THE FEYNMAN PROPAGATOR ON CURVED SPACETIME

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On many curved spacetimes one can define four natural Green functions of the Klein-Gordon equation:

- the **retarded** or **forward propagator** $G^\vee$,
- the **advanced** or **backward propagator** $G^\wedge$,
- the (distinguished) **Feynman propagator** $G^F$,
- the (distinguished) **antiFeynman propagator** $G^{\overline{F}}$.

The first two are well-known. The last two are less obvious.

Feynman and antiFeynman propagators are key ingredients of perturbative Quantum Field Theory. I will discuss their various possible definitions and properties.
I. FLAT SPACETIME.

Consider first the *Klein-Gordon equation* on the flat *Minkowski space* $\mathbb{R}^{1,d-1}$:

$$(-\Box + m^2)\psi = 0. \quad (1)$$

We will say that $G(x, y)$ is a *Green function* of (1) if

$$(-\Box_x + m^2)G(x, y) = \delta(x - y),$$

$$(-\Box_y + m^2)G(x, y) = \delta(x - y).$$
There are four Green functions invariant wrt the restricted Poincaré group:

- the forward/backward propagator
  \[ G^\lor_\land(x, y) := \frac{1}{(2\pi)^4} \int \frac{e^{-i(x-y)\cdot p}}{p^2 + m^2 \pm i0 \operatorname{sgn} p_0} \, dp, \]

- the Feynman/anti-Feynman propagator
  \[ G^\text{F/F}(x, y) := \frac{1}{(2\pi)^4} \int \frac{e^{-i(x-y)\cdot p}}{p^2 + m^2 \mp i0} \, dp. \]

\( G^\lor \) and \( G^\land \) are related to the classical \textbf{Cauchy problem}, because their support is in the forward, resp. backward cone. \( G^\text{F/F} \) are used in QFT to compute \textbf{Feynman diagrams}.

They satisfy the identity \( G^\text{F/F} + G^\text{F/F} = G^\lor + G^\land \).
Using the above Green functions we can define the following useful bisolutions of the Klein-Gordon operator:

- the Pauli–Jordan propagator or commutator function
  \[ G^{\text{PJ}}(x, y) := G^\vee - G^\wedge, \]

- the positive frequency or Wightman 2-point function
  \[ G^{(+)}(x, y) := \frac{1}{i}(G^F - G^\wedge) = \frac{1}{i}(-G^\overline{F} + G^\vee), \]

- the negative frequency or anti-Wightman 2-point function
  \[ G^{(-)}(x, y) := \frac{1}{i}(-G^\overline{F} + G^\wedge) = \frac{1}{i}(G^F - G^\vee). \]
The Feynman and anti-Feynman propagator has an interesting “operator-theoretic” interpretation:

1. The Klein-Gordon operator \( K = -\Box + m^2 \) is \textit{essentially self-adjoint} on \( C_c^\infty(\mathbb{R}^{1,3}) \) in the sense of \( L^2(\mathbb{R}^{1,3}) \).

2. For \( s > \frac{1}{2} \), as an operator \( \langle t \rangle^{-s}L^2(\mathbb{R}^{1,3}) \rightarrow \langle t \rangle^sL^2(\mathbb{R}^{1,3}) \), the Feynman propagator is the \textit{boundary value of the resolvent of the Klein-Gordon operator}:

\[
\begin{align*}
\lim_{\epsilon \downarrow 0} (K \mp i\epsilon)^{-1} &= G^F / F.
\end{align*}
\]

Here \( \langle t \rangle \) denotes the so-called “Japanese bracket”

\[
\langle t \rangle := \sqrt{1 + t^2}.
\]
After quantization, we obtain an operator–valued distribution \( \mathbb{R}^{1,d-1} \ni x \mapsto \psi^*(x) = \psi(x)^* \) satisfying the Klein-Gordon equation and commutation relations

\[
(-\Box + m^2)\psi^*(x) = 0,
\]
\[
[\hat{\psi}(x), \hat{\psi}^*(y)] = -iG^{PJ}(x, y).
\]

