = 0$ at $\mathscr{I}^+$, all the terms that contribute to $\dot{D}$ on BH horizons Δ now vanish at $\mathscr{I}^+$ and terms that vanish at Δ now survive! Thus, at $\mathscr{I}^+$ we have:

$$\dot{\hat{D}}_a \hat{\ell}_b = \tfrac{1}{2} \hat{S}_{\underleftarrow{ab}} + \alpha\, \hat{q}_{ab} \quad \text{or, in a Bondi conformal frame,} \quad \mathsf{TF}\ (\dot{\hat{D}}_a \hat{\ell}_b) = \tfrac{1}{2} \hat{N}_{ab} \quad (3).$$

• While time-dependence of D was driven by the Weyl curvature of g_{ab} at Δ, at $\mathscr{I}^+$ is is driven by the Ricci curvature of $\hat{g}_{ab}$, the conformally invariant part of which is just the Bondi news! Therefore, $\hat{D}$ has 2-degrees of freedom per point of $\mathscr{I}^+$. These are 3-d degrees of freedom, representing the radiative modes. Put differently, while the WIH geometry of Δ, carries only 'coulombic' information contained in the 'corner data', that at $\mathscr{I}^+$ carries 'radiative' information contained on all of $\mathscr{I}^+$.

This diametrically opposite physics emerges from the same equation, (1), because Einstein's and conformal Einstein's equations set complementary terms to zero!

Organization

0. Preamble ✓
1. Weakly Isolated Horizons (WIHs) ✓
2. Black Hole (and Cosmological) Horizons ✓
3. Null Infinity as a WIH ✓
4. Fluxes (and Charges) associated with Symmetries
5. Discussion, Summary and References

References for parts ✓ covered so far: 1,2 (and 3,4) in the source material listed at the end.

4A. Symmetries of WIHs

• Let's return to geometrical WIHs $\mathfrak{h}$. Fields they are equipped with, $(\bar{q}_{ab}, [\bar{k}^a])$, vary from one WIH to another. However, each $\mathfrak{h}$ carries a 3-parameter family of round, unit metrics $\mathring{q}_{ab}$, conformally related to its $\bar{q}_{ab}$, hence a 3-parameter family of pairs $(\mathring{q}_{ab} := \mathring{\psi}^2 \bar{q}_{ab}, [\mathring{k}^a] = \mathring{\psi}^{-1} \bar{k}^a)$. (Rescaling of $[\bar{k}^a]$, which is natural rescaling for vectors, plays an essential role in the multipole moment considerations.)

• While the conformal factor $\mathring{\psi}$ in $\mathring{q}_{ab} := \mathring{\psi}^2 \bar{q}_{ab}$ varies from one WIH to another, the relative conformal factors $\mathring{\alpha}$ between two unit round metrics, $\mathring{q}'_{ab} = \mathring{\alpha}^2 \mathring{q}_{ab}$, are universal: (normalized) linear combinations of the first 4 spherical harmonics of $\mathring{q}_{ab}$. Thus, the universal structure of WIHs consists of conformally related pairs $(\mathring{q}_{ab}, [\mathring{k}^a])$, where $(\mathring{q}'_{ab}, [\mathring{k}'^{\,a}]) = (\mathring{\alpha}^2 \mathring{q}_{ab}, [\mathring{\alpha}^{-1} \mathring{k}^a])$ on $\mathfrak{h}$.

• The WIH symmetry group $\mathfrak{G}$ is the subgroup of diffeos of $\mathfrak{h}$ that preserve this universal structure. It has a semi-direct product structure: $\mathfrak{G} = \mathfrak{B} \ltimes \mathfrak{D}$, where $\mathfrak{B}$ is the BMS group and $\mathfrak{D}$ is a 1-d group of dilations. Action of the infinitesimal generators is given by: $\mathcal{L}_\xi \, \mathring{q}_{ab} = 2\mathring{\beta} \, \mathring{q}_{ab}$ and $\mathcal{L}_\xi \mathring{k}^a = -(\mathring{\beta} + \mathring{\varpi})\mathring{k}^a$
where, as in the BMS case, $\mathring{\beta}$ is a linear combination of the $\ell = 1$ spherical harmonics, while the constant $\mathring{\varpi}$ is new, representing the action of dilation which only rescales each $\mathring{k}^a$ in $[\mathring{k}]$ by a constant, leaving $\mathring{q}_{ab}$ unchanged.

