NONLINEAR STABILITY OF SLOWLY ROTATING KERR BLACK HOLES

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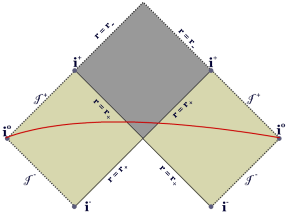
STABILITY OF KERR. $\mathcal{K}(a, m)$, $|a| < m$

General EV perturbations of a given $\mathcal{K}(a, m)$ have MFGHD approaching the exterior of another $\mathcal{K}(a_f, m_f)$.



THEOREM "True" if $|a|/m \ll 1$.

- MAIN[K-Szeftel(2021)]
- GCM PAPERS[K-Szeftel(2019), Shen(2022)]
- WAVE PAPER[Giorgi-K-Szeftel(2022)]

KERR FAMILY $\mathcal{K}(a, m)$

2-parameter family of stationary, asympt. flat (AF), solutions of

$$\mathbf{Ric(g) = 0}. \qquad \textbf{(EVE)}$$

- MINKOWSKI (1907). $a = m = 0.$
- SCHWARZSCHILD (1916). $a = 0, m \neq 0$
- KERR(1963). $0 < |a| \leq m.$

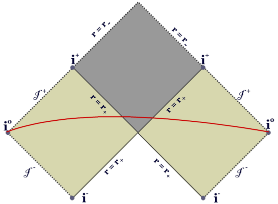
KERR FAMILY $\mathcal{K}(a, m)$

$$-\frac{|q|^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ |q|^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

STATIONARY, AXISYMMETRIC. $\partial_t, \ \partial_\varphi$ KILLING

ASYMPTOTICALLY FLAT. Approaches Minkowski as $r \to \infty$.



KERR FAMILY $\mathcal{K}(a, m)$

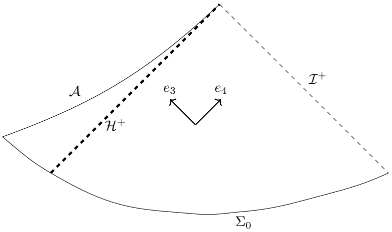
PRINCIPAL NULL PAIR. $\{e_3, e_4\}$.

- Diagonalizes the curvature tensor.
- Horizontal structure $\mathcal{H} = \{e_3, e_4\}^\perp$ is non-integrable if $a \neq 0$.

$$\underline{\chi}(e_a, e_b) = \mathbf{g}(\mathbf{D}_{e_a} e_3, e_b), \qquad \chi(e_a, e_b) = \mathbf{g}(\mathbf{D}_{e_a} e_4, e_b)$$

$$\mathrm{tr}\, \underline{\chi} = \delta^{ab} \underline{\chi}_{ab}, \qquad \mathrm{tr}\, \chi = \delta^{ab} \chi_{ab},$$

$$^{(a)}\mathrm{tr}\, \underline{\chi} = \in^{ab} \underline{\chi}_{ab}, \qquad ^{(a)}\mathrm{tr}\, \chi = \in^{ab} \chi_{ab},$$

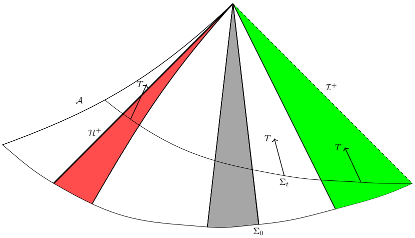


KERR FAMILY $\mathcal{K}(a, m)$

- HORIZON. $\Delta(r) = 0, \quad r{=}r_+ = m + \sqrt{m^2 - a^2}.$
- ERGOREGION. $\mathbf{g}(T, T) = \frac{\Delta - a^2 \sin^2\theta}{|q|^2} > 0,$
- TRAPPING. $\mathcal{T} = \mathcal{T} = r^3 - 3mr^2 + a^2 r + ma^2$

$$\mathcal{M}_{trap} := \mathcal{M} \cap \left\{ \frac{|\mathcal{T}|}{r^3} \leq \delta_{trap} \right\}.$$

