Sheaves for spacetime

Pierre Schapira

Sorbonne University

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We shall study the Cauchy problem on globally hyperbolic manifolds but with a totally different approach to traditional ones. Instead of using distributions and functional analysis, we will use sheaves, more precisely microlocal sheaf theory, and $\mathscr{D}$-modules. The unique tool from analysis will be the precise Cauchy-Kowalevska theorem.

Some historical comments

Microlocal analysis : introduced by Mikio Sato in 1970 with the tools of complex analysis and sheaf theory. He defined first the analytic wave front set of hyperfunctions, hence in particular, of distributions. He was soon followed by Lars Hörmander who used Fourier analysis.

$\mathscr{D}$**-module theory:** initiated by Sato in the 60s, developed by Masaki Kashiwara in his master's thesis in 1970 and independently by Joseph Bernstein in 1971.

Microlocal sheaf theory: Kashiwara-S in 1982 developed in 1985 and giving rise to the book "Sheaves on Manifolds" 1990.
The theory of analytic linear PDE (i.e., generalized holomorphic solutions of $\mathscr{D}$-modules) becomes series of exercises of microlocal sheaf theory.

Plan of my talk

Introduction: sheaves and $\mathscr{D}$-modules
Microlocal sheaf theory (with Masaki Kashiwara)
Applications to causal manifolds (with Benoît Jubin)
Sheaves for the Big Bang

Basic geometrical notions

Let X be a real manifold of class C^∞.

- $\tau\colon TX \to X$ the tangent bundle, $\pi\colon T^*X \to X$ the cotangent bundle,
- for M a submanifold of X, we associate the normal bundle T_MX and the conormal bundle T^*_MX:

$$0 \to TM \to M \times_X TX \to T_MX \to 0,$$
$$0 \to T^*_MX \to M \times_X T^*X \to T^*M \to 0.$$

- X is identified with T^*_XX, the zero-section of T^*X.

For example, if M is the hypersurface $\{x \in X; \varphi(x) = 0\}$ with $d\varphi \neq 0$ on M, then

$$T^*_MX = \{(x; \lambda \cdot d\varphi(x)); \varphi(x) = 0, \lambda \in \mathbb{R}\}.$$

Let $f\colon X \to Y$ a morphism of manifolds. We get the maps

$$
\begin{array}{ccccc}
TX & \xrightarrow{f'} & X \times_Y TY & \xrightarrow{f_\tau} & TY \\
& \searrow^{\tau_X} & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_Y} \\
& & X & \xrightarrow{f} & Y
\end{array}
$$

Recall that $X \times_Y TY = \{(x;(y,v)), x \in X, (y,v) \in TY, f(x) = y\}$.
By duality, we get the maps

$$
\begin{array}{ccccc}
T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\
& \searrow^{\pi_X} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\
& & X & \xrightarrow{f} & Y
\end{array}
$$

Sheaves

Let X be a real manifold of class C^∞. We fix a field $\mathbf{k}$ (in practice, $\mathbf{k} = \mathbb{C}$). We work in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves on X. Hence, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ is represented by a bounded complex of sheaves. For example, the de Rham complex is isomorphic to the constant sheaf $\mathbb{C}_X$ or, on a complex manifold, the sheaf $\mathscr{O}_X$ is isomorphic to the Dolbeault complex.

For A a locally closed subset of X, one denotes by $\mathbf{k}_A$ the constant sheaf on A extended by 0 on $X \setminus A$.

We shall define later the "singular support" or micro-support of F, denoted $\mathrm{SS}(F)$, a closed $\mathbb{R}^+$-conic subset of T^*X.

