THE FEYNMAN PROPAGATOR ON CURVED SPACETIMES

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On many curved spacetimes one can define four natural *Green functions* of the Klein-Gordon equation:

- the *retarded* or *forward propagator* $G^{\vee}$,
- the *advanced* or *backward propagator* $G^{\wedge}$,
- the (distinguished) *Feynman propagator* G^{F},
- the (distinguished) *antiFeynman propagator* $G^{\overline{\mathrm{F}}}$.

The first two are well-known. The last two are less obvious.

Feynman and antiFeynman propagators are key ingredients of perturbative *Quantum Field Theory*. I will discuss their various possible definitions and properties.

I. FLAT SPACETIME.

Consider first the *Klein-Gordon equation* on the flat *Minkowski space* $\mathbb{R}^{1,d-1}$:

$$(-\Box + m^2)\psi = 0. \tag{1}$$

We will say that $G(x, y)$ is a *Green function* of (1) if

$$(-\Box_x + m^2)G(x, y) = \delta(x - y),$$
$$(-\Box_y + m^2)G(x, y) = \delta(x - y).$$

There are four Green functions invariant wrt the restricted Poincaré group:

- the *forward/backward propagator*

$$G^{\vee/\wedge}(x,y) := \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2 + m^2 \pm \mathrm{i}0 \operatorname{sgn} p_0} \, \mathrm{d}p,$$

- the *Feynman/anti-Feynman propagator*

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) := \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2 + m^2 \mp \mathrm{i}0} \, \mathrm{d}p.$$

$G^{\vee}$ and $G^{\wedge}$ are related to the classical *Cauchy problem*, because their support is in the forward, resp. backward cone. G^{F} and $G^{\overline{\mathrm{F}}}$ are used in QFT to compute *Feynman diagrams*.

They satisfy the identity $G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}$.

Using the above Green functions we can define the following useful *bisolutions of the Klein-Gordon operator*:

- the *Pauli–Jordan propagator* or *commutator function*

$$G^{\mathrm{PJ}}(x,y) \coloneqq G^{\vee} - G^{\wedge},$$

- the *positive frequency* or *Wightman* 2-point function

$$G^{(+)}(x,y) \coloneqq \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\wedge}) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\vee}),$$

- the *negative frequency* or *anti-Wightman* 2-point function

$$G^{(-)}(x,y) \coloneqq \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\wedge}) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\vee}).$$

The Feynman and antiFeynman propagator has an interesting "operator-theoretic" interpretation:

(1) The Klein-Gordon operator $K = -\Box + m^2$ is *essentially self-adjoint* on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^{1,3})$ in the sense of $L^2(\mathbb{R}^{1,3})$.
(2) For $s > \frac{1}{2}$, as an operator $\langle t\rangle^{-s}L^2(\mathbb{R}^{1,3}) \to \langle t\rangle^{s}L^2(\mathbb{R}^{1,3})$, the Feynman propagator is the *boundary value of the resolvent of the Klein-Gordon operator*:

$$\operatorname*{s-lim}_{\epsilon \searrow 0}(K \mp \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}/\overline{\mathrm{F}}}.$$

Here $\langle t\rangle$ denotes the so-called "Japanese bracket"

$$\langle t\rangle := \sqrt{1+t^2}.$$

After *quantization*, we obtain an operator–valued distribution $\mathbb{R}^{1,d-1} \ni x \mapsto \psi^*(x) = \psi(x)^*$ satisfying the Klein-Gordon equation and commutation relations

$$(-\Box + m^2)\psi^*(x) = 0,$$

$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x,y).$$

We also have a state $(\Omega| \cdot \Omega)$ such that

$$(\Omega \,|\, \hat{\psi}(x)\hat{\psi}^*(y)\Omega) = G^{(+)}(x,y),$$

$$(\Omega \,|\, \hat{\psi}^*(x)\hat{\psi}(y)\Omega) = G^{(-)}(x,y),$$

$$\big(\Omega \,\big|\, \mathrm{T}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = -\mathrm{i}G^{\mathrm{F}}(x,y),$$

$$\big(\Omega \,\big|\, \overline{\mathrm{T}}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = \mathrm{i}G^{\overline{\mathrm{F}}}(x,y).$$

