based on 2206.05073, 2310.19655 and 2403.09261
with P. Hintz and D. Häfner

# The quantum scalar field on Kerr-de Sitter

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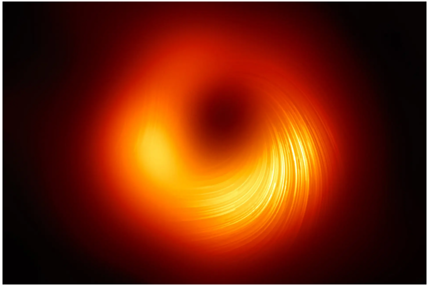
# Outline

# MOTIVATION

# Black holes - a stage for quantum physics



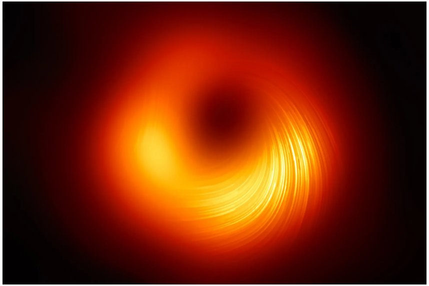
Picture of the black hole in M87

[EHT Collaboration et al.: 2021]

– Observational tests of GR

$\Rightarrow$ Rotating black holes and cosmological expansion

# Black holes - a stage for quantum physics



Picture of the black hole in M87

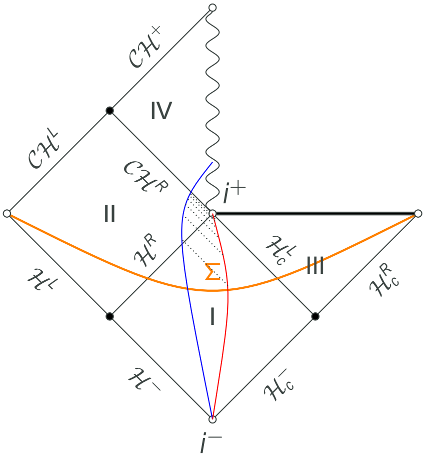
[EHT Collaboration et al.: 2021]

- Observational tests of GR

$\Rightarrow$ Rotating black holes and cosmological expansion

- Black hole evaporation
- Black hole stability
- Strong Cosmic Censorship conjecture

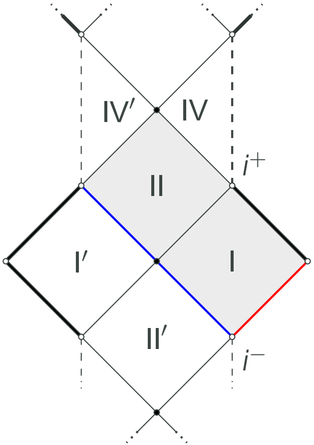
# Strong cosmic censorship




- Cauchy horizon beyond which events not determined by initial data on $\Sigma$
- Signals reaching $\mathcal{CH}^R$ infinitely blueshifted [Penrose: 1974] $\Leftrightarrow$ Cosmological redshift
- Strong cosmic censorship conjecture (sCC): For generic initial data, metric is inextendible across $\mathcal{CH}^R$ with certain regularity [Christodoulou: 2008]
- Scalar test field: sCC violated in RNdS for large charge [Hintz, Vasy: 2017, Cardoso et al: 2018]

# KERR(-DE SITTER)

# The Kerr spacetime

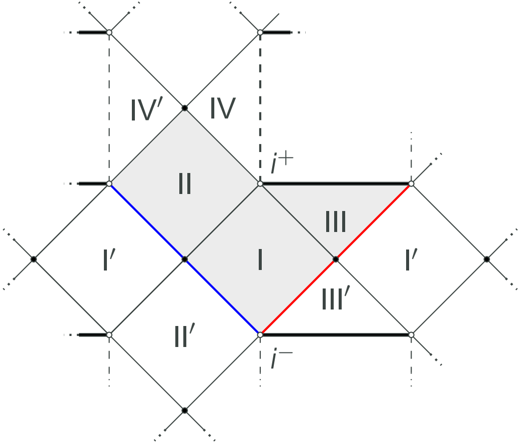



$$g = \frac{\sin^2\theta}{\rho^2}(a\,\mathrm{d}t - (r^2 + a^2)\,\mathrm{d}\varphi)^2 + \frac{-\Delta_r}{\rho^2}(\,\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2 + \rho^2\left(\frac{\mathrm{d}r^2}{\Delta_r} + \mathrm{d}\theta^2\right),$$