$\mathscr{D}$-modules

Let $(X, \mathscr{O}_X)$ be a complex manifold, $\mathscr{D}_X$ the sheaf of rings of finite order differential operators. An object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is locally isomorphic to a bounded complex where the $\cdot P_j$'s are matrices of differential operators which operate on the right.

$$\mathscr{M} \simeq 0 \to \mathscr{D}_X^{N_r} \to \cdots \to \mathscr{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathscr{D}_X^{N_0} \to 0.$$

Then $\mathrm{Sol}_X(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is given by

$$\mathrm{Sol}_X(\mathscr{M}) \simeq 0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0 \cdot} \mathscr{O}_X^{N_1} \to \cdots \to \mathscr{O}_X^{N_r} \to 0,$$

where now the $P_j\cdot$'s operate on the left.
One can define the characteristic variety of $\mathscr{M}$, denoted $\mathrm{char}(\mathscr{M})$, a closed complex analytic **co-isotropic** (S-K-K, Gabber) subset of T^*X, conic with respect to the action of $\mathbb{C}^\times$ on T^*X.

The Cauchy–Kowalevska-Kashiwara theorem

Let Y be a complex submanifold of the complex manifold X and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. One defines $\mathscr{M}_Y \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$, the induced system by $\mathscr{M}$ on Y. In general, $\mathscr{M}_Y$ is not coherent. One says that

Y is non-characteristic for $\mathscr{M}$ if $\mathrm{char}(\mathscr{M}) \cap T^*_Y X \subset T^*_X X$.

Recall that $\mathrm{Sol}_X(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$.

Theorem (Kashiwara 70)

Assume Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M}_Y$ is a coherent $\mathscr{D}_Y$-module and one has the CKK isomorphism:

$$\mathrm{Sol}_X(\mathscr{M})|_Y \xrightarrow{\sim} \mathrm{Sol}_Y(\mathscr{M}_Y).$$

The proof is deduced from the classical Cauchy-Kowalevska theorem by purely algebraic arguments.

An example

Let P be a differential operator on X open in $\mathbb{C}^n$. Set $\mathscr{M} = \mathscr{D}_X / \mathscr{D}_X \cdot P$. Then

$$\mathrm{char}(\mathscr{M}) = \{(z;\zeta) \in T^*X; \sigma(P)(z;\zeta) = 0\},$$

where $\sigma(P)$ denotes the principal symbol of P.

$$\mathrm{Sol}_X(\mathscr{M}) \text{ is the complex } 0 \to \mathscr{O}_X \xrightarrow{P\cdot} \mathscr{O}_X \to 0.$$

If P has order m and Y is a non-characteristic hypersurface, then $\mathscr{M}_Y \simeq \mathscr{D}_Y^m$ and the CKK theorem gives

$$\mathrm{Ker}\, P|_Y \simeq \mathscr{O}_Y^m, \quad coker\, P|_Y \simeq 0.$$

Sato's hyperfunctions

Let M be a real analytic manifold, X a complexification of M. Set

$$\mathrm{D}'_X F = \mathrm{R}\mathscr{H}om(F, \mathbb{C}_X).$$

Then

$$\mathscr{A}_M = \mathbb{C}_M \otimes \mathscr{O}_X,$$
the sheaf of real analytic functions,
$$\mathscr{B}_M = \mathrm{R}\mathscr{H}om(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X),$$
the sheaf of Sato's hyperfunctions.

Since $\mathbb{C}_M \simeq \mathrm{R}\mathscr{H}om(\mathrm{D}'_X\mathbb{C}_M, \mathbb{C}_X)$, we get the natural map

$$\mathscr{A}_M \simeq \mathrm{R}\mathscr{H}om(\mathrm{D}'_X\mathbb{C}_M, \mathbb{C}_X) \otimes \mathscr{O}_X, \to \mathscr{B}_M.$$

Note that if $\dim M = n$, then $\mathrm{D}'_X\mathbb{C}_M \simeq \mathrm{or}_M[-n]$ and thus

$$\mathscr{B}_M = \mathrm{R}\mathscr{H}om(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X) \simeq \mathrm{R}\Gamma_M(\mathscr{O}_X) \otimes \mathrm{or}_M[n] \simeq H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M.$$

Elliptic systems

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and set $F = \mathrm{Sol}_X(\mathscr{M})$.