- NULL INFINITY. $r \to \infty$
- NON-INTEGRABILITY. ${}^{(a)}\mathrm{tr}\,\chi, \ {}^{(a)}\mathrm{tr}\,\underline{\chi} \neq 0.$



MAIN DIFFICULTIES

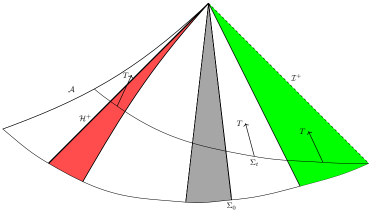
- E.V. strongly coupled, tensorial, hyperbolic, nonlinear.
- GAUGE GROUP= All diffeomorphisms $\mathbf{g} \equiv \Phi^* \mathbf{g}$.
- MODULATION. $\mathbf{R}\big(\mathbf{g}_{a,m}\big) = 0, \quad \mathbf{R}\big(\Phi_\lambda^* \mathbf{g}_{a,m}\big) = 0$
- NON-TRIVIAL CHARACTER OF $\mathcal{K}(a, m)$
  - horizon,
  - ergoregion,
  - trapping,
  - null infinity,
  - non-integrability.
- DECAY. Decay of waves in Kerr
- FINAL PARAMETERS + GAUGE Emerge in the limit!
- LOW RATES OF DECAY TO THE FINAL STATE.

WAVE EQ. IN KERR. $\square_{a,m}\psi = 0$

1. NONTRIVIAL ERGOREGION. **T** space-like
2. NONTRIVIAL TRAPPED SET.
3. LIMITED SYMMETRIES. $\mathbf{T} = \partial_t$, $\mathbf{Z} = \partial_\varphi$

STANDARD SYMMETRY. $[\mathbf{T}, \square_{a,m}] = [\mathbf{Z}, \square_{a,m}] = 0.$

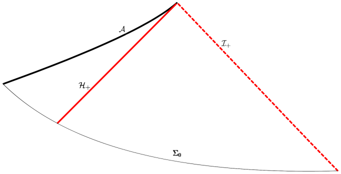
CARTER OPERATOR. $[\mathcal{K}, \square_{a,m}] = 0.$



STABILITY OF SLOWLY ROTATING KERR

THEOREM[Kl-Szeftel(2021)] *The MFD of a general IDS, close to the IDS of a* $\mathcal{K}(a_0, m_0)$, $|a_0|/m_0 \ll 1$

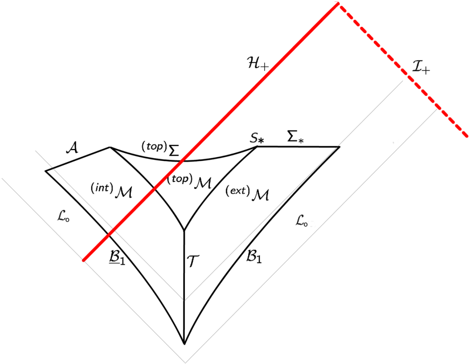
- *Has a complete future null infinity* $\mathcal{I}^+$
- *Converges in* $\mathcal{J}^{-1}(\mathcal{I}^+)$ *to a nearby* $\mathcal{K}(a_\infty, m_\infty)$ *with* (a_∞, m_∞) *close to* (a_0, m_0).
- *Has an event horizon* $\mathcal{H}^+$.
- Recoil.



- $\mathcal{K}(a_\infty, m_\infty)$ - limit of finite GCM admissible spacetimes.

GCM ADMISSIBLE $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$

- $\mathcal{L}_0$- **Data**
- S_* - **GCM** sphere.
  - (a, m), "**axis**".
- Σ_*- **GCM** hypers.
  - **Initializes** Φ_f.
- **PG**-structures
  - $({}^{(ext)}\mathcal{M}, u, r)$
  - $({}^{(int)}\mathcal{M}, \underline{u}, r)$
- **PT**-structures
- **Bootstrap.**



$(a, m, u, \underline{u}, r,$ **axis**$, \Phi_f : \mathcal{M} \to \mathcal{M})$ are continuously **upgraded**.