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\Delta_r = r^2 + a^2 - 2Mr$$

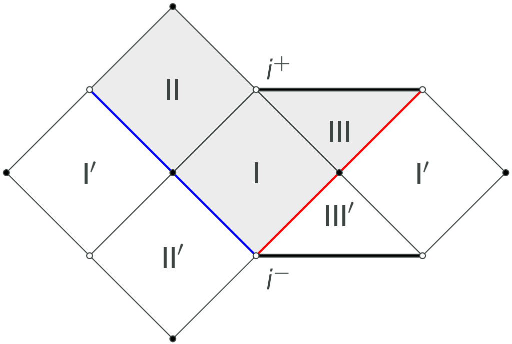
# The Kerr-de Sitter spacetime




$$
\begin{aligned}
g =& \frac{\Delta_\theta \sin^2\theta}{\chi^2\rho^2}(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\varphi)^2 \\
&+ \frac{-\Delta_r}{\chi^2\rho^2}(\,\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2 \\
&+ \rho^2\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}\theta^2}{\Delta_\theta}\right),
\end{aligned}
$$

$$
\begin{aligned}
\rho^2 =& r^2 + a^2\cos^2\theta, \\
\Delta_r =& (1 - r^2\Lambda/3)(r^2+a^2) - 2Mr, \\
\chi =& 1 + a^2\Lambda/3, \\
\Delta_\theta =& 1 + a^2\cos^2\theta\Lambda/3
\end{aligned}
$$

# The Kerr-de Sitter spacetime - Coordinates




- Horizons: $r_- \sim \mathcal{CH}$, $r_+ \sim \mathcal{H}$ and $r_c \sim \mathcal{H}_c$

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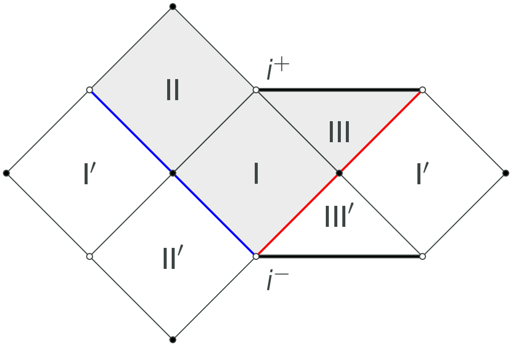


- Horizons: $r_- \sim \mathcal{CH}$, $r_+ \sim \mathcal{H}$ and $r_c \sim \mathcal{H}_c$
- $\varphi_\pm = \varphi \pm \int \frac{\chi a}{\Delta_r} dr$, $\varphi_i = \varphi - \Omega_i t$

  with $\Omega_i = \frac{a}{r_i^2 + a^2}$, $i \in \{-, +, c\}$.

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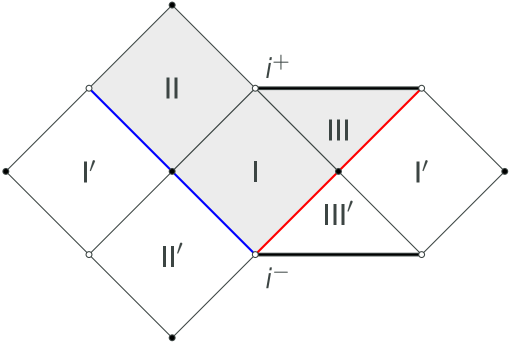


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- $v = t + r_*$, $u = t - r_*$,
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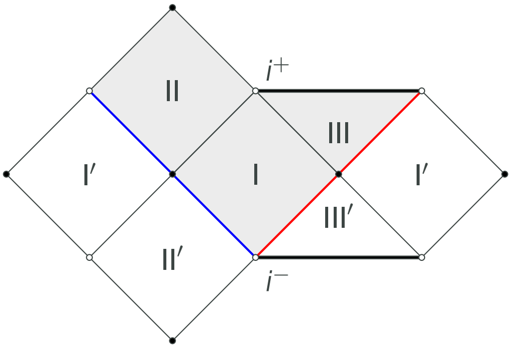