We also have a state \((\Omega \cdot \Omega)\) such that

\[
(\Omega \mid \hat{\psi}(x)\hat{\psi}^*(y)\Omega) = G^{(+)}(x, y),
\]
\[
(\Omega \mid \hat{\psi}^*(x)\hat{\psi}(y)\Omega) = G^{(-)}(x, y),
\]
\[
(\Omega \mid T(\hat{\psi}(x)\hat{\psi}^*(y))\Omega) = -iG^{F}(x, y),
\]
\[
(\Omega \mid \overline{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega) = iG^{\overline{F}}(x, y).
\]
II. CURVED SPACETIMES.

Consider a curved spacetime $M$ with the metric tensor $g_{\mu\nu}$. Define the d’Alembertian and the Klein-Gordon operator

\[-\Box := -|g|^{-\frac{1}{2}} \partial_{\mu} g^{\frac{1}{2}} g^{\mu\nu} \partial_{\nu}, \quad K := -\Box + m^2.\]

(One could also replace the term $m^2$ with an $x$-dependent scalar potential). How to generalize the well-known propagators from $\mathbb{R}^{1,d-1}$ to generic spacetimes?
As is well-known, if $M$ is *globally hyperbolic*, then the *forward/backward propagators* have natural generalizations. Namely, there exist unique distributions $G^\vee$ and $G^\wedge$ such that

$$(-\Box + m^2)\zeta^\vee/\wedge = f,$$

$$\text{supp}\,\zeta^\vee/\wedge \subset \text{future/past shadow of suppf}$$

is uniquely solved by

$$\zeta^\vee/\wedge(x) := \int G^\vee/\wedge(x, y)f(y)\,g^{\frac{1}{2}}(y)\,dy.$$
Note that $-\Box$ is obviously Hermitian (symmetric) on $C_c^\infty(M)$ in the sense of the Hilbert space $L^2(M, |g|^{\frac{1}{2}})$. Assume it is essentially self-adjoint. Then its resolvent $(-\Box + m^2)^{-1}$ is well defined for complex $m^2$. For real $m^2$, not eigenvalues of $\Box$, we define the operator-theoretic Feynman/antiFeynman propagator as the integral kernel of

$$G_{\text{op}}^F := \lim_{\epsilon \downarrow 0} \frac{1}{(-\Box + m^2 - i\epsilon)}, \quad G_{\text{op}}^{\overline{F}} := \lim_{\epsilon \downarrow 0} \frac{1}{(-\Box + m^2 + i\epsilon)}.$$

I believe that the following argument justifies this definition. Here is an elementary fact about Fresnel integrals (with $x \in \mathbb{R}$):

$$\int \frac{e^{\pm i\left(\frac{c}{2}x^2 + Jx\right)}}{\int e^{\pm i\frac{c}{2}x^2} \, dx} \, dx = \exp \left( \mp \frac{iJ^2}{2(c \pm i0)} \right).$$
If we use *path integrals*, the generating function formally is

\[
Z(J) := \frac{\int e^{iS(\psi,\psi^*) + i\psi J^* + i\psi^* J} \mathcal{D}\psi \mathcal{D}\psi^*}{\int e^{iS(\psi,\psi^*)} \mathcal{D}\psi \mathcal{D}\psi^*}.
\]

If the action is *quadratic*

\[
S(\psi, \psi^*) = -\int \left( \partial_\mu \psi^*(x) \partial^\mu \psi(x) + m^2 \psi^*(x) \psi(x) \right) \sqrt{|g|} \, dx
\]

\[
= - \left( \psi|(-\Box + m^2)\psi \right),
\]

then the path integral can be *rigorously defined* as

\[
Z(J) = \exp \left( i \int \int J(x) G_{\text{op}}^F(x, y) J(y) \sqrt{|g|(x)} \sqrt{|g|(y)} \, dx \, dy \right)
\]

\[
= \exp i \left( J|(-\Box + m^2 - i0)^{-1} J \right).
\]
Essential self-adjointness of the d’Alembertian is easy in some special cases:

- *stationary* spacetimes;
- *Friedmann-Lemaitre-Robertson-Walker* (FLRW) spacetimes;
- *1+0-dimensional* spacetimes;
- *deSitter* and (the universal covering of) *anti-deSitter spacetime*, (which follows from general properties of symmetric spaces).