GEOMETRIC FRAMEWORK

1. Null Pair $(e_3, e_4), \quad \mathbf{g}(e_3, e_4) = -2.$

2. Horizontal structure $\mathcal{H} := \{e_3, e_4\}^\perp.$

3. Connection coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}.$

$$\begin{aligned}
\chi_{ab} &= \mathbf{g}(\nabla_a e_4, e_a), \qquad \underline{\chi}_{ab} = \mathbf{g}(\nabla_a e_3, e_b) \\
\chi_{ab} &= \hat{\chi}_{ab} + \frac{1}{2}\mathrm{tr}\,\chi\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\chi \in_{ab} \\
\underline{\chi}_{ab} &= \underline{\hat{\chi}}_{ab} + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\,\underline{\chi} \in_{ab}
\end{aligned}$$

${}^{(a)}\mathrm{tr}\,\chi = {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0 \Rightarrow$ Integrability.

4. Curvature coefficients $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, {}^*\rho$

$$\alpha_{ab} = \mathbf{R}(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} = \mathbf{R}(e_a, e_3, e_b, e_3)$$

GEOMETRIC FRAMEWORK

5. Complexification. $P = \rho + i\,{}^*\rho$.

$$X = \chi + i\,{}^*\chi, \quad \underline{X} = \underline{\chi} + i\,{}^*\underline{\chi}, \quad A = \alpha + i\,{}^*\alpha, \ldots$$

6. Complete set of null decompositions.
   - Connection $\Gamma = \left\{X, \underline{X}, \Xi, \underline{\Xi}, H, \underline{H}, Z, \omega, \underline{\omega}, \right\}$
   - Curvature $R = \{A, B, P, \underline{B}, \underline{A}\}$

7. Cartan-Bianchi. Tensorial character!

$$d\Gamma + [\Gamma, \Gamma] = R, \qquad dR + [R, \Gamma] = 0.$$

9. Comparison to NP, GHP, CK

10. Null frame transformations

$$(e_3, e_4, \mathcal{H}) \to (e_3', e_4', \mathcal{H}'), \qquad (\Gamma, R) \to (\Gamma', R')$$

KERR FAMILY $\mathcal{K}(a, m)$

PR. DIRECTIONS. $e_4, e_3 = \frac{r^2+a^2}{q\sqrt{\Delta}}\partial_t \pm \frac{\sqrt{\Delta}}{q}\partial_r + \frac{a}{q\sqrt{\Delta}}\partial_\varphi.$

CANONICAL BASIS. $e_1 = \frac{1}{|q|}\partial_\theta, \; e_2 = \frac{a\sin\theta}{|q|}\partial_t + \frac{1}{|q|\sin\theta}\partial_\phi.$

CRUCIAL FACT.

1. In Kerr, relative to a principal null pair,

$$A, \underline{A}, B, \underline{B} = 0, \quad P = -\frac{2m}{q^3}, \qquad \hat{\chi}, \underline{\hat{\chi}}, \xi, \underline{\xi} = 0.$$

2. In Schwarzschild

   - $\{e_3, e_4\}^\perp$ is integrable, $\quad {}^{(a)}\mathrm{tr}\,\chi = {}^{(a)}\mathrm{tr}\,\underline{\chi} = 0.$
   - ${}^*\rho = \Im(P) = 0$

3. In Minkowski $\quad \rho = \Re(P) = 0.$

$O(\epsilon)$ - PERTURBATIONS

ASSUME. $(e_3, e_4, \mathcal{H}, r, \theta)$-structure on $\mathcal{M}$.

$$\check{\Gamma} := \Gamma - \Gamma_{Kerr}, \quad \check{R} = R - R_{Kerr} =: O(\epsilon)$$

FRAME DEPENDENCE. $(f, \underline{f})_{a=1,2}, \ \lambda = 1 + O(\epsilon)$

$$\begin{aligned} e'_4 &= \lambda\Big(e_4 + f_a e_a + O(\epsilon^2)\Big) \\ e'_3 &= \lambda^{-1}\Big(e_3 + \underline{f}_a e_a + O(\epsilon^2)\Big) \\ e'_a &= e_a + \frac{1}{2}\underline{f}_a e_4 + \frac{1}{2} f_a e_3 + O(\epsilon^2) \end{aligned}$$

FACT!