Theorem

Assume $\mathscr{M}$ is elliptic, that is, $\mathrm{char}(\mathscr{M}) \cap T^*_M X \subset T^*_X X$. *Then*

$$\mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, \mathbb{C}_X) \otimes F \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, F).$$

In other words

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M).$$

This is a particular case of the following result for which X is real. (No complex structure, no $\mathscr{D}_X$-modules.)
The notation $\mathrm{SS}(F)$ will be defined in the sequel.

Theorem (Petrowsky theorem for sheaves)

Let $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ *and assume* $\mathrm{SS}(G) \cap \mathrm{SS}(F) \subset T^*_X X$. *Then*

$$\mathrm{R}\mathscr{H}om(G, \mathbf{k}_X) \otimes F \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(G, F).$$

$G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ means that G is $\mathbb{R}$-constructible, that is, there exists a "subanalytic" stratification such that all cohomology groups $H^j(G)$ are locally constant on the strata and of finite rank.
Example, $= (X \setminus M) \sqcup M$, $G = \mathrm{D}'_X \mathbb{C}_M$.

Microlocal sheaf theory

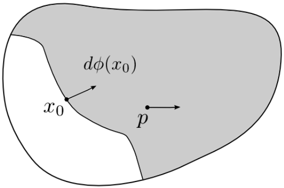
Let X be a real manifold.

Definition

Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. The microsupport $\mathrm{SS}(F)$ of F is the closed $\mathbb{R}^+$-conic subset of T^*X defined as follows: for an open subset $W \subset T^*X$, one has $W \cap \mathrm{SS}(F) = \emptyset$ if and only if for any $x_0 \in X$ and any real $\mathscr{C}^1$-function φ on X with $(x_0; d\varphi(x_0)) \in W$, one has $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

The micro-support is the set of co-directions of non propagation.



Set $U = \{x \in X; \varphi(x) < \varphi(x_0)\}$. Then $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} \simeq 0$ if and only if

$$\varinjlim_{V \ni x_0} H^j(U \cup V; F) \xrightarrow{\sim} H^j(U; F) \text{ for all } j \in \mathbb{Z}.$$

Any section of $H^j(U; F)$ will extend through the boundary in a neighborhood of x_0.

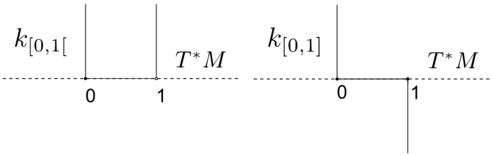
- By its construction, the microsupport is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*X.
- $\mathrm{SS}(F) \cap T^*_X X = \pi(\mathrm{SS}(F)) = \mathrm{supp}(F)$.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.
- The microsupport $\mathrm{SS}(F)$ is **co-isotropic**.

Examples

(i) $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ is a local system (i.e., F is locally constant) if and only if $\mathrm{SS}(F) \subset T^*_X X$.

(ii) If M is a closed submanifold of X and $F = \mathbf{k}_M$, then $\mathrm{SS}(F) = T^*_M X$, the conormal bundle to M in X.

(iii) When $X = \mathbb{R}$ (denoted M on the picture!), $F = \mathbf{k}_I$ where I is one of the intervals $[0, 1[$ or $[0, 1]$:



Let X be a complex manifold and let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Then

$$\mathsf{SS}(\mathrm{Sol}_X(\mathscr{M})) = \mathsf{char}(\mathscr{M}).$$

The proof only uses the Cauchy-Kowalevska theorem, in a precise form due to Petrowsky, Leray, Zerner.

Let $f\colon X \to Y$ a morphism of manifolds. Recall the maps

$$\begin{array}{ccccc} T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ & \searrow^{\pi_X} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\ & & X & \xrightarrow{f} & Y \end{array}$$

Theorem

Let $f\colon X \to Y$ be a morphism of manifolds, let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$.