On a class of *asymptotically Minkowskian* spacetimes essential self-adjointness was recently proven by Vasy and Nakamura-Taira. Essential self-adjointness is destroyed by (space-like or time-like) *boundaries*—this can be repaired by imposing boundary conditions.
There exists also a different definition of Feynman propagators based on a time-ordered expectation of quantum fields in a state. Let $\hat{\psi}(x)$ be the quantum field satisfying

$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -iG_{PJ}^{\alpha}(x, y).$$

Let $\Omega \alpha$ be any Fock vacuum (in other words, pure quasifree state). Set

$$G_{\alpha}^{(+)\alpha}(x, y) = (\Omega \alpha | \hat{\psi}(x)\hat{\psi}^*(y)\Omega \alpha), \quad G_{\alpha}^{(-)\alpha}(x, y) = (\Omega \alpha | \hat{\psi}^*(x)\hat{\psi}(y)\Omega \alpha),$$

$$-iG_{\alpha}^{F\alpha} = (\Omega \alpha | T(\hat{\psi}(x)\hat{\psi}^*(y))\Omega \alpha), \quad iG_{\alpha}^{\overline{F}\alpha} = (\Omega \alpha | \overline{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega \alpha).$$

We have

$$G_{\alpha}^{F\alpha}(x, y) + G_{\alpha}^{\overline{F}\alpha}(x, y) = G_{\alpha}^{\vee}(x, y) + G_{\alpha}^{\wedge}(x, y).$$

We say that $\Omega \alpha$ is Hadamard if the singularities of $G_{\alpha}^{(+)\alpha}$ are similar to those on the Minkowski space.
It useful to extend the above definitions of 2-point functions, Feynman and antiFeynman propagators to *pairs of vacua* $\Omega_\alpha$ and $\Omega_\beta$:

$$G_{\alpha\beta}^{(+)}(x, y) = \frac{(\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$G_{\alpha\beta}^{(-)}(x, y) = \frac{(\Omega_\alpha|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$-iG_{\alpha\beta}^{F}(x, y) = \frac{(\Omega_\alpha|T(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$iG_{\alpha\beta}^{\bar{F}}(x, y) = \frac{(\Omega_\alpha|\overline{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)}.$$
Note that they satisfy

\[(\Box - m^2) G^{(+)}_{\alpha\beta}(x, y) = 0,\]

\[(\Box - m^2) G^{F/F\overline{F}}_{\alpha\beta}(x, y) = \delta(x, y),\]

\[i(G^{(+)}_{\alpha\beta} - G^{(-)}_{\alpha\beta}) = G^{\text{PJ}} = G^\lor - G^\land,\]

\[G^{F}_{\alpha\beta} + G^{\overline{F}}_{\alpha\beta} = G^\lor + G^\land,\]

\[G^{(+)}_{\alpha\beta}(x, y) = G^{(-)}_{\beta\alpha}(y, x), \quad G^{F}_{\alpha\beta}(x, y) = G^{\overline{F}}_{\beta\alpha}(y, x).\]