Reduction of $\mathfrak{G}$ to $\mathfrak{B}$ on asymptotic WIHs $\mathscr{I}^+$

• So far we consider geometrical WIHs $\mathfrak{h}$. Nothing changes on BH WIHs Δ. The symmetry group is again $\mathfrak{G}$. Could we add fields to the universal structure to eliminate the 1-d dilation group $\mathfrak{D}$ on $\mathfrak{h}$? The answer is in the negative: For example, on the Schwarzschild WIH Δ, the static Killing field is a dilation. So if one were to add structure to eliminate $\mathfrak{D}$, the time translation symmetry would fail to belong to the resulting WIH symmetry group!

• Let us now consider $\mathscr{I}^+$ as a WIH in any given conformal completion $\hat{M}, \hat{g}_{ab}$. This WIH comes with a preferred null normal $\hat{n}^a \hat{=} \hat{\nabla}^a \Omega$, rather than an equivalence class $[\hat{n}^a]$. Therefore, from the WIH perspective, the universal structure now reduces to the pairs $(\mathring{\hat{q}}'_{ab},\, \mathring{\hat{n}}^a) \approx (\mathring{\alpha}^2 \mathring{\hat{q}}'_{ab},\, \mathring{\alpha}^{-1} \mathring{\hat{n}}^a)$ –i.e. Bondi conformal frames. The subgroup of Diff($\mathscr{I}^+$) that preserves this universal structure is precisely the BMS group $\mathfrak{B}$! Thus, $\mathfrak{G}$ reduces to $\mathfrak{B}$ because we no longer have the freedom to rescale the null normal to $\mathscr{I}^+$. The dilation disappears. What about the Killing field in, say, Schwarzschild? While it is a dilation on Δ, it is a supertranslation on $\mathscr{I}^+$. Interesting and rather subtle interplay between Δ and $\mathscr{I}^+$ that allows us to treat both in the general WIH framework.

• A physical space-time (M, g_{ab}) admits infinitely many divergence-free completions $(\hat{M}, \hat{g}_{ab})$. $\mathscr{I}^+$ is a WIH in each completion, but each completion endows it with a distinct WIH geometry $(\hat{q}_{ab}, \hat{n}^a, \hat{D})$, all with the same universal structure. Although distinct, these geometries carry the same physics (BMS fluxes and charges) because of conformal invariance of observables.

4.B Phase spaces of local DOF: Strategy

• Example: KG field satisfying $(\Box - \mu^2)\phi = 0$ in Minkowski space. The covariant phase space $\mathbf{\Gamma}_{\rm cov}$ is the space of (suitably regular) solutions ϕ, equipped with the symplectic structure:

$$\boldsymbol{\omega}|_\phi\,(\delta_1, \delta_2) = \int_\Sigma \big((\delta_1\phi)\nabla_a(\delta_2\phi) \,-\, (\delta_2\phi)\nabla_a(\delta_1\phi)\big)\,\epsilon^a{}_{bcd}\,\mathrm{d}S^{bcd} \;\equiv\; \int_\Sigma \mathfrak{J}_{bcd}\,\mathrm{d}S^{bcd}$$

Every Killing field ξ^a defines a Hamiltonian H_ξ via $\delta H_\xi = \boldsymbol{\omega}|_\phi(\delta_\xi, \delta)$. As is well-known, $H_\xi = F_\xi := \int_\Sigma T_{ma}\,\xi^m \epsilon^a{}_{bcd}\,\mathrm{d}S^{bcd}$. Thus, the usual flux F_ξ associated with ξ^a –which is conserved because Σ is a Cauchy surface– is naturally recovered from Hamiltonian considerations.

• Let us introduce a 'local' phase space $\mathbf{\Gamma}_\mathcal{R}$ associated with an open 3-d region $\mathcal{R}$ of Σ, representing degrees of freedom that are 'reside' in $\mathcal{R}$. $\mathbf{\Gamma}_\mathcal{R}$ consists of initial data $\gamma \equiv (\varphi,\, \pi)_\mathcal{R}$, restricted to $\mathcal{R}$, equipped with the topology given by the norm:

$$||\gamma||^2_\mathcal{R} \;:= \int_\mathcal{R} \left(\pi^2 + D_a\varphi D^a\varphi + \mu^2\varphi^2\right)\mathrm{d}^3x \,<\, \infty.$$