II. CURVED SPACETIMES.

Consider a curved spacetime M with the *metric tensor* $g_{\mu\nu}$. Define the *d'Alembertian* and the *Klein-Gordon operator*

$$-\Box := -|g|^{-\frac{1}{2}}\partial_\mu |g|^{\frac{1}{2}} g^{\mu\nu}\partial_\nu, \qquad K := -\Box + m^2.$$

(One could also replace the term m^2 with an x-dependent *scalar potential*). How to generalize the well-known propagators from $\mathbb{R}^{1,d-1}$ to generic spacetimes?

As is well-known, if M is *globally hyperbolic*, then the *forward/backward propagators* have natural generalizations. Namely, there exist unique distributions $G^{\vee}$ and $G^{\wedge}$ such that

$$\begin{aligned}(-\Box + m^2)\zeta^{\vee/\wedge} &= f,\\ \operatorname{supp}\zeta^{\vee/\wedge} &\subset \text{future/past shadow of} \operatorname{supp} f\end{aligned}$$

is uniquely solved by

$$\zeta^{\vee/\wedge}(x) := \int G^{\vee/\wedge}(x,y) f(y) |g|^{\frac{1}{2}}(y)\,\mathrm{d}y.$$

Note that $-\Box$ is obviously *Hermitian* (symmetric) on $C_c^\infty(M)$ in the sense of the Hilbert space $L^2(M, |g|^{\frac{1}{2}})$. Assume it is *essentially self-adjoint*. Then its resolvent $(-\Box + m^2)^{-1}$ is well defined for complex m^2. For real m^2, not eigenvalues of $\Box$, we define the *operator-theoretic Feynman/antiFeynman propagator* as the integral kernel of

$$G_{\mathrm{op}}^{\mathrm{F}} := \lim_{\epsilon \searrow 0} \frac{1}{(-\Box + m^2 - \mathrm{i}\epsilon)}, \quad G_{\mathrm{op}}^{\overline{\mathrm{F}}} := \lim_{\epsilon \searrow 0} \frac{1}{(-\Box + m^2 + \mathrm{i}\epsilon)}.$$

I believe that the following argument justifies this definition. Here is an elementary fact about *Fresnel integrals* (with $x \in \mathbb{R}$):

$$\frac{\int \mathrm{e}^{\pm\mathrm{i}(\frac{c}{2}x^2 + Jx)}\,\mathrm{d}x}{\int \mathrm{e}^{\pm\mathrm{i}\frac{c}{2}x^2}\,\mathrm{d}x} = \exp\Big(\mp \frac{\mathrm{i}J^2}{2(c \pm \mathrm{i}0)}\Big).$$

If we use *path integrals*, the generating function formally is

$$Z(J) := \frac{\int \mathrm{e}^{\mathrm{i}S(\psi,\psi^*)+\mathrm{i}\psi J^*+\mathrm{i}\psi^* J}\mathcal{D}\psi\mathcal{D}\psi^*}{\int \mathrm{e}^{\mathrm{i}S(\psi,\psi^*)}\mathcal{D}\psi\mathcal{D}\psi^*}.$$

If the action is *quadratic*

$$\begin{aligned} S(\psi,\psi^*) &= -\int \big(\partial_\mu\psi^*(x)\partial^\mu\psi(x) + m^2\psi^*(x)\psi(x)\big)\sqrt{|g|}(x)\,\mathrm{d}x \\ &= -\,\big(\psi|(-\Box + m^2)\psi\big), \end{aligned}$$

then the path integral can be *rigorously defined* as

$$\begin{aligned} Z(J) &= \exp\Big(\mathrm{i}\int\int \overline{J(x)}G^{\mathrm{F}}_{\mathrm{op}}(x,y)J(y)\sqrt{|g|}(x)\sqrt{|g|}(y)\,\mathrm{d}x\,\mathrm{d}y\Big) \\ &= \exp\mathrm{i}\big(J|(-\Box + m^2 - \mathrm{i}0)^{-1}J\big). \end{aligned}$$