(Let us stress that $G^{(+)}_{\alpha\beta}, G^{(-)}_{\beta\alpha}, G^{F}_{\alpha\beta}, G^{\overline{F}}_{\beta\alpha}$ can be also defined in a purely operator-theoretic way, without invoking QFT).
Suppose that the Klein-Gordon equation is *stationary* (does not depend on time) and *stable* (the classical Hamiltonian is positive). Then there is a distinguished vacuum $\Omega$, given by the space of *positive frequency modes* of the generator of dynamics. It is then easy to show (D. Siemssen and JD) that $-\Box + m^2$ is essentially self-adjoint and the operator-theoretic Feynman propagator corresponds to $\Omega$: 

\[
-iG^F_{\text{op}} = (\Omega | T(\hat{\psi}(x)\hat{\psi}^*(y))\Omega), \\
iG^{F\dagger}_{\text{op}} = (\Omega | \overline{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega).
\]
If $M$ is *asymptotically stationary and stable in the future and past* then we have two natural states: the *in-vacuum* $\Omega_-$ and the *out-vacuum* $\Omega_+$. As proven by Gérard and Wrochna, they are Hadamard.

Introduce the *out-in Feynman propagator* $G^{\text{F}}_{+-}$ and the *in-out antiFeynman propagator* $\bar{G}^{\text{F}}_{-+}$

$$-iG^{\text{F}}_{+-}(x, y) = \frac{\langle \Omega_+ | T(\hat{\psi}(x)\hat{\psi}^*(y)) \rangle \Omega_-)}{\langle \Omega_+ | \Omega_- \rangle},$$

$$i\bar{G}^{\text{F}}_{-+}(x, y) = \frac{\langle \Omega_- | \bar{T}(\hat{\psi}(x)\hat{\psi}^*(y)) \rangle \Omega_+)}{\langle \Omega_- | \Omega_+ \rangle}.$$  

By the *Wick Theorem*, they appear in the evaluation of *Feynman diagrams* for the scattering operator resp. its inverse.
One can heuristically derive (D. Siemssen and JD), and under some technical assumptions prove rigorously (Vasy and Nakamura–Taira) that they coincide with the operator-theoretic propagators:

\[ G_{\text{op}}^F = G^F_{++}, \]
\[ \overline{G}^F_{\text{op}} = \overline{G}^F_{--}. \]
Assume now that $M$ is *globally hyperbolic* and $-\Box$ is essentially self-adjoint. (If not, choose a self-adjoint extension).

We will say that $-\Box + m^2$ is *special* if

$$G_{\text{op}}^F(x, y) + G_{\text{op}}^\bar{F}(x, y) = G^\vee(x, y) + G^\wedge(x, y).$$

Equivalently, it is special if

$$\text{supp} \left( G_{\text{op}}^F(\cdot, y) + G_{\text{op}}^\bar{F}(\cdot, y) \right) \subset \text{causal shadow of}\{y\}.$$
Special Klein-Gordon equations are *superconvenient*! There exist good techniques to compute the Feynman and antiFeynman propagators (because they are defined in the framework of operator theory). The forward/backward propagators can then be computed as

\[ G^{\vee/\wedge}(x, y) := \theta(\pm x^0 \mp y^0)(G_{\text{op}}^F(x, y) + G_{\text{op}}^\overline{F}(x, y)). \]

As usual, we then set \( G^{\text{PJ}} := G^{\vee} - G^{\wedge} \). More interestingly, we have a natural candidate for the *two-point function of a distinguished state*:

\[
(\Omega | \hat{\psi}(x)\hat{\psi}^*(y)\Omega) = \frac{1}{i}(G_{\text{op}}^F - G^{\wedge}) = \frac{1}{i}(-G_{\text{op}}^\overline{F} + G^{\vee}),
\]

\[
(\Omega | \hat{\psi}^*(x)\hat{\psi}(y)\Omega) = \frac{1}{i}(-G_{\text{op}}^\overline{F} + G^{\wedge}) = \frac{1}{i}(G_{\text{op}}^F - G^{\vee}).
\]
Recall that for any state $\alpha$

$$G^F_\alpha(x, y) + \bar{G}^F_\alpha(x, y) = G^\lor(x, y) + G^\land(x, y).$$

Hence if $\Omega_- = \Omega_+$,

then $-\Box + m^2$ is special.