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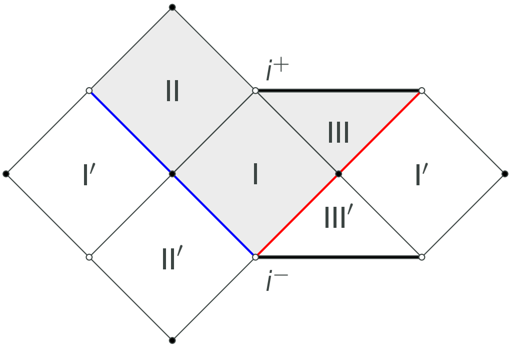
- $v = t + r_*$, $u = t - r_*$,

  with $dr_* = \frac{\chi(r^2+a^2)}{\Delta_r} dr$
- $U_+ = -e^{-\kappa_+ u}$, $V_+ = e^{\kappa_+ v}$,

  $U_c = e^{\kappa_c u}$ and $V_c = -e^{-\kappa_c v}$,

  with $\kappa_i = \frac{|\partial_r \Delta_r|_{r=r_i}}{2\chi(r_i^2 + a^2)}$

# The Kerr-de Sitter spacetime - Some results




Spacetime $M$ (gray) and its extension $\tilde{M}$

**Lemma ([CK: 2023])**

*$M$ and $\tilde{M}$ are globally hyperbolic.*

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*If $a$, $\Lambda$ sufficiently small, inextendible null geodesic not crossing $\mathcal{H}$ or $\mathcal{H}_c$ pass through region where $\partial_t + \Omega_+ \partial_\varphi$ and $\partial_t + \Omega_c \partial_\varphi$ are both timelike.*

[Gérard, Häfner, Wrochna: 2020]

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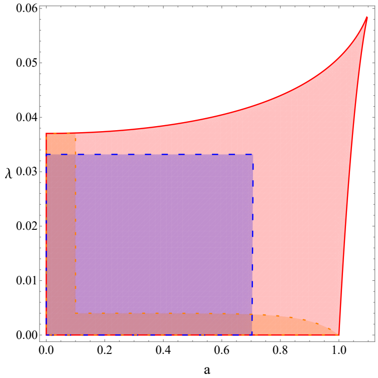


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[Gérard, Häfner, Wrochna: 2020]

# THE UNRUH STATE

# The scalar field theory

- Scalar field $\phi$: $\mathcal{K}\phi = (\Box_g - m^2)\phi = 0$
- Algebra $\mathcal{A}(M)$: free unital $*$-algebra generated by $\phi(f)$, $f \in C_0^\infty(M)$ and $\mathbf{1}$ and subject to relations
  - Linearity: $\phi(f + \alpha h) = \phi(f) + \alpha\phi(h)$,
  - Hermiticity: $\phi(f)^* = \phi(\bar{f})$,
  - KGE: $\phi(\mathcal{K}f) = 0$,
  - Commutation relation: $[\phi(f), \phi(h)] = iE(f, h)\mathbf{1}$

  for all $f, h \in C_0^\infty(M)$, $\alpha \in \mathbb{C}$.

[Fewster, Verch: 2015]

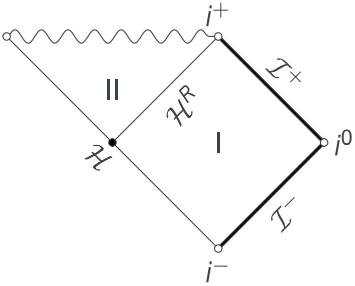
# Quasi-free states for the real scalar

- Quasi-free state: determined by its two-point function $w(f,h) = \langle \phi(f)\phi(h) \rangle$, satisfying
  - Bi-distribution: $w \in \mathcal{D}'(M \times M)$
  - Bi-solution: $w(\mathcal{K}f, h) = w(f, \mathcal{K}h) = 0$
  - Commutator property: $w(f,h) - w(h,f) = iE(f,h)$
  - Positivity: $w(\bar{f}, f) \geq 0$

  for all $f, h \in C_0^\infty(M)$
- Hadamard property: [Radzikowski:1996]

  $\mathrm{WF}(w) = \{(x,k;y,-l) \in T^*(M \times M)\backslash o : (x,k) \sim (y,l) \text{ and } k \text{ future pointing}\}$