Essential self-adjointness of the d'Alembertian is easy in some special cases:

- *stationary* spacetimes;
- *Friedmann-Lemaitre-Robertson-Walker* (FLRW) spacetimes;
- $1{+}0$-*dimensional* spacetimes;
- *deSitter* and (the universal covering of) *anti-deSitter spacetime*, (which follows from general properties of symmetric spaces).

On a class of *asymptotically Minkowskian* spacetimes essential self-adjointness was recently proven by Vasy and Nakamura-Taira. Essential self-adjointness is destroyed by (space-like or time-like) *boundaries*—this can be repaired by imposing boundary conditions.

There exists also a different definition of Feynman propagators based on a *time-ordered expectation* of quantum fields in a state. Let $\hat{\psi}(x)$ be the quantum field satisfying

$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x, y).$$

Let Ω_α be any *Fock vacuum* (in other words, *pure quasifree state*). Set

$$G_\alpha^{(+)} = (\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\alpha), \quad G_\alpha^{(-)} = (\Omega_\alpha|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\alpha),$$

$$-\mathrm{i}G_\alpha^{\mathrm{F}} = (\Omega_\alpha|\mathrm{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\alpha), \quad \mathrm{i}G_\alpha^{\overline{\mathrm{F}}} = (\Omega_\alpha|\overline{\mathrm{T}}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\alpha).$$

We have $$G_\alpha^{\mathrm{F}}(x, y) + G_\alpha^{\overline{\mathrm{F}}}(x, y) = G^{\vee}(x, y) + G^{\wedge}(x, y).$$

We say that Ω_α is *Hadamard* if the singularities of $G_\alpha^{(+)}$ are similar to those on the Minkowski space.

It useful to extend the above definitions of 2-point functions, Feynman and antiFeynman propagators to *pairs of vacua* Ω_α and Ω_β:

$$G^{(+)}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$G^{(-)}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$-\mathrm{i}G^{\mathrm{F}}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\mathrm{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$\mathrm{i}G^{\overline{\mathrm{F}}}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\overline{\mathrm{T}}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)}.$$

Note that they satisfy

$$\begin{aligned}
(-\Box_x + m^2) G^{(+)/(-)}_{\alpha\beta}(x,y) &= 0, \\
(-\Box_x + m^2) G^{\mathrm{F}/\overline{\mathrm{F}}}_{\alpha\beta}(x,y) &= \delta(x,y), \\
\mathrm{i}(G^{(+)}_{\alpha\beta} - G^{(-)}_{\alpha\beta}) &= G^{\mathrm{PJ}} = G^{\vee} - G^{\wedge}, \\
G^{\mathrm{F}}_{\alpha\beta} + G^{\overline{\mathrm{F}}}_{\alpha\beta} &= G^{\vee} + G^{\wedge}, \\
G^{(+)}_{\alpha\beta}(x,y) = \overline{G^{(-)}_{\beta\alpha}(y,x)}, &\qquad G^{\mathrm{F}}_{\alpha\beta}(x,y) = \overline{G^{\overline{\mathrm{F}}}_{\beta\alpha}(y,x)}.
\end{aligned}$$

(Let us stress that $G^{(+)}_{\alpha\beta}$, $G^{(-)}_{\beta\alpha}$, $G^{\mathrm{F}}_{\alpha\beta}$, $G^{\overline{\mathrm{F}}}_{\beta\alpha}$ can be also defined in a purely operator-theoretic way, without invoking QFT).