This is in particular true if $M$ is stationary and stable—hence they are special.
III. EXAMPLES OF SPACETIMES AND THEIR PROPAGATORS

Stationary stable Klein-Gordon equations are special, as we discussed above. This includes the Minkowski space. Recall that stability means that the Hamiltonian is positive definite (which corresponds to $m^2 \geq 0$).

For tachyonic stationary Klein-Gordon equations, that is with $m^2 < 0$, we can also define all four Green’s functions. However they are not special! (And, of course, we do not have a physical state).
Consider a $1 + 0$ dimensional spacetime. In view of applications to FLRW spacetimes, assume that it is perturbed by a time-dependent potential. Thus the Klein-Gordon operator has the form of a 1-dimensional Schrödinger operator

$$K = -H + m^2, \quad H := -\partial_t^2 + V(t).$$

Then it is special if $H$ is reflectionless at the energy $m^2$. For instance, the symmetric Scarf Hamiltonian

$$-\partial_t^2 - \frac{\alpha^2 - \frac{1}{4}}{\cosh^2 t}$$

is reflectionless at all energies for $\alpha \in \mathbb{Z} + \frac{1}{2}$. 
The *deSitter space* is defined as the submanifold of the \(d + 1\)-dimensional Minkowski *ambient space*:

\[
dS^d := \{ X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = 1 \}.
\]

One can look for the Feynman propagator by solving the equation

\[
(-\Box_x + m^2)G^F(x, y) = \delta(x - y),
\]

and requiring that \(G^F(x, y) = G^F(w)\), where \(w = x \cdot y\) is the product of the vectors in the ambient space. We obtain the *Gegenbauer equation*

\[
\left( (1 - w^2)\partial_w^2 - dw \partial_w - \left(\frac{d-1}{2}\right)^2 + m^2 \right) G^F(w) = 0.
\]

We demand the singularities of \(G^F\) are similar to those of the Feynman propagator on the Minkowski space.
Assuming $m > \frac{d-1}{2}$ and setting $\nu := \sqrt{m^2 - (\frac{d-1}{2})^2}$ we obtain

$$G_{E}^{F/\bar{F}}(w) = \pm i \frac{\Gamma(\frac{d-1}{2} + i\nu)\Gamma(\frac{d-1}{2} - i\nu)}{(4\pi)^{\frac{d}{2}}} S_{\frac{d}{2} - 1, i\nu}( - w \pm i0).$$

Above, $S_{\alpha,\nu}$ is the **Gegenbauer function** regular at 1 and equal \frac{1}{\Gamma(\alpha+1)} there. It satisfies

$$G_{E}^{F} + G_{E}^{\bar{F}} = G^{\vee} + G^{\wedge}.$$

We can compute forward/backward propagators, and the distinguished two-point function, called the **Euclidean state** (because it is obtained by the Wick rotation from the Euclidean sphere). It is the unique deSitter invariant Hadamard state.
The d’Alembertian on $C_c^\infty(dS^d)$ is essentially self-adjoint and thus one can define the operator-theoretic Feynman and antiFeynman propagator. However, it is different from the “Euclidean” one:

$$G_E^F \neq G_{\text{op}}^F, \quad G_E^\text{F} \neq G_{\text{op}}^\text{F}.$$  

Note that the deSitter space is quite pathological—in particular it is not asymptotically stationary, and the Euclidean state is neither the \textit{in-state} nor the \textit{out-state}. 
There exists a family of deSitter invariant states parametrized by a complex parameter, called *alpha-vacua*. Among them there is the Euclidean state, an in-state and an out-state. The out-in Feynman propagators coincides with the operator-theoretic Feynman propagators and is given by