# The Unruh state - Previous results



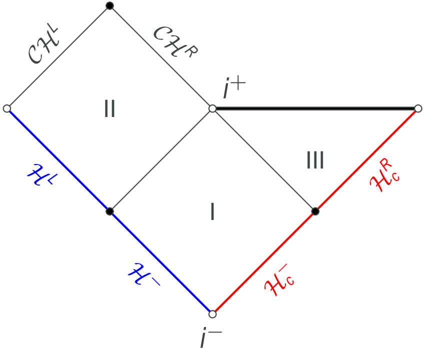

- Stationary, but non-equilibrium state
- Empty at $\mathcal{I}^-$, thermal energy flux at $\mathcal{I}^+$

⇒ Captures late-time behaviour in collapse [Häfner: 2009]

- Hadamard on
  - Schwarzschild [Dappiaggi, Moretti, Pinamonti: 2011]
  - Schwarzschild-de Sitter [Brum, Jorás: 2014]
  - Reissner-Nordström-de Sitter [Hollands, Wald, Zahn: 2019]
  - Kerr with small *a* for massless free fermions

    [Gérard, Häfner, Wrochna: 2020]

# The Unruh state - Idea




- Formally: $\phi(f) = \int_M \phi(x) f(x) \, \mathrm{d}vol_g(x)$
- Expand:

$$\phi(x) = \sum_J \int_0^\infty d\omega \, \psi_{\omega J}(x) a_{\omega J} + \overline{\psi}_{\omega J}(x) a^\dagger_{\omega J}$$

⇒ Unruh state: $a_{\omega J}|0\rangle = 0$

⇒ Two-point function:

$$\langle \phi(x)\phi(y)\rangle_U = \sum_J \int_0^\infty d\omega \psi_{\omega J}(x) \overline{\psi}_{\omega J}(y)$$

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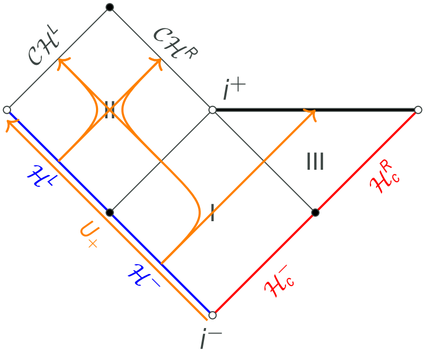


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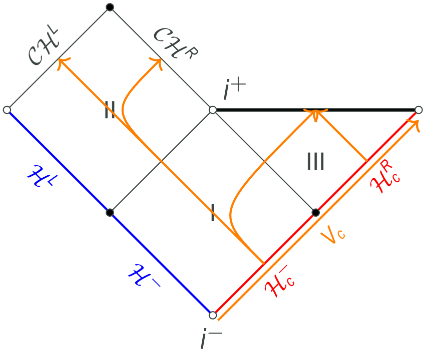


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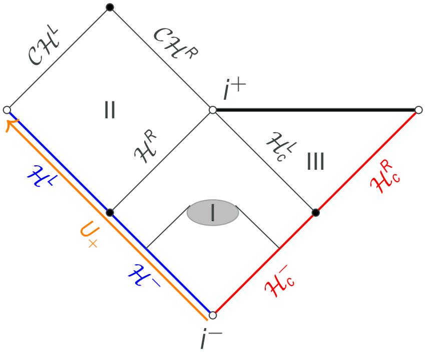
# The Unruh state

- Two-point function $w(f, h)$
- Map to functional on space-compact solutions to $\mathcal{K}\phi = 0$ using $E$
- Use a null boundary-to-bulk construction [Hollands: 2000, Moretti: 2008, Gérard, Wrochna: 2016]
- Use Kay-Wald two-point function at $\mathcal{H}$ and $\mathcal{H}_c$ [Kay, Wald: 1988]

$$\Rightarrow w(f,h) = -\lim_{\epsilon\to 0^+} \frac{r_+^2 + a^2}{\chi\pi} \int \frac{E(f)|_{\mathcal{H}}(U_+, \Omega_+) E(h)|_{\mathcal{H}}(U'_+, \Omega_+)}{(U_+ - U'_+ - i\epsilon)^2} \mathrm{d}U_+ \mathrm{d}U'_+ \mathrm{d}^2\Omega_+$$
$$- \lim_{\epsilon\to 0^+} \frac{r_c^2 + a^2}{\chi\pi} \int \frac{E(f)|_{\mathcal{H}_c}(V_c, \Omega_c) E(h)|_{\mathcal{H}_c}(V'_c, \Omega_c)}{(V_c - V'_c - i\epsilon)^2} \mathrm{d}V_c \mathrm{d}V'_c \mathrm{d}^2\Omega_c$$