Suppose that the Klein-Gordon equation is *stationary* (does not depend on time) and *stable* (the classical Hamiltonian is positive). Then there is a distinguished vacuum Ω, given by the space of *positive frequency modes* of the generator of dynamics. It is then easy to show (D. Siemssen and JD) that $-\Box + m^2$ is essentially self-adjoint and the operator-theoretic Feynman propagator corresponds to Ω:

$$-\mathrm{i}G_{\mathrm{op}}^{\mathrm{F}} = (\Omega|\mathrm{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega),$$
$$\mathrm{i}G_{\mathrm{op}}^{\overline{\mathrm{F}}} = (\Omega|\overline{\mathrm{T}}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega).$$

If M is *asymptotically stationary and stable in the future and past* then we have two natural states: the *in-vacuum* Ω_- and the *out-vacuum* Ω_+. As proven by Gérard and Wrochna, they are Hadamard

Introduce the *out-in Feynman propagator* G^{F}_{+-} and the *in-out antiFeynman propagator*

$$
\begin{aligned}
-\mathrm{i}G^{\mathrm{F}}_{+-}(x,y) &= \frac{\big(\Omega_+|\mathrm{T}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega_-\big)}{\big(\Omega_+|\Omega_-\big)},\\
\mathrm{i}G^{\overline{\mathrm{F}}}_{-+}(x,y) &= \frac{\big(\Omega_-|\overline{\mathrm{T}}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega_+\big)}{\big(\Omega_-|\Omega_+\big)}.
\end{aligned}
$$

By the *Wick Theorem*, they appear in the evaluation of *Feynman diagrams* for the scattering operator resp. its inverse.

One can heuristically derive (D.Siemssen and JD), and under some technical assumptions prove rigorously (Vasy and Nakamura–Taira) that they coincide with the operator-theoretic propagators:

$$G_{\rm op}^{\rm F} = G_{+-}^{\rm F},$$
$$G_{\rm op}^{\overline{\rm F}} = G_{-+}^{\overline{\rm F}}.$$

Assume now that M is *globally hyperbolic* and $-\Box$ *is essentially self-adjoint*. (If not, choose a self-adjoint extension).

We will say that $-\Box + m^2$ is *special* if

$$G^{\mathrm{F}}_{\mathrm{op}}(x,y) + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y) = G^{\vee}(x,y) + G^{\wedge}(x,y).$$

Equivalently, it is special if

$$\mathrm{supp}\left(G^{\mathrm{F}}_{\mathrm{op}}(\cdot,y) + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(\cdot,y)\right) \subset \text{causal shadow of}\{y\}.$$

Special Klein-Gordon equations are *superconvenient*! There exist good techniques to compute the Feynman and antiFeynman propagators (because they are defined in the framework of operator theory). The forward/backward propagators can then be computed as

$$G^{\vee/\wedge}(x,y) := \theta(\pm x^0 \mp y^0)\big(G^{\mathrm{F}}_{\mathrm{op}}(x,y) + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y)\big).$$

As usual, we then set $G^{\mathrm{PJ}} := G^{\vee} - G^{\wedge}$. More interestingly, we have a natural candidate for the *two-point function of a distinguished state*:

$$(\Omega\,|\,\hat{\psi}(x)\hat{\psi}^*(y)\Omega) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}}_{\mathrm{op}} - G^{\wedge}) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}}_{\mathrm{op}} + G^{\vee}),$$

$$(\Omega\,|\,\hat{\psi}^*(x)\hat{\psi}(y)\Omega) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}}_{\mathrm{op}} + G^{\wedge}) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}}_{\mathrm{op}} - G^{\vee}).$$

Recall that for any state α

$$G_{\alpha}^{\mathrm{F}}(x,y)+G_{\alpha}^{\overline{\mathrm{F}}}(x,y)=G^{\vee}(x,y)+G^{\wedge}(x,y).$$

Hence if

$$\Omega_{-}=\Omega_{+},$$

then $-\Box+m^2$ is special.

This is in particular true if M is stationary and stable—hence they are special.