The symplectic structure $\boldsymbol{\omega}_\mathcal{R}$ on $\mathbf{\Gamma}_\mathcal{R}$ is obtained by just restricting the integral to $\mathcal{R}$:

$$\boldsymbol{\omega}_\mathcal{R}|_\gamma\,(\delta_1, \delta_2) \;=\; \int_\mathcal{R} \mathfrak{J}_{abc}\,\mathrm{d}S^{abc} \;=\; \int_\mathcal{R} \big(\delta_1\varphi\,\delta_2\pi \,-\, \delta_2\varphi\,\delta_1\pi\big)\,\mathrm{d}^3x.$$

It is continuous and weakly non-degenerate on $\mathbf{\Gamma}_\mathcal{R}$. Given a Killing vector field ξ^a tangential to $\mathcal{R}$, the vector field $\delta_\xi := (\mathcal{L}_\xi\varphi,\, \mathcal{L}_\xi\pi)$ is Hamiltonian on a dense subspace of $\mathbf{\Gamma}_\mathcal{R}$ with the Hamiltonian given by $H_\xi = \int_\mathcal{R} \left(\mathcal{L}_\xi\varphi\right)\pi\,\mathrm{d}^3x$. Clearly, it admits a continuous extension to all of $\mathbf{\Gamma}_\mathcal{R}$. Furthermore, $H_\xi = F_\xi\,|_R$, so that it has the interpretation of the flux across $\mathcal{R}$ associated with ξ^a. Thus, (as is common in infinite dimensional phase spaces) the Hamiltonian VF is only densely defined. But the the Hamiltonian is a continuous function on full $\mathbf{\Gamma}_\mathcal{R}$ and equals $F_\xi|_R$.

Phase spaces of local DOF at $\mathscr{I}^+$

- Let us now consider a massless KG field and extend local phase spaces to regions $\hat{R}$ of $\mathscr{I}^+$, bounded by any two 2-sphere cross-sections. Now the degrees of freedom of ϕ residing in $\hat{\mathcal{R}}$ is encoded in the 'radiation field' $\hat{\phi}|_{\hat{\mathcal{R}}} = \Omega^{-1}\phi|_{\hat{\mathcal{R}}}$. The local phase space $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$ has the topology induced by the norm:

$$||\hat{\phi}||^2_{\mathcal{R}} \;=\; \int_{\mathcal{R}} \left(|\hat{n}^a D_a \hat{\phi}|^2 + |{}^2\hat{D}_a \hat{\phi}|^2 + \tfrac{1}{l^2}|\hat{\phi}|^2\right) \mathrm{d}^3\mathscr{I}^+ \;<\infty$$

Topology is insensitive to the extra structure used to define the norm. Pull-back of the symplectic current $\mathfrak{J}_{abc}$ to $\hat{\mathcal{R}}$ gives

$$\boldsymbol{\omega}_{\hat{\mathcal{R}}}|_{\hat{\phi}}\,(\delta_1, \delta_2) = \int_{\hat{\mathcal{R}}} \left((\delta_1\hat{\phi})\hat{\nabla}_a(\delta_2\hat{\phi}) \,-\, (\delta_2\hat{\phi})\hat{\nabla}_a(\delta_1\phi)\right)\, \hat{\epsilon}^a{}_{bcd}\;\mathrm{d}\hat{S}^{bcd}$$

It is continuous and weakly non-degenerate on $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$.

- For any BMS vector field ξ^a on $\mathscr{I}^+$, the phase space vector field $\delta_\xi\hat{\phi} = (\xi^a\hat{D}_a + \hat{\beta})\hat{\phi}$ is again well-defined and a Hamiltonian VF on a dense subspace of $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$. The Hamiltonian $H_\xi = \int_{\hat{\mathcal{R}}}(\delta_\xi\hat{\phi})(\mathcal{L}_{\hat{n}}\hat{\phi})$ admits a continuous extension to all of $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$. As before, $H_\xi = F_\xi\,|_{\hat{\mathcal{R}}}$. Thus, the Hamiltonians now represent the BMS fluxes across the region $\hat{\mathcal{R}}$ of $\mathscr{I}^+$.