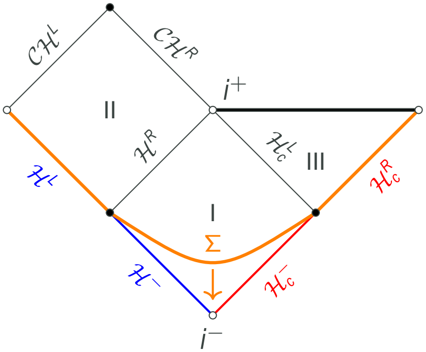
# Well-definedness, Commutator property, Positivity




- Near $i^-$: $|E(f)|_{\mathcal{H}}(U_+, \Omega_+)| \lesssim C|U_+|^{-\frac{\alpha}{\kappa_+}}$

[Hintz, Vasy: 2017]

# Well-definedness, Commutator property, Positivity

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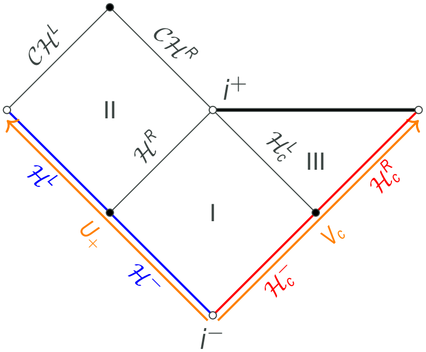
- Commutator: $E(f,h) = \int_\Sigma E(f) \overleftrightarrow{\nabla}_n E(h) d\Sigma$

$\Rightarrow$ Take limit $\Sigma \to \mathcal{H} \cup \mathcal{H}_c$

# Well-definedness, Commutator property, Positivity

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– Commutator: $E(f,h) = \int_\Sigma E(f) \overleftrightarrow{\nabla}_n E(h) d\Sigma$

⇒ Take limit $\Sigma \to \mathcal{H} \cup \mathcal{H}_c$

– $w(f,h) = \sum_{i=+,c} \left\langle K_i(\bar{f}), K_i(h) \right\rangle_{L^2(\mathbb{R}_+ \times \mathbb{S}^2; \nu_i)}$

⇒ $\nu_i = 2\eta(r_i^2 + a^2)\chi^{-1} \mathrm{d}\eta \mathrm{d}\Omega_i$

⇒ $K_i(f)(\eta, \Omega_i) = \mathcal{F}_i(E(f)|_{\mathcal{H}_i})|_{\eta \geq 0}(\eta, \Omega_i)$

[Dappiaggi, Moretti, Pinamonti: 2011]

# Hadamard property - Sketch of proof

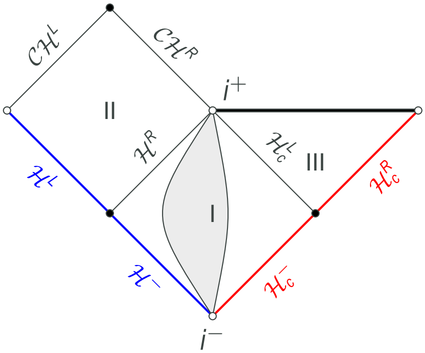
**Theorem (Propagation of Singularities [Duistermaat, Hörmander: 1972])**

*Let* $w \in \mathcal{D}'(M \times M)$ *be a weak bisolution to the Klein-Gordon-equation. If* $(x_1, k_1; x_2, k_2) \in T^*(M \times M)\backslash o$ *is in* $\mathrm{WF}'(w)$*, then* $k_1$ *and* $k_2$ *are null covectors or zero and* $\{(x_1', k_1'; x_2', k_2') : (x_j', k_j') \sim (x_j, k_j)\} \subset \mathrm{WF}'(w)$.