III. EXAMPLES OF SPACETIMES AND THEIR PROPAGATORS

Stationary stable Klein-Gordon equations are special, as we discussed above. This includes the Minkowski space. Recall that stability means that the Hamiltonian is positive definite (which corresponds to $m^2 \geq 0$).

For *tachyonic* stationary Klein-Gordon equations, that is with $m^2 < 0$, we can also define all four Green's functions. However they are not special! (And, of course, we do not have a physical state).

Consider a $1+0$ *dimensional spacetime*. In view of applications to FLRW spacetimes, assume that it is perturbed by a time-dependent potential. Thus the Klein-Gordon operator has the form of a *1-dimensional Schrödinger operator*

$$K = -H + m^2, \quad H := -\partial_t^2 + V(t).$$

Then it is special if H is *reflectionless* at the energy m^2.

For instance, the *symmetric Scarf Hamiltonian*

$$-\partial_t^2 - \frac{\alpha^2 - \frac{1}{4}}{\cosh^2 t}$$

is reflectionless at all energies for $\alpha \in \mathbb{Z} + \frac{1}{2}$.

The *deSitter space* is defined as the submanifold of the $d+1$-dimensional Minkowski *ambient space*:

$$\mathrm{dS}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}.$$

One can look for the Feynman propagator by solving the equation

$$(-\Box_x + m^2) G^{\mathrm{F}}(x,y) = \delta(x-y),$$

and requiring that $G^{\mathrm{F}}(x,y) = G^{\mathrm{F}}(w)$, where $w = x \cdot y$ is the product of the vectors in the ambient space. We obtain the *Gegenbauer equation*

$$\Big((1-w^2)\partial_w^2 - dw\partial_w - (\tfrac{d-1}{2})^2 + m^2\Big) G^{\mathrm{F}}(w) = 0.$$

We demand the singularities of G^{F} are similar to those of the Feynman propagator on the Minkowski space.

Assuming $m > \frac{d-1}{2}$ and setting $\nu := \sqrt{m^2 - (\frac{d-1}{2})^2}$ we obtain

$$G_{\mathrm{E}}^{\mathrm{F}/\overline{\mathrm{F}}}(w) = \pm \mathrm{i} \frac{\Gamma(\frac{d-1}{2} + \mathrm{i}\nu)\Gamma(\frac{d-1}{2} - \mathrm{i}\nu)}{(4\pi)^{\frac{d}{2}}} \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- w \pm \mathrm{i}0\big).$$

Above, $\mathbf{S}_{\alpha,\nu}$ is the *Gegenbauer function* regular at 1 and equal $\frac{1}{\Gamma(\alpha+1)}$ there. It satisfies

$$G_{\mathrm{E}}^{\mathrm{F}} + G_{\mathrm{E}}^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}.$$

We can compute forward/backward propagators, and the distinguished two-point function, called the *Euclidean state* (because it is obtained by the Wick rotation from the Euclidean sphere). It is the unique deSitter invariant Hadamard state.

The d'Alembertian on $C_{\rm c}^\infty(\mathrm{dS}^d)$ is essentialy self-adjoint and thus one can define the operator-theoretic Feynman and antiFeynman propagator. However, it is different from the "Euclidean" one:

$$G_{\rm E}^{\rm F} \neq G_{\rm op}^{\rm F}, \qquad G_{\rm E}^{\overline{\rm F}} \neq G_{\rm op}^{\overline{\rm F}}.$$

Note that the deSitter space is quite pathological—in particular it is not asymptotically stationary, and the Euclidean state is neither the *in-state* nor the *out-state*.