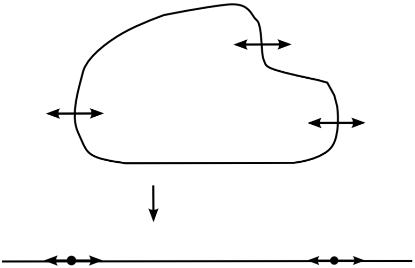
- (a) *Assume that f is proper on* $\mathrm{supp}(F)$. *Then* $\mathrm{SS}(\mathrm{R}f_!F) \subset f_\pi f_d^{-1}\,\mathrm{SS}(F)$.
- (b) *Assume that f is non characteristic for G, that is, f_d is proper on* $f_\pi^{-1}\,\mathrm{SS}(G)$. *Then* $\mathrm{SS}(f^{-1}G) \subset f_d(f_\pi^{-1}\,\mathrm{SS}(G))$. *Moreover* $f^{-1}G \otimes \omega_{X/Y} \xrightarrow{\sim} f^!F$.

Direct image

On the example below, $X = \mathbb{R}^2$, f is the projection to $\mathbb{R} = Y$. Recall

$$T^*X \xleftarrow{f_d} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y.$$

Hence, f is submersive, $f_d \colon X \times_Y T^*Y \hookrightarrow T^*X$ is an embedding and $X \times_Y T^*Y \subset T^*X$ is the set of "horizontal" co-vectors. The theorem says that the micro-support of the direct image is contained in $f_\pi(SS(F) \cap X \times_Y T^*Y)$, but the inclusion may be strict. There is a similar phenomena for inverse images.



Microsupport along a submanifold

Let M be a closed submanifold of the real manifold X. The projection $T^*_M X \to M$ (a submersive map) defines the embedding $T^*M \times_M T^*_M X \hookrightarrow T^* T^*_M X$, hence the embedding:

$$T^*M \hookrightarrow T^*T^*_M X.$$

In local coordinates $(x, y; \xi, \eta) \in T^*X$, $M = \{y = 0\}$, $T^*_M X = \{y = \xi = 0\}$, $T^*M \hookrightarrow T^*T^*_M X$ is given by

$$(x; \xi) \mapsto (x, 0; \xi, 0).$$

Let $A \subset T^*X$. We shall use the Whitney normal cone of A along $T^*_M X$

$$C_{T^*_M X}(A) \subset T_{T^*_M X} T^*X \simeq T^*T^*_M X.$$

$$(x_0; \xi_0) \in T^*M \cap C_{T^*_M X}(A) \quad \Leftrightarrow \quad \text{there exists } (x_n, y_n; \xi_n, \eta_n)_n \subset A,$$
$$(x_n; \xi_n) \xrightarrow{n} (x_0; \xi_0), |y_n| \xrightarrow{n} 0, |y_n||\eta_n| \xrightarrow{n} 0.$$

Let X be a real manifold, M a submanifold. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one defines the micro-support of F along M as

$$\mathsf{SS}_M(F) := T^*M \cap C_{T^*_M X}(\mathsf{SS}(F)).$$

Theorem

One has

$$\mathsf{SS}(\mathrm{R}\Gamma_M F) \cup \mathsf{SS}(F|_M) \subset \mathsf{SS}_M(F).$$

Now let X be a complex manifold, M a real submanifold. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. One sets

$$\mathrm{char}_M(\mathscr{M}) := T^*M \cap C_{T^*_M X}(\mathrm{char}(\mathscr{M})).$$

Assume that X is a complexification of M. Let $N \hookrightarrow M$ be a closed submanifold of M and $Y \hookrightarrow X$ a complexification of N in X. Assume Y is non characteristic for $\mathscr{M}$. Applying the preceding result to $F = \mathrm{Sol}(\mathscr{M})$, using $\mathrm{R}\Gamma_N(F) \simeq \mathrm{R}\Gamma_N \mathrm{R}\Gamma_Y(F) \simeq \mathrm{R}\Gamma_N(F|_Y \otimes \omega_{X/Y})$, we get

Corollary

Assume $T^*_N M \cap \mathrm{char}_M(\mathscr{M}) \subset T^*_M M$. Then the CKK-isomorphism induces the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

In other words, the Cauchy problem is well-posed in the framework of Sato's hyperfunctions for (weakly) hyperbolic systems.