- Topology: The Hamiltonian framework does not have direct knowledge of T_{ab}. But continuity of the Hamiltonian H_ξ generating δ_ξ in the above topology ensures that H_ξ equals the flux $F_\xi = \int_{\hat{\mathcal{R}}} T_{ab}\xi^a\hat{n}^b\, d^3\mathscr{I}^+$. Gravitational waves have no T_{ab} but we can carry over the topology to define BMS fluxes across $\hat{\mathcal{R}}$.

4.C Gravitational field at $\mathscr{I}^+$

- To construct the phase space, let us consider conformal completions $(\hat{M}, \hat{g}_{ab})$ of asymptotically flat vacuum solutions. What are the local DOF in an open region $\hat{\mathcal{R}}$ of $\mathscr{I}^+$ bounded by 2 cross-sections? As our discussion in part 3 suggests, these are captured in connection $\hat{D}$ restricted to $\hat{\mathcal{R}}$, or rather, in certain equivalence classes $\{\hat{D}\}$ of connections on $\hat{\mathcal{R}}$, after removing redundant conformal freedom. Information in the curvature of $\{\hat{D}\}$: the Bondi news $\hat{N}_{ab}$, and, part of Weyl curvature that captures purely radiative information, ${}^\star\hat{K}^{ab}$, (or, the NP $(\Psi_4^\circ, \Psi_3^\circ, \mathrm{Im}\Psi_2^\circ)$, or the CK $(\underline{\alpha}, \underline{\beta}, \sigma)$) .

- There are connections $\{\hat{\mathfrak{D}}\}$ with 'trivial curvature' (i.e., for which $\hat{N}_{ab} = 0$) and ${}^\star\hat{K}^{ab} = 0$. They serve as origins in the affine space of $\{\hat{D}\}$. Recall that the non-trivial content in any connection $\hat{D}$ is captured by $\hat{D}_a \hat{\ell}_b$, for any $\hat{\ell}_a$ satisfying $\hat{\ell}_a \hat{n}^a = -1$. So we can label any $\{\hat{D}\}$ by $\hat{\gamma}_{ab} = \mathrm{TF}(\hat{D}_a - \hat{\mathfrak{D}}_a)\hat{\ell}_b$; they provide a natural chart on the affine space of $\{\hat{D}\}$ (as Cartesian coordinates do on the Minkowski affine space.) The two components of the STT $\hat{\gamma}_{ab}$ encode radiative DOF in $\{\hat{D}\}$.

- The 'local' phase space $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$ is the space of $\{\hat{D}_a\}$ on $\hat{\mathcal{R}}$, equipped with topology given by the norm $||\hat{\gamma}||^2_{\hat{\mathcal{R}}} := \int_{\hat{\mathcal{R}}} \left[|\hat{n}^a \hat{D}_a \hat{\gamma}_{bc}|^2 + |{}^2\hat{D}_a \hat{\gamma}_{bc}|^2 + \frac{1}{l^2} |\hat{\gamma}_{ab}|^2 \right] \mathrm{d}^3\mathscr{I}^+$. Again, the topology is insensitive to the extra structure introduced to define the norm. $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$ is the phase space of 'true' or radiative DOF that reside in $\hat{\mathcal{R}}$.

Fluxes associated with BMS symmetries

- As before, the symplectic structure on $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$ is obtained by pulling back to $\hat{\mathcal{R}}$ the symplectic current $\mathfrak{J}_{abc}$ of the full covariant phase space $\mathbf{\Gamma}_{\rm cov}$ of GR. Result:

$$\boldsymbol{\omega}|_{\hat{\gamma}}(\delta_1, \delta_2) = \frac{1}{8\pi G} \int_{\hat{\mathcal{R}}} \Big[(\delta_1\, \hat{\gamma}_{ab})\, (\mathcal{L}_{\hat{n}}\, \delta_2\, \hat{\gamma}_{cd}) \,-\, (\delta_2\, \hat{\gamma}_{ab})(\mathcal{L}_{\hat{n}}\, \delta_1\, \hat{\gamma}_{cd})\Big]\, \hat{q}^{ac}\, \hat{q}^{bd}\, \mathrm{d}^3\mathscr{I}^+$$

Action of the BMS group on $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$: First consider ξ^a for which $\mathcal{L}_\xi \hat{q}_{ab} = 0$, $\mathcal{L}_\xi \hat{n}^a = 0$. The infinitesimal motion generated by ξ^a induces a densely defined VF δ_ξ on $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$: $\delta_\xi \hat{\gamma}_{ab} = \mathrm{TF}(\mathcal{L}_\xi \hat{D}_a - \hat{D}_a \mathcal{L}_\xi)\hat{\ell}_b$.