- Curvature components $A, \underline{A}$ are $O(\epsilon^2)$ invariant.
- $A, \underline{A}$ verify $O(\epsilon^2)$-decoupled Teukolsky wave equations!

TEUKOLSKY EQTS.

$$\dot{\square}_{\mathbf{g}} A + \mathcal{L}[A] = \text{err}(\check{\Gamma}, \check{R})$$
$$\dot{\square}_{\mathbf{g}} \underline{A} + \underline{\mathcal{L}}[\underline{A}] = \underline{\text{err}}(\check{\Gamma}, \check{R})$$

CHANDRASEKHAR. $A \to \mathfrak{q}, \quad \underline{A} \to \underline{\mathfrak{q}}, \quad q = r + ia\cos\theta$

$$\mathfrak{q} = q\overline{q}^3 \left(\nabla_3\nabla_3 A + C\cdot\nabla_3 A + D\cdot A\right),$$
$$\underline{\mathfrak{q}} = \overline{q}q^3 \left(\nabla_4\nabla_4 \underline{A} + \underline{C}\cdot\nabla_4 \underline{A} + \underline{D}\cdot\underline{A}\right). \quad .$$

gRW EQUATIONS. Coupled systems $(A, \mathfrak{q}), \quad (\underline{A}, \underline{\mathfrak{q}})$

$$\dot{\square}_2 \mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} = aL_{\mathfrak{q}}[A] + \text{err}[\dot{\square}_2\mathfrak{q}],$$
$$\dot{\square}_2 \underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - \underline{V}\underline{\mathfrak{q}} = aL_{\underline{\mathfrak{q}}}[\underline{A}] + \text{err}[\dot{\square}_2\underline{\mathfrak{q}}],$$

REMAINING QUANTITIES. Gauge dependent $\Gamma, B, \check{P}, \underline{B}$

GCM ADMISSIBLE $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$

- $\mathcal{L}_0$- **Data**
- S_* - **GCM** sphere.
  - (a, m), "**axis**".
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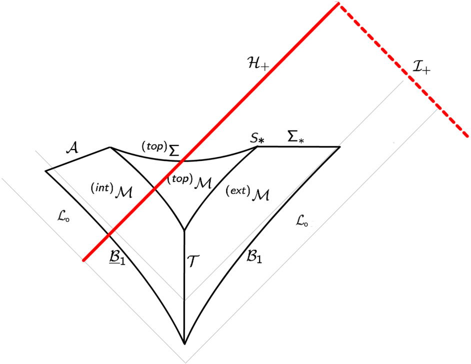


$(a, m, u, \underline{u}, r,$ **axis**$, \Phi_f : \mathcal{M} \to \mathcal{M})$ are continuously **upgraded**.

GCM ADMISSIBLE. $\quad$ GCM$(\epsilon_0, \epsilon, k_{large})$

$$\Gamma_g := \left\{\widetilde{\mathrm{tr}X}, \quad \widehat{X}, \quad \widetilde{\mathrm{tr}\underline{X}}, \quad \underline{\check{H}}, \quad \check{Z}, \quad \Xi\right\} \cup \left\{r(A, B, \check{P})\right\},$$
$$\Gamma_b := \left\{\underline{\widehat{X}}, \quad \check{H}, \quad \underline{\Xi}\right\} \cup \left\{r\underline{B}, \underline{A}\right\}.$$

BOOTSTRAP.