**Lemma ([Strohmaier, Verch, Wollenberg: 2002])**

*It suffices to show that* $w$ *satisfies* $\mathrm{WF}'(w) \cap \Delta_{T^*(M\times M)} \subset \mathcal{N}^+ \times \mathcal{N}^+$*, where* $\mathcal{N}^+ = \{(x, k) : k \text{ null and future pointing}\}$.

# Hadamard property - Sketch of proof

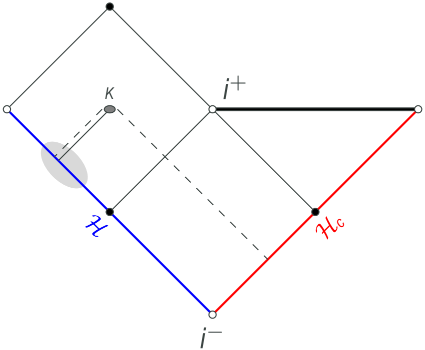



- Start with region $\mathcal{O}$ in which both $\partial_t + \Omega_+\partial_\varphi$ and $\partial_t + \Omega_c\partial_\varphi$ are timelike
- Utilize that $w_i$ is KMS-like [Dappiaggi, Moretti, Pinamonti: 2011] with respect to $\partial_t + \Omega_i\partial_\varphi$

$\Rightarrow$ Apply proof for passive states [Sahlmann, Verch: 2001]

- This covers all null geodesics not passing through one of the horizons

# Hadamard property - Sketch of proof



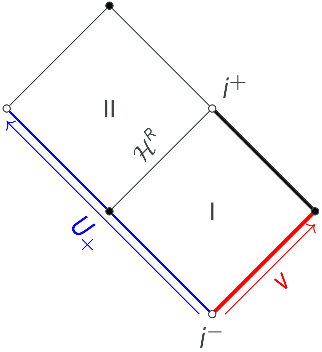

- Consider $(x_0, k_0) \in T^*M$ with $k_0 \neq 0$ null
- Splitting: $w = ((h \otimes h)A_+)(E|_{\mathcal{H}} \otimes E|_{\mathcal{H}})$ + rest terms

$\Rightarrow$ Direct computation for first term

$\Rightarrow$ Show that $(x_0, k_0; x_0, k_0) \notin \mathrm{WF}'(\text{rest terms})$

# THE UNRUH STATE ON KERR

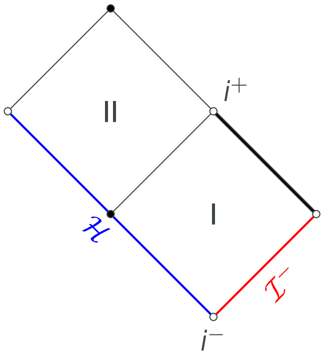
# Unruh state for massless fermions on Kerr




Kerr spacetime $\mathcal{M}_K$

- Consider massless, free fermions on Kerr: $\mathbb{D}\psi = 0$
- For sufficiently small $|a|$, the Unruh state is Hadamard [Gérard, Häfner, Wrochna: 2020]
- Smallness of $|a|$ enters in proof of Hadamard property for geodesics not ending at $\mathcal{H}$ or $\mathcal{I}^-$
- Sufficient if on these geodesics $k$ future directed if $k(\partial_t) > 0$ or $k(\partial_t + \Omega_+\partial_\varphi) > 0$

# The Kerr spacetime - backwards trapped set



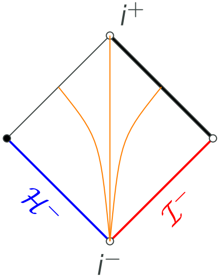

Kerr spacetime $\mathcal{M}_K$

**Proposition ([Häfner, CK: 2024])**

*Let $(x, k) \in T^*\mathcal{M}_K$ specify a null geodesic not reaching $\mathcal{H}$ or $\mathcal{I}^-$. Then if $k(\partial_t + \Omega_+\partial_\varphi) > 0$ or $k(\partial_t) > 0$, $k$ is future directed, i.e. $k(v) > 0$ for any timelike future directed $v \in T_x\mathcal{M}_K$.*

$\Rightarrow$Hadamard property holds for all $|a| < M$.