- Again, it is Hamiltonian, i.e. satisfies $\boldsymbol{\omega}|_{\hat{\gamma}}(\delta_\xi, \delta) = \delta H_\xi$ with

$$H_\xi = \frac{1}{8\pi G} \int_{\hat{\mathcal{R}}} (\delta_\xi \hat{\gamma}_{ab})(\mathcal{L}_{\hat{n}} \hat{\gamma}_{cd})\, \hat{q}^{ac} \hat{q}^{bd}\, \mathrm{d}^3\mathscr{I}^+ = \frac{1}{16\pi G} \int_{\hat{\mathcal{R}}} \big[(\mathcal{L}_\xi \hat{D}_a - \hat{D}_a \mathcal{L}_\xi)\hat{\ell}_b\big]\, \hat{N}_{cd}\, \hat{q}^{ac} \hat{q}^{bd}\, \mathrm{d}^3\mathscr{I}^+$$

on a dense subspace of $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$. The first form makes it clear that it has a continuous extension from the dense subspace to full $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$. This H_ξ represents the BMS flux associated with ξ^a across a general open region $\hat{\mathcal{R}}$ of $\mathscr{I}^+$. The second form brings out geometrical meaning. For supertranslations, $\xi^a = \hat{\mathfrak{s}} \hat{n}^a$ with $\mathcal{L}_{\hat{n}} \hat{\mathfrak{s}} = 0$ and the expression simplifies:

$$H_{\hat{\mathfrak{s}}}(\{\hat{D}\}) = \frac{1}{32\pi G} \int_{\hat{\mathcal{R}}} \big[\hat{\mathfrak{s}}\, \hat{N}_{ab} \,+\, 2\hat{D}_a \hat{D}_b\, \hat{\mathfrak{s}} + \hat{\mathfrak{s}}\, \hat{\rho}_{ab}\big]\, \hat{N}_{cd}\, \hat{q}^{ac} \hat{q}^{bd}\, \mathrm{d}^3\mathscr{I}^+$$

providing both the hard and the soft terms.

- For a general BMS VF, $\mathcal{L}_\xi \hat{q}_{ab} = 2\beta\, \hat{q}_{ab}$, $\mathcal{L}_\xi \hat{n}^a = -\beta\, \hat{n}^a$. The only difference is that (as in the scalar field case) there is now an extra term in the expression of the VF δ_ξ on $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$: $\delta_\xi \hat{\gamma}_{ab} = \mathrm{TF}\big[(\mathcal{L}_\xi \hat{D}_a - \hat{D}_a \mathcal{L}_\xi)\hat{\ell}_b \,+\, 2\hat{\ell}_{(a} \hat{D}_{b)} \beta\big]$.

4.D Fluxes on BH (and cosmological) horizons Δ

• What fluxes do we obtain if we apply this procedure to open regions $\mathcal{R}$ of Black hole horizons Δ using infinitesimal generators ξ^a of $\mathfrak{G}$? Space-time metrics induce fields (q_{ab}, D) on Δ (and hence on regions $\mathcal{R}$) that constitute $\mathbf{\Gamma}_{\mathcal{R}}$. They can again be labelled by freely specifiable fields (analogs of $\hat{\gamma}_{ab}$), but now on any 2-d cross section of $\mathcal{R}$; now the DOF are only 2-d! By pulling-back the symplectic current $\mathfrak{J}_{abc}$ to $\mathcal{R}$ we can ask if $\boldsymbol{\omega}|_{(q,D)}(\delta_\xi, \delta) = \delta H_\xi$ for some H_ξ.

• It turns out that all fluxes vanish identically now, because the pull-back of the symplectic current $\mathfrak{J}_{abc}$ (of the full covariant phase space of GR) to Δ itself vanishes. Of course this is what we expect physically. But we now see explicitly that one and the same strategy –of using local phase spaces that capture the DOF that reside in regions $\hat{\mathcal{R}}$ of $\mathscr{I}^+$ and $\mathcal{R}$ of Δ– yield strikingly different physical results although both are WIHs!