- BA(decay). For $k \le k_{small} = \left\lfloor \frac{1}{2} k_{large} \right\rfloor + 1.$

$$\left|\mathfrak{d}^{\le k}\Gamma_g\right| \le \epsilon \min\left\{r^{-2}\tau^{-1/2-\delta_{dec}}, \; r^{-1}\tau^{-1-\delta_{dec}}\right\},$$
$$\left|\mathfrak{d}^{\le k}\Gamma_b\right| \lesssim \epsilon r^{-1}\tau^{-1-\delta_{dec}},$$
$$\left|\mathfrak{d}^{\le k}(A, B)\right| \lesssim \epsilon r^{-7/2-\delta_{dec}}.$$

with τ appropriately defined time function.,

- BA(energy). For $k \le k_{large}$. Integral norms

STRATEGY. MAIN CONTINUATION ARGUMENT

Given a $GCM(\epsilon_0, \epsilon, k_{large})$-admissible spacetime $(\mathcal{M}, \mathbf{g})$

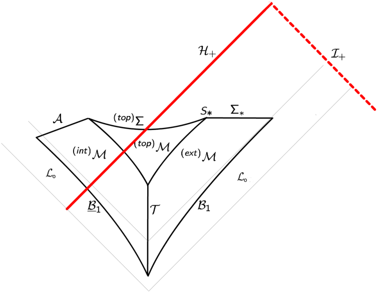
PART A. Show that the BA assumptions BAd, BAe can be improved by replacing ϵ with ϵ_0.

Part B. Extend the $GCM(\epsilon_0, \epsilon, k_{large})$ to a strictly larger GCM.

- Extend $(\mathcal{M}, \mathbf{g})$ to a strictly larger EV-spacetime $(\mathcal{M}', \mathbf{g}')$.
- Construct new GCM boundary Σ_*. K-Sz(2019), Shen(2022).
  - Construct GCM spheres and hypersurfaces
  - Define suitable (m, a) converging to (m_f, a_f)
  - Track the "axis"

MAIN NEW IDEAS

- General Covariant Modulated (GCM) spheres.
- Choice of the last slice Σ_*.
- Non integrability if $a \neq 0$.
- Control of Teukolsky variables $A, \underline{A}$.

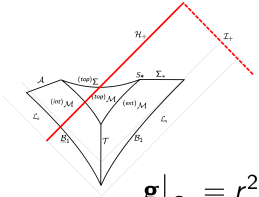


GCM SPHERES AND GAUGE FIXING

MODULATION I = choice of S_*, constants (m, a), last slice Σ_*.

- GCM1[Kl-Szeftel 19].
  - Fix 5 geometric quantities to agree with their Kerr values.
- GCM2[Kl-Szeftel 19].
  - Effective (stable) version of the uniformization theorem on S_*.
  - Intrinsic definitions of (m, a) and " axis of rotation" on S_*.
- GCM-Σ_*[Kl-Szeftel18], [Shen22]. Extend to Σ_*- more GCM conditions.

GCM SPHERE S_*. INTRINSIC CONDITIONS.



$$\mathbf{g}\big|_{S_*} = r^2 e^{2\phi}\Big((d\theta)^2 + \sin^2\theta (d\varphi)^2\Big)$$

$$J^{(0)} := \cos\theta, \quad J^{(-)} := \sin\theta\sin\varphi, \quad J^{(+)} := \sin\theta\cos\varphi,$$

EFFECTIVE UNIFORMIZATION. Aubin, K-Szeftel(2019)

$$\int_{S_*} J^- = \int_{S_*} J^0 = \int_{S_*} J^+ = 0$$

- Existence and uniqueness.
- Stability.

GCM SHERE S_*- EXTRINSIC CONDITIONS

$$\mathrm{tr}\,\chi = \frac{2}{r}, \quad \mathrm{tr}\,\underline{\chi} = -\frac{2(1-\frac{2m}{r})}{r}, \quad \left(\mathrm{div}\,\zeta - \rho - \frac{2m}{r^3}\right)_{\ell\geq 2} = 0,$$

$$\int_{S_*} J^{(p)}\,\mathrm{div}\,\beta \quad = \quad 0, \qquad p = 0, +, -,$$

$$\int_{S_*} J^{(+)} curl\,\beta \quad = \quad 0, \qquad \int_{S_*} J^{(-)} curl\,\beta = 0.$$