There exists a family of deSitter invariant states parametrized by a complex parameter, called *alpha-vacua*. Among them there is the Euclidean state, an in-state and an out-state. The out-in Feynman propagators coincides with the operator-theoretic Feynman propagators and is given by

$$G^{\mathrm{F}}_{+-}(w) = \frac{\Gamma(\frac{d-1}{2} + \mathrm{i}\nu)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}} \sinh \pi\nu} \Big(\mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- w - \mathrm{i}0\big) - \mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- w + \mathrm{i}0\big)\Big), \quad \text{odd } d;$$

$$G^{\mathrm{F}}_{+-}(w) = \frac{\Gamma(\frac{d-1}{2} + \mathrm{i}\nu)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}} \cosh \pi\nu} \Big(\mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- w - \mathrm{i}0\big) + \mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(- w + \mathrm{i}0\big)\Big), \quad \text{even } d.$$

where $\mathbf{Z}_{\alpha,\lambda}$ is the *Gegenbauer function* behaving as $\frac{w^{-\frac{1}{2}-\alpha-\lambda}}{\Gamma(\lambda+1)}$ at $w \to +\infty$.

In *odd dimensions* and with $m^2 > (\frac{d-1}{2})^2$, the deSitter space is special and the *out and in vacua coincide*. This is not the case in even dimensions!

There is an alternative approach to the deSitter space based on global coordinates

$$X_0 = \sinh t, \quad X_i = \cosh t \hat{x}_i, \quad \hat{x} \in \mathbb{S}^{d-1}$$

yielding the metric $-\,\mathrm{d}t^2 + \cosh^2 t\, \mathrm{d}\Omega^2$. This has a FLRW form and yields the Schrödinger operator

$$-\partial_t^2 - \frac{\left(\frac{d-2}{2}\right)^2 - \frac{1}{4} - \Delta_{\mathbb{S}^{d-1}}}{\cosh^2 t} + \left(\frac{d-1}{2}\right)^2.$$

The spectrum of $-\Delta_{\mathbb{S}^{d-1}}$ is $\{l(l+d-2) \ : \ l = 0, 1, 2, \dots\}$, hence we obtain the symmetric Scarf potential with $\alpha = \frac{d-2}{2} + l$. Thus all modes are reflectionless iff d is odd. Consequently, *all modes are special iff d is odd, and they are not if d is even*.

The *Anti-deSitter space* is defined as

$$\mathrm{AdS}^d := \{(X,Y) \in \mathbb{R}^2 \times \mathbb{R}^{d-1} \;:\; -X_1^2 - X_2^2 + Y_1^2 + \cdots + Y_{d-1}^2 = -1\}.$$

It is stationary, however has timelike loops. Introduce the coordinates

$$X_1 = \frac{\cos t}{\cos \rho}, \quad X_2 = \frac{\sin t}{\cos \rho}, \quad Y_i = \tan \rho \hat{y}_i;$$

$$\text{with the metric} \quad \frac{1}{\cos^2 \rho}(-\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2 \rho \,\mathrm{d}\Omega^2).$$

where $t \in]-\pi, \pi]$. By taking the *universal covering* of the Anti-deSitter space we remove timelike loops. In coordinates this means $t \in \mathbb{R}$,

The d'Alembertian is essentially self-adjoint. We again set $w := x \cdot y$ from the ambient space. For $m^2 > -(\frac{d-1}{2})^2$, with $\nu := \sqrt{(\frac{d-1}{2})^2 + m^2}$, we obtain

$$G_{\mathrm{AdS}}^{\mathrm{F}/\overline{\mathrm{F}}}(w) = \pm \mathrm{i} \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2} + \nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}} 2^{\nu}} \mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-w \pm \mathrm{i}0(-1)^n\big), \quad n = \left\lfloor \frac{|t|}{\pi} \right\rfloor.$$

In the following, it will be useful to know some properties of the *trigonometric Pöschl-Teller Hamiltonian*:

$$H := -\partial_\rho^2 + \frac{\alpha^2 - \frac{1}{4}}{\sin^2 \rho} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 \rho}.$$

This Hamiltonian, as an operator on $L^2[0, \frac{\pi}{2}]$, is essentially self-adjoint iff $\alpha^2 \geq 1$ and $\beta^2 \geq 1$, and has a positive Friedrichs extension if $\alpha^2 \geq 0$ and $\beta^2 \geq 0$. If $\alpha^2 < 0$ or $\beta^2 < 0$, then all its extensions are unbounded from below.