\[ G^F_{+-}(w) = \frac{\Gamma(d/2 - 1/2 + i\nu)}{2^{2+\nu}(2\pi)^{d/2}} \sinh \pi \nu \left( Z_{d/2-1,i\nu}^d (-w - i0) - Z_{d/2-1,i\nu}^d (-w + i0) \right), \quad \text{odd } d; \]

\[ G^F_{+-}(w) = \frac{\Gamma(d/2 - 1/2 + i\nu)}{2^{2+\nu}(2\pi)^{d/2}} \cosh \pi \nu \left( Z_{d/2-1,i\nu}^d (-w - i0) + Z_{d/2-1,i\nu}^d (-w + i0) \right), \quad \text{even } d. \]

where \(Z_{\alpha,\lambda}\) is the *Gegenbauer function* behaving as \(\frac{w^{-1/2 - \alpha - \lambda}}{\Gamma(\lambda + 1)}\) at \(w \to +\infty\).

In *odd dimensions* and with \(m^2 > (d-1/2)^2\), the deSitter space is special and the *out and in vacua coincide*. This is not the case in even dimensions!
There is an alternative approach to the deSitter space based on global coordinates

\[ X_0 = \sinh t, \quad X_i = \cosh t \hat{x}_i, \quad \hat{x} \in S^{d-1} \]

yielding the metric \(-dt^2 + \cosh^2 t \, d\Omega^2\). This has a FLRW form and yields the Schrödinger operator

\[ -\partial_t^2 - \frac{(d-2)^2}{\cosh^2 t} \frac{1 + \Delta_{S^{d-1}}}{4} + \left(\frac{d-1}{2}\right)^2. \]

The spectrum of \(-\Delta_{S^{d-1}}\) is \(\{l(l + d - 2) : l = 0, 1, 2, \ldots\}\), hence we obtain the symmetric Scarf potential with \(\alpha = \frac{d-2}{2} + l\). Thus all modes are reflectionless iff \(d\) is odd. Consequently, \textit{all modes are special iff }d\textit{ is odd, and they are not if }d\textit{ is even.}
The *Anti-deSitter space* is defined as

\[
\text{AdS}^d := \{(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^{d-1} : -X_1^2 - X_2^2 + Y_1^2 + \cdots + Y_{d-1}^2 = -1\}.
\]

It is stationary, however has timelike loops. Introduce the coordinates

\[
X_1 = \frac{\cos t}{\cos \rho}, \quad X_2 = \frac{\sin t}{\cos \rho}, \quad Y_i = \tan \rho \hat{y}_i;
\]

with the metric

\[
\frac{1}{\cos^2 \rho}(-dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2).
\]

where \( t \in ]-\pi, \pi] \). By taking the *universal covering* of the Anti-deSitter space we remove timelike loops. In coordinates this means \( t \in \mathbb{R} \),

The d’Alembertian is essentially self-adjoint. We again set \( w := x \cdot y \) from the ambient space. For \( m^2 > -(d-1)^2/2 \), with \( \nu := \sqrt{(d-1)^2/2 + m^2} \), we obtain

\[
G^{F/F}_{\text{AdS}}(w) = \pm i \frac{\sqrt{\pi} \Gamma(d-1/2 + \nu)}{\sqrt{2}(2\pi)^d 2\nu} \mathbb{Z}^{d-1,\nu}( -w \pm i0(-1)^n ), \quad n = \left\lfloor \frac{|t|}{\pi} \right\rfloor.
\]
In the following, it will be useful to know some properties of the trigonometric Pöschl-Teller Hamiltonian:

\[ H := -\partial^2_\rho + \frac{\alpha^2 - \frac{1}{4}}{\sin^2 \rho} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 \rho}. \]

This Hamiltonian, as an operator on \( L^2[0, \pi/2] \), is essentially self-adjoint iff \( \alpha^2 \geq 1 \) and \( \beta^2 \geq 1 \), and has a positive Friedrichs extension if \( \alpha^2 \geq 0 \) and \( \beta^2 \geq 0 \). If \( \alpha^2 < 0 \) or \( \beta^2 < 0 \), then all its extensions are unbounded from below.
The Anti-deSitter space, even after taking its universal covering, is still not globally hyperbolic: it has trajectories that escape to infinity in finite time.