• Why did the pull-back of $\mathfrak{J}_{abc}$ not vanish at $\mathscr{I}^+$? After all, it too is a WIH. The terms that vanish on Δ are multiplied by inverse powers of the conformal factor Ω relating g_{ab} to $\hat{g}_{ab}$ making the pull-back is non-zero. Again, it is the fact that Δ is a WIH in (M, g_{ab}) while $\mathscr{I}^+$ is a WIH in the conformally completed $(\hat{M}, \hat{g}_{ab})$ that leads to drastically different results for fluxes associated with $\mathcal{R}$ and $\hat{R}$!

5. Discussion: Remarks on results at $\mathscr{I}^+$

1. BMS Fluxes are physical observables on radiative phase spaces $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$. All fluxes vanish if $\hat{N}_{ab} = 0$. (Thus, there is no flux of angular momentum if the energy flux vanishes.)

2. Fluxes require only radiative degrees of freedom that are encoded *intrinsically* in $\hat{\mathcal{R}}$. They have the form $F_\xi = \int_{\hat{\mathcal{R}}} \mathcal{F}^{(\xi)}_{abc}$. Question: Are the 3-forms $\mathcal{F}^{(\xi)}_{abc}$ on $\mathscr{I}^+$ exact? If so, there would be corresponding charges Q_ξ.

Now, full conformal Einstein's equations and Bianchi identities at $\mathscr{I}^+$ satisfied by $\hat{g}_{ab}$ relate the radiative DOF to the Coulombic DOF contained in the asymptotic Weyl curvature (in particular, the NP $(\mathrm{Re}\Psi_2^\circ,\, \Psi_1^\circ)$ and the CK (ρ, β)). These relations imply existence of unique 2-forms $\mathcal{Q}^{(\xi)}_{ab}$ on $\mathscr{I}^+$, constructed locally from fields, such that $d\mathcal{Q}^{(\xi)} = \mathcal{F}^{(\xi)}$, and $\mathcal{Q}^{(\xi)}_{ab} = 0$ in Minkowski space. Therefore, for any cross-section S of $\mathscr{I}^+$, we obtain charges Q_ξ satisfying the balance law $Q_\xi[S_1] - Q_\xi[S_2] = F_\xi(\hat{\mathcal{R}})$ for the region $\hat{\mathcal{R}}$ of $\mathscr{I}^+$ bounded by S_1, S_2. The Charges and fluxes are the standard ones.

For fluxes, the 4-metric and the full field equations are excess baggage. Charges on the other hand refer to Coulombic aspects that are intertwined with the radiative aspects through field equations. To obtain them one needs to step out of $\mathbf{\Gamma}_{\hat{\mathcal{R}}}$ and use information from full $\mathbf{\Gamma}_{\mathrm{cov}}$.

3. Unlike in other approaches, this framework does not require one to extend symmetry vector fields ξ^a away from $\mathscr{I}^+$ or choose specific symplectic potentials, and encompasses $\mathscr{I}^+$ with $\Lambda > 0$. But so far, developed only for 4-d GR.

Summary

• Null infinity $\mathscr{I}$ of asymptotically flat space-times, and horizons Δ of BHs in equilibrium, are null 3-manifolds but have very different physical connotations. Typically Δ lies in the strong field region and there is no radiation flux across it, while $\mathscr{I}$ lies in the asymptotic, weak field regime with arbitrarily large fluxes of radiation across it! Yet, they share a large number of geometric properties, making them both WIHs $\mathfrak{h}$, (but not an Isolated Horizon!!) Consequently symmetry groups of Δ and $\mathscr{I}$ are almost the same. The origin of the drastic differences in their physics is remarkably simple: Einstein's equations hold on Δ, while conformal Einstein's equations hold on $\mathscr{I}$! (These considerations apply also for cosmological horizons).

• The geometry $(\bar{q}_{ab}, \bar{D})$ of general WIHs $\mathfrak{h}$, captures their constraint-free DOF. Local phase spaces of these DOF lead us fluxes (and charges) associated with the WIH symmetries. It reproduces the standard results at $\mathscr{I}$ without having to extend symmetry vector fields to the space-time interior, or to find preferred symplectic potentials, even though $\mathscr{I}$ is a 'leaky' boundary. At Δ, on the other hand, it yields zero fluxes, just as one would physically expect.

• This interplay between $\mathscr{I}$ and Δ discussed in parts 1 -4 is reminiscent of a musical fugue with 4 voices.

Source Material

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- Further details on infinite dimensional phase spaces

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