DEFINE:

$$\frac{2m}{r} \quad = \quad 1 + \frac{1}{16\pi}\int_{S_*} \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi},$$

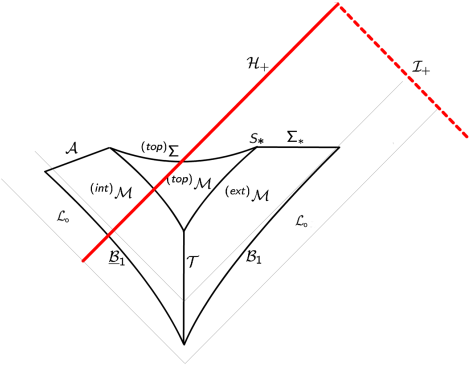
$$a \quad = \quad \frac{r^3}{8\pi m}\int_{S_*} J^{(0)} curl\,\beta.$$

GCM PAPERS[K-Szeftel(2019)]

GCM SPHERES AND GAUGE FIXING

MODULATION II

- Extend to $^{(ext)}\mathcal{M}$ - outgoing geodesic foliation.
- Extend to $^{(int)}\mathcal{M}$ - ingoing geodesic foliation.



REMARK. Gauge is initialized from the **future** with no reference to the initial data! This induces a new foliation of $\mathcal{L}_0$. **Recoil**!

LACK OF INTEGRABILITY $a \neq 0$

RECALL. Kerr possesses principal null directions e_3, e_4:

- Diagonalize the Riemann curvature tensor
- The horizontal bundle $\{e_3, e_4\}^\perp$ is non-integrable!

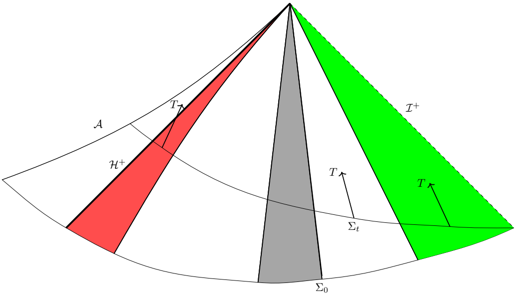
Rely on a geometric, non integrable, formalism.

- Construct two non-integrable gauges adapted to decay (PG) and energy estimates (PT).
- Control of Hodge type systems on spheres? - construct associated integrable structures.
- Replaces the Newman-Penrose formalism.

CONTROL OF TEUKOLSKY. A-B METHOD

STANDARD VECTORFIELD METHOD FAILS!

- Nontrivial ergoregion
- Degeneracy at the trapped set: $\mathcal{M}_{trap} = \mathcal{M} \cap \left\{ \frac{|\mathcal{T}|}{r^3} \leq \delta \right\}$
$$\mathcal{T} = r^3 - 3mr^2 + a^2 r + ma^2.$$
- Limited symmetries in Kerr. $\mathbf{T} = \partial_t$, $\mathbf{Z} = \partial_\varphi$



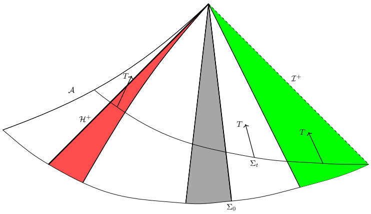
CONTROL OF TEUKOLSKY. A-B METHOD

STANDARD SYMMETRY. $[\mathbf{T}, \Box_{a,m}] = [\mathbf{Z}, \Box_{a,m}] = 0.$

CARTER OPERATOR. $[\mathcal{K}, \Box_{a,m}] = 0.$

MODIFIED LAPLACIAN $[\mathcal{O}, |q|^2\Box_{a,m}] = 0.$

$$\mathcal{O} := \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 + 2a\partial_t\partial_\phi + a^2\sin^2\theta\,\partial_t^2$$