# Sketch of proof

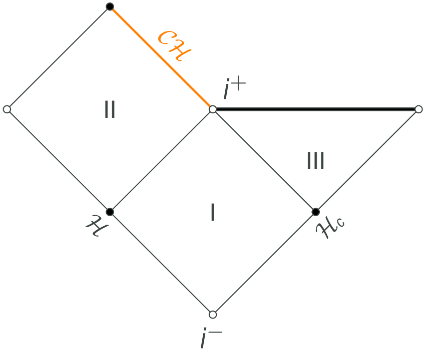



Region I

- restrict to exterior region I
- In BL-coordinates: $k$ future/past directed if $k_t \gtrless f(r,\theta)ak_\varphi$, with $|\Omega_+| > |a|f(r,\theta)$
- Trapped set:
  $\{\rho^2 g^{-1}(k,k) = \partial_r \rho^2 g^{-1}(k,k) = k_r = 0\} \subset T^*\mathrm{I} \setminus o$ [Dyatlov: 2015]
- Trapped set $\cap \{k_t \leq 0\} \cap \{k(\partial_t + \Omega_+ \partial_\varphi) > 0\} = \emptyset$
- Use relation of past-trapped geodesics and trapped set

# UNIVERSALITY AT THE CAUCHY HORIZON

# Universality of divergence




- Divergence of stress-energy tensor
- In certain RNdS spacetimes, leading divergence is independent of choice of Hadamard state [Hollands, Wald, Zahn:2020]

⇒ Does this hold on Kerr-de Sitter?

⇒ Does it hold for general observables of the form $A(x) = \mathcal{D}_1\phi(x)\mathcal{D}_2\phi(x)$?

# Universality on KdS

**Theorem ([Hintz, CK: 2023])**

*Let $\omega_{1,2}$ Hadamard states on $M$, $x_0 \in \mathcal{CH}$ a point on the Cauchy horizon, and let us denote*

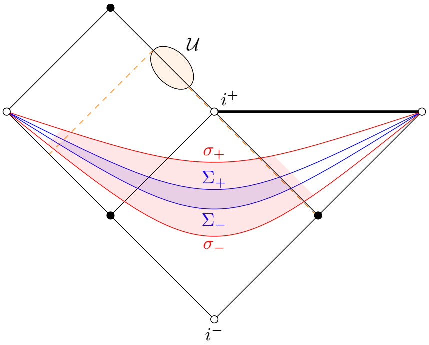
$$A[\omega_1, \omega_2](x) = \lim_{x' \to x} \left[ g(x, x') \mathcal{D}_1(x) \mathcal{D}_2(x') \left( \omega_1(\phi(x)\phi(x')) - \omega_2(\phi(x)\phi(x')) \right) \right] .$$

*Let $\mathcal{U}$ be a compact neighbourhood of $x_0$ in the analytic extension of $M$, and let us fix coordinates $(r - r_-, u, \theta, \varphi_-)$. Then, if $\alpha > 0$, there is a constant $C > 0$ so that*

$$\left| (r - r_-)^{2-\beta'} A[\omega_1, \omega_2]^{\mu_1, \dots, \mu_k}_{\nu_1, \dots, \nu_l}(x) \right| \leq C$$

*uniformly in $(u, \theta, \varphi_-)$ on $\mathcal{U} \cap M$ for some $0 < \beta' < \min\left(\frac{\alpha}{\kappa_-}, 1\right)$.*

# Sketch of Proof




[Hintz, CK:2023]

- $\omega_1(\phi(x)\phi(x')) - \omega_2(\phi(x)\phi(x'))$ smooth bisolution of $\mathcal{K}$

$\Rightarrow$ Write as series of (classical) forward-solutions $\psi_i$ with compactly supported source terms $b_i$

[Verch:1994, Hollands, Wald, Zahn: 2020]

$\Rightarrow$ Show $\left|\partial_r^j \partial_u^l \partial_y^\gamma \psi_i(r, u, \theta, \varphi_-)\right| \leq C_{jl\gamma\epsilon}(r - r_-)^{\min(\beta',1)-\epsilon-j} \|b_i\|_{C^{m'}}$ near the Cauchy horizon

- The Unruh state for scalar fields on Kerr-de Sitter is Hadamard for small $a$ and $\Lambda$
- The small-$a$ condition can be lifted for the Unruh state of massless fermions on Kerr
- The leading divergence of quadratic observables at the Cauchy horizon is universal

# THANK YOU FOR YOUR ATTENTION

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