Consider now the Klein-Gordon operator on Anti-deSitter:

\[
(tan \rho)^{d-2} (-\Box + m^2)(tan \rho)^{-d/2}
= \cos^2 \rho \left( \partial_t^2 - \partial^2_\rho + \frac{(d-3)^2}{2} - \frac{1}{4} - \Delta_{S^{d-2}} \right) + \frac{(d-1)^2 - \frac{1}{4} + m^2}{\cos^2 \rho}
= \cos^2 \rho \left( \partial_t^2 + H \right),
\]

where \( H \) is the trigonometric Pöschl-Teller Hamiltonian. \( \rho = 0 \) is a coordinate singularity. \( \rho = \frac{\pi}{2} \) is the spatial infinity, where classical particles may escape. Following Wald-Ishibashi, we note that \( H \) is self-adjoint for \( m^2 \geq 1 - (d-1)^2 \). For \( m^2 \geq -(d-1)^2 \), we need to take the Friedrichs extension of \( H \). In all these cases the Anti-deSitter space is special! Only for \( m^2 < -(d-1)^2 \) we do not have distinguished forward and backward propagators (and of course the specialty breaks down).
Thank you for your attention
Purely operator-theoretic approach to 2-point functions.
Consider the space of space-compact solutions to the Klein-Gordon equation
\[ (-\Box + m^2)\zeta = 0. \] (2)
It is equipped with the (indefinite) Klein-Gordon scalar product
\[ (\zeta_1,\zeta_2)_{\text{KG}} := \int_S \bar{\zeta_1}(x) \leftrightarrow_\sigma^{\mu}(x) \partial_\mu \zeta_2(x) \] (3)
where
\[ \bar{\zeta_1}(x) \leftrightarrow_\sigma^{\mu}(x) \partial_\mu \zeta_2(x) = \partial_\mu \bar{\zeta_1}(x) \zeta_2(x) - \bar{\zeta_1}(x) \partial_\mu \zeta_2(x). \] (4)
and \( S \) is an arbitrary Cauchy surface. It is convenient to assume that this space can be completed to a Krein space \( \mathcal{W}_{\text{KG}} \) (this is typically possible).
Let $G^\bullet(x, y)$ be a \textit{bisolution to the Klein-Gordon equation}, that is
\[(−\square_x + m^2)G^\bullet(x, y) = (−\square_y + m^2)G^\bullet(x, y) = 0.\] (5)

Then $G^\bullet$ defines a linear map $T^\bullet$ on $\mathcal{W}_{KG}$:
\[
(T^\bullet \zeta)(x) := \int_S G^\bullet(x, y) \overset{\leftrightarrow}{\partial_\mu} \zeta(y) \sigma^\mu(y).
\] (6)

We then say that $G^\bullet$ is the \textit{Klein-Gordon kernel} of $T^\bullet$.