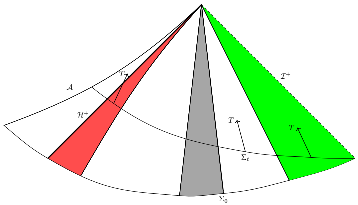


CONTROL OF $(\mathfrak{q}, A)$, $(\underline{\mathfrak{q}}, \underline{A})$

1a. Estimates for reduced gRW.

$$
\begin{aligned}
\dot{\square}_2 \mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} &= N,\\
\dot{\square}_2 \underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - \underline{V}\,\underline{\mathfrak{q}} &= \underline{N},
\end{aligned}
$$

- Morawetz-Energy estimates for $\mathfrak{q}, \underline{\mathfrak{q}}$ Extended VM.
- r^p-weighted estimates in ${}^{(ext)}\mathcal{M}$.
- Flux decay estimates
- Red shift estimates near horizon



1b. Estimates for the full linear gRW.

$$\begin{aligned} \dot{\square}_2 \mathfrak{q} - i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\mathfrak{q} - V\mathfrak{q} &= L_{\mathfrak{q}}[A] + \text{err}, \\ \nabla_3\nabla_3 A + C\cdot\nabla_3 A + D\cdot A &= \left(q\overline{q}^3\right)^{-1}\mathfrak{q} \end{aligned}$$

$$\begin{aligned} \dot{\square}_2 \underline{\mathfrak{q}} + i\frac{4a\cos\theta}{|q|^2}\nabla_{\mathbf{T}}\underline{\mathfrak{q}} - \underline{V}\underline{\mathfrak{q}} &= L_{\underline{\mathfrak{q}}}[\underline{A}] + \text{err} \\ \nabla_4\nabla_4 \underline{A} + \underline{C}\cdot\nabla_4 \underline{A} + \underline{D}\cdot\underline{A} &= \left(\overline{q}q^3\right)^{-1}\underline{\mathfrak{q}} \end{aligned}$$

1c. Estimates for the Error terms.

- Use BA for all $\check{\Gamma}, \check{R}$,
- Use special structure of the nonlinear terms.

SHORT HISTORY

1. Discovery of Kerr[1963].
2. Linear mode stability[1963-1975].
   - Regge-Wheeler[1957]. metric perturbations
   - Newmann-Penrose[1962]. **curvature perturbations**
   - Teukolsky equations[1973]. curvature perturbations
   - **Chandrasekhar transform**[1975].
   - Whiting[1989].
3. Global Stability of Minkowski space [1993]
   - **Vectorfield method**
     - Local Energy Decay[1961].
     - Pointwise Decay[1985].
   - **Null condition**[1983, 1986].

SHORT HISTORY

4. Robust decay for scalar waves [2003-2014]
   - $a = 0$, $m > 0$. **Soffer, Blue-S.** Morawetz monotonicity!
   - $a = 0$, $m > 0$. B-Sterbenz, Daf-Rodn, Marzuola-Metcalf-Tataru-Tohaneanu
   - $a \ll m$. D-R, T-T, **Andersson-Blue**

5. Robust decay for spin-2 waves [2016-2019]
   - $a = 0$. **D-Holzegel-R**.
   - $|a| \ll m$. **Ma**, D-H-R

6. Linear stability
   - $a = 0$. D-H-R[2016], Hung-Keller- M.T.Wang.
   - $a \ll m$. A-Bäckdahl-Blue-Ma[2019], Hintz-Vasy[2021].

SHORT HISTORY

6. Nonlinear stability of Schwarzschild
   - Polarized case **K-Szeftel**[2018].
   - Codim 3 Data DHR+Taylor[2021].
7. GCM Spheres and hypersurfaces in perturbations of Kerr
   - Construction of GCM spheres [K-S(2018)].
   - Intrinsic GCM spheres [K-S(2019)].
   - Construction of GCM hypersurfaces [Shen(2022)].
8. Nonlinear stability of slowly rotating Kerr
   - Kerr stability for small angular momentum[K-S(2021)].
   - Wave equations estimates and the nonlinear stability of slowly rotating Kerr black holes[Giorgi-K-S(2022)].