Example

Assume X open in $\mathbb{C}^n$, $M = X \cap \mathbb{R}^n$.
Let $(x + \sqrt{-1}y; \xi + \sqrt{-1}\eta)$ denotes the coordinates on T^*X, hence $T^*_M X = \{y = \xi = 0\}$.
Let $\mathscr{M} = \mathscr{D}_X / \mathscr{D}_X \cdot P$ for a differential operator P. Let $(x_0; \theta) \in T^*M$ with $\theta \neq 0$. Then

$$(x_0; \theta) \notin \mathrm{char}_M(\mathscr{M}) \Leftrightarrow \sigma(P)(x; \theta + \sqrt{-1}\eta) \neq 0$$

for all $\eta \in \mathbb{R}^n$ and x in a neighborhood of x_0.
For example, the Cauchy problem on $X = \mathbb{R}^2$ with coordinates (t, x) is well-posed for $P = \partial_t^2 - \partial_x$ on the hypersurface $\{t = 0\}$ in the space of hyperfunctions. This is is no more true for distributions (Hadamard).

Causal manifolds

The part of the talk concerned with spacetime is based on a joint paper with Benoît Jubin published at LMP (2016).

Definition

(a) A causal manifold (M, λ) is a nonempty connected manifold M equipped with a closed convex proper cone $\lambda \subset T^*M$ satisfying $T^*_M M \subset \lambda$ and $\lambda = \overline{\mathrm{Int}(\lambda)}$.

(b) A morphism of causal manifolds $f\colon (M, \lambda_M) \to (N, \lambda_N)$ is a morphism of manifolds $f\colon M \to N$ satisfying $f_d f_\pi^{-1} \lambda_N \subset \lambda_M$. Equivalently, $Tf(\lambda_M^\circ) \subset \lambda_N^\circ$, where λ° denotes the polar cone.

The polar cone $\lambda^\circ \subset TX$ is given by

$$\lambda^0 = \{v \in TX; \langle v, \xi \rangle \geq 0\} \text{ for all } \xi \in \lambda.$$

Definition

A locally closed set $A \subset M$ is a λ-set if $\mathrm{SS}(\mathbf{k}_A) \subset \lambda$. A λ-open set (resp. a λ-closed set) is an open set (resp. a closed set) which is also a λ-set.

An open set $U \subset M$ is λ-open if in a local chart $V \subset \mathbb{R}^n$ in a neighborhood of $x_0 \in \partial U$, for any open convex cone $\gamma \subset \mathbb{R}^n$ with $V \times \overline{\gamma} \subset \mathrm{Int}\lambda^{\circ a}$, one has $(x + \gamma) \subset U$ in a neighborhood of $x \in \partial U$.

Theorem

The family of λ-open subsets of M defines a (classical) topology on M.

- (a) For $A \subset M$, we denote by $J_\lambda^+(A)$ the closure of A for the λ-topology. In other words

$$J_\lambda^+(A) = \bigcap Z \text{ with } A \subset Z \text{ and } Z \text{ is } \lambda\text{-closed.}$$

- (b) For $x \in M$, we set $J_\lambda^+(x) = J_\lambda^+(\{x\})$.
- (c) We call $J_\lambda^+(A)$ the future of A.
- (d) One defines the pre-order $\preceq_\lambda$ by

$$x \preceq_\lambda y \text{ if and only if } y \in J_\lambda^+(x).$$

Let I be the interval $[0,1]$ of the real line $\mathbb{R}$ with the coordinate t.

Definition

- (a) A path $c\colon I \to M$ is a piecewise smooth (ps for short) map.
- (b) If (M,λ) is a causal manifold, the path c is *causal* if $c'_l(t), c'_r(t) \in (\lambda^\circ)_{c(t)}$ for any $t \in I$. Here, $c'_l(t)$ and $c'_r(t)$ are the left and right derivative.
- (c) One denotes by $\underset{ps}{\preceq}$ the preorder given by $x \underset{ps}{\preceq} y$ if there exists a causal path $c\colon I \to M$ with $c(0) = x$ and $c(1) = y$.