Example: $G_{\text{PJ}}^\bullet(x, y)$ is the Klein-Gordon kernel of the identity.
Let
\[ \mathcal{W}_{KG} = Z_{\alpha}^{(+)} \oplus Z_{\alpha}^{(-)} \] (7)
be an orthogonal decomposition of the Krein space \( \mathcal{W}_{KG} \) into a maximal positive and maximal negative subspace. Such a decomposition defines a Fock representation with the vacuum \( \Omega_{\alpha} \). Let \( \Pi_{\alpha}^{(+)} \) and \( \Pi_{\alpha}^{(-)} \) be the corresponding projections. Then
\[ G_{\alpha}^{(+)}(x, y) = (\Omega_{\alpha}|\hat{\psi}(x)\hat{\psi}^{*}(y)\Omega_{\alpha}), \]
\[ G_{\alpha}^{(-)}(x, y) = (\Omega_{\alpha}|\hat{\psi}^{*}(x)\hat{\psi}(y)\Omega_{\alpha}). \]
are the Klein-Gordon kernels of \( \Pi_{\alpha}^{(+)} \) and \( \Pi_{\alpha}^{(-)}. \)
Let
\[ \mathcal{W}_{KG} = \mathcal{Z}_\beta^+ \oplus \mathcal{Z}_\beta^- \] (8)
be another orthogonal decomposition of the Krein space \( \mathcal{W}_{KG} \) into a maximal positive and maximal negative subspace, defining the vacuum \( \Omega_\beta \). One can show that we have a (non-orthogonal) direct sum
\[ \mathcal{W}_{KG} = \mathcal{Z}_\beta^+ \oplus \mathcal{Z}_\alpha^- \] (9)

Let \( \Pi_{\beta\alpha}^+, \Pi_{\alpha\beta}^- \) be the corresponding projections. Then

\[ G_{\alpha\beta}^+(x, y) = \frac{\langle \Omega_\alpha | \hat{\psi}(x) \hat{\psi}^*(y) \Omega_\beta \rangle}{\langle \Omega_\alpha | \Omega_\beta \rangle}, \] (10)

\[ G_{\beta\alpha}^-(x, y) = \frac{\langle \Omega_\beta | \hat{\psi}^*(x) \hat{\psi}(y) \Omega_\alpha \rangle}{\langle \Omega_\beta | \Omega_\alpha \rangle} \] (11)

are the Klein-Gordon kernels of \( \Pi_{\beta\alpha}^+ \) and \( \Pi_{\alpha\beta}^- \).
Green functions of 1-dimensional Schrödinger operators.

Suppose $\psi_1, \psi_2$ solve

\[(H + k^2)\psi_i(t) = 0, \quad i = 1, 2.\]

Then their Wronskian

\[\mathcal{W}(\psi_1, \psi_2) := \psi_1(t)\psi_2'(t) - \psi_1'(t)\psi_2(t)\]

does not depend on $t$.

The function

\[G^{\leftrightarrow}(-k^2; t, s) := \frac{1}{\mathcal{W}(\psi_1, \psi_2)}(\psi_1(t)\psi_2(s) - \psi_2(t)\psi_1(s))\]

does not depend on the choice of $\psi_1, \psi_2$ and defines the so-called canonical bisolution, the analog of the Pauli Jordan propagator. From $G^{\leftrightarrow}$ we can define the forward and backward Green functions:

\[G^{\rightarrow}(-k^2; t, s) := G^{\leftrightarrow}(-k^2, t, s)\theta(t - s),\]

\[G^{\leftarrow}(-k^2; t, s) := -G^{\leftrightarrow}(-k^2, t, s)\theta(s - t).\]
For \( \Re k > 0 \) we define the left and right \textit{Jost solutions} to be the unique solutions of
\[
(H + k^2)\psi_{\pm}(t, k) = 0, \quad \psi_{\pm}(t, k) \sim e^{\mp tk}, \quad \pm t \to \infty.
\]
We also introduce the \textit{Jost function}
\[
\mathcal{W}(k) := \mathcal{W}(\psi_+(\cdot, k), \psi_-(\cdot, k)).
\]
The \textit{resolvent} of \( H \), denoted \( G(-k^2) := (H + k^2)^{-1} \) has the integral kernel
\[
G(-k^2; t, s) = \frac{1}{\mathcal{W}(k)}(\theta(t - s)\psi_+(t, k)\psi_-(s, k) + \theta(s - t)\psi_-(t, k)\psi_+(s, k)).
\]
We say that \( H \) is \textit{reflectionless} if there exist functions \( T(\pm ip) \) such that
\[
\psi_+(\pm ip) = T(\pm ip)\psi_-(\mp ip).
\]