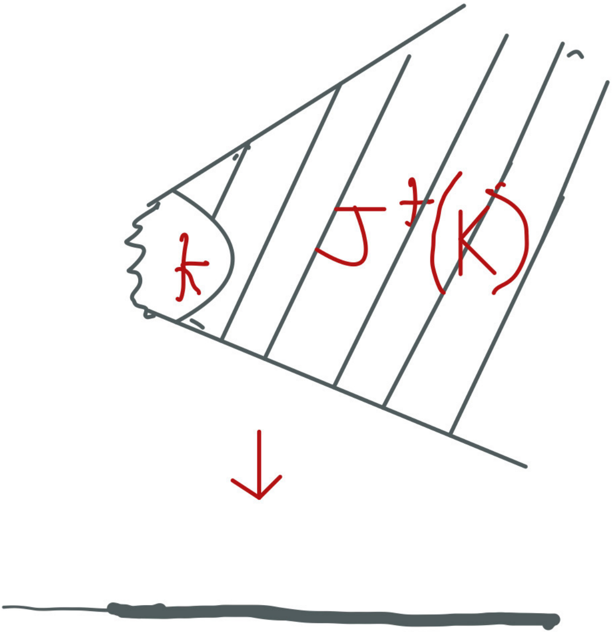
Theorem

The preorder $\underset{ps}{\preceq}$ *is causal, that is,* $x \preceq_\lambda y$ *implies* $x \underset{ps}{\preceq} y$.

Let $(t;\tau)$ be the coordinates on $T^*\mathbb{R}$. We denote for short by $(\mathbb{R},+)$ the causal manifold associated with the cone $\{\tau \geq 0\}$.

Definition (Jubin-S)

a. Let (M,λ) be a causal manifold. A Cauchy time function $q\colon (M,\lambda) \to (\mathbb{R},+)$ is a surjective and submersive morphism of causal manifolds such that for any compact set $K \subset M$, the map q is proper both on $J_\lambda^+(K)$ and on $J_\lambda^-(K)$.
b. A G-causal (G for Geroch) manifold (M,λ,q) is a causal manifold endowed with a Cauchy time function q.


k
J†(k)

A classical causal manifold (M, g) is globally hyperbolic if diamonds are compact and there are no causal loops.

Theorem

If (M, λ) is globally hyperbolic then there exists a Cauchy time function q and thus (M, λ, q) is G-causal.

See Geroch70, Bernal-Sanchez 2005, the survey paper Minguzi-Sanchez08. See also Fathi-Siconolli for a more general version.

Lemma (Jubin-S)

Let (M, λ, q) be a G-causal manifold and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Assume that $\mathrm{SS}(F) \cap \lambda \subset T^*_M M$. Then

$$\mathrm{SS}(\mathrm{R}q_* F) \cap \{\tau \geq 0\} \subset T^*_{\mathbb{R}}\mathbb{R}.$$

Proof. Set $Z = J^-_\lambda(K)$. Then $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^a$ and thus $\mathrm{SS}(F_Z) \cap \lambda \subset T^*_M M$. Since q is proper on Z, $\mathrm{SS}(\mathrm{R}q_* F_Z) \cap \{\tau \geq 0\} \subset T^*_{\mathbb{R}}\mathbb{R}$. Then one covers (locally on $\mathbb{R}$) the space M with an increasing family of such Z.

Theorem (Jubin-S)

Let (M, λ, q) be a G-causal manifold. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ satisfying $\mathrm{SS}(F) \cap (\lambda \cup \lambda^a) \subset T^*_M M$.
Set $M_0 = q^{-1}(0)$, a Cauchy hypersurface. Then the natural restriction morphism below is an isomorphism:

$$\mathrm{R}\Gamma(M; F) \xrightarrow{\sim} \mathrm{R}\Gamma(M_0; F|_{M_0}).$$

In particular, for all $j \in \mathbb{Z}$,

$$H^j(M; F) \xrightarrow{\sim} H^j(M_0; F|_{M_0}).$$

Now consider an analytic G-causal manifold (M, λ, q), that is, M and q are real analytic. Let X be a complexification of M and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Applying the preceding results, we get:

Theorem (Jubin-S16)

Let (M, λ, q) be an analytic G-causal manifold. Let $N = q^{-1}(0)$ and let Y be a complexification of N in X. Assume

a. *Y is non characteristic for $\mathscr{M}$, i.e.,* $\mathrm{char}(\mathscr{M}) \cap T^*_Y X \subset T^*_X X$,
b. *N is hyperbolic for $\mathscr{M}$, i.e.,* $\mathrm{char}_M(\mathscr{M}) \cap T^*_N M \subset T^*_M M$,
c. $\mathrm{char}_M(\mathscr{M}) \cap \lambda \subset T^*_M M$.

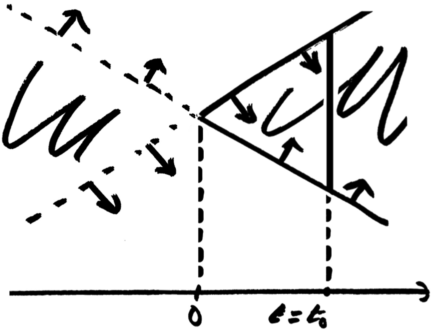
Then one has the natural isomorphism

$$\mathrm{R}\Gamma(M; \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)) \xrightarrow{\sim} \mathrm{R}\Gamma(N; \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N)).$$

In other words, the Cauchy problem for hyperfunctions with data on N is *globally* well-posed on M.
The theorem applies when $\mathscr{M} = \mathscr{D}_X / \mathscr{D}_X \cdot P$ for P a wave operator.

Before the Big Bang

Let us represent the universe as a closed ball in $\mathbb{R}^n$ whose radius grows linearly with the time t. What happens for $t < 0$? If one replaces the spacetime with the constant sheaf supported by it, the sheaf $\mathbf{k}_{\{|x| \leq t\}}$ defined on $t \geq 0$, we need to extend it naturally for $t < 0$. The micro-support of this sheaf at the boundary is the interior conormal. If we extend it naturally for $t < 0$ we get the exterior conormal which is the micro-support of the constant sheaf on the open cone.



With Guillermou and Kashiwara, we have constructed a “distinguished triangle” as follows. Set $X = \mathbb{R}^n_x \times \mathbb{R}_t$. The morphism $\mathbf{k}_{\{|x|\le -t\}} \to \mathbf{k}_{\{0\}}$ and the isomorphisms

$$\mathrm{D}'_X \mathbf{k}_{\{0\}} \simeq \mathbf{k}_{\{0\}}\,[-n-1], \quad \mathrm{D}'_X \mathbf{k}_{\{|x|\le -t\}} \simeq \mathbf{k}_{\{|x|<-t\}}$$

induce by duality the morphism

$$\mathbf{k}_{\{0\}}[-n-1] \to \mathbf{k}_{\{|x|<-t\}}.$$

Composing with $\mathbf{k}_{\{|x|\le t\}} \to \mathbf{k}_{\{0\}}$, we get the morphism $\mathbf{k}_{\{|x|\le t\}} \xrightarrow{\psi} \mathbf{k}_{\{|x|<-t\}}\,[n+1]$ hence a distinguished triangle

$$\mathbf{k}_{\{|x|<-t\}}[n] \to K \to \mathbf{k}_{\{|x|\le t\}} \xrightarrow[\psi]{+1}$$

The micro-support of K outside the zero-section is the smooth Lagrangian manifold, the image of $T^*_{\{0\}}\mathbb{R}^n$ by the Hamiltonian isotopy

$$(x;\xi) \mapsto (x - t\xi/|\xi|;\xi).$$

One can modify the Lorentzian case encountered above and replace $\mathbb{R}^n_x$ with a Riemannian manifold (with convexity radius and injectivity radius > 0) using the Hamiltonian isotopy associated with $||\xi||_x$.

In particular, one can consider the n-dimensional unit sphere $M = \mathbb{S}^n$ ($n \geq 2$) endowed with the canonical Riemannian metric. In this case, the sheaf obtained has a shift which jumps by the dimension when $t \in \pi\mathbb{Z}$, that is, at each pole.