The Anti-deSitter space, even after taking its universal covering, is still not globally hyperbolic: it has trajectories that *escape to infinity in finite time*.

Consider now the Klein-Gordon operator on Anti-deSitter:

$$
\begin{aligned}
&(\tan\rho)^{\frac{d-2}{2}}(-\Box + m^2)(\tan\rho)^{-\frac{d-2}{2}} \\
&= \cos^2\rho\Big(\partial_t^2 - \partial_\rho^2 + \frac{\left(\frac{d-3}{2}\right)^2 - \frac{1}{4} - \Delta_{\mathbb{S}^{d-2}}}{\sin^2\rho} + \frac{\left(\frac{d-1}{2}\right)^2 - \frac{1}{4} + m^2}{\cos^2\rho}\Big) \\
&= \cos^2\rho\big(\partial_t^2 + H\big),
\end{aligned}
$$

where H is the trigonometric Pöschl-Teller Hamiltonian. $\rho = 0$ is a coordinate singularity. $\rho = \frac{\pi}{2}$ is the spatial infinity, where classical particles may escape. Following Wald-Ishibashi, we note that H is self-adjoint for $m^2 \geq 1 - (\frac{d-1}{2})^2$. For $m^2 \geq -(\frac{d-1}{2})^2$, we need to take the Friedrichs extension of H. In all these cases the *Anti-deSitter space is special!* Only for $m^2 < -(\frac{d-1}{2})^2$ we do not have distinguished forward and backward propagators (and of course the specialty breaks down).

Thank you for your attention

Purely operator-theoretic approach to 2-point functions.
Consider the space of *space-compact solutions to the Klein-Gordon equation*

$$(-\Box + m^2)\zeta = 0. \tag{2}$$

It is equipped with the (indefinite) *Klein-Gordon scalar product*

$$(\zeta_1|\zeta_2)_{\mathrm{KG}} := \int_{\mathcal{S}} \overline{\zeta_1(x)} \overleftrightarrow{\partial}_\mu \zeta_2(x) \sigma^\mu(x) \tag{3}$$

$$\text{where} \quad \overline{\zeta_1(x)} \overleftrightarrow{\partial}_\mu \zeta_2(x) = \partial_\mu \overline{\zeta_1(x)} \zeta_2(x) - \overline{\zeta_1(x)} \partial_\mu \zeta_2(x). \tag{4}$$

and $\mathcal{S}$ is an arbitrary Cauchy surface. It is convenient to assume that this space can be completed to a *Krein space* $\mathcal{W}_{\mathrm{KG}}$ (this is typically possible).

Let $G^\bullet(x, y)$ be a *bisolution to the Klein-Gordon equation*, that is

$$(-\Box_x + m^2)G^\bullet(x, y) = (-\Box_y + m^2)G^\bullet(x, y) = 0. \tag{5}$$

Then $G^\bullet$ defines a linear map $T^\bullet$ on $\mathcal{W}_{\mathrm{KG}}$:

$$(T^\bullet\zeta)(x) := \int_{\mathcal{S}} G^\bullet(x, y)\overset{\leftrightarrow}{\partial}_\mu \zeta(y)\sigma^\mu(y). \tag{6}$$

We then say that $G^\bullet$ is the *Klein-Gordon kernel* of $T^\bullet$.

Example: $G^{\mathrm{PJ}}(x, y)$ is the Klein-Gordon kernel of the identity.

Let

$$\mathcal{W}_{\mathrm{KG}} = \mathcal{Z}_\alpha^{(+)} \oplus \mathcal{Z}_\alpha^{(-)} \tag{7}$$

be an orthogonal decomposition of the Krein space $\mathcal{W}_{\mathrm{KG}}$ into a maximal positive and maximal negative subspace. Such a decomposition defines a Fock representation with the vacuum Ω_α. Let $\Pi_\alpha^{(+)}$ and $\Pi_\alpha^{(-)}$ be the corresponding projections. Then

$$G_\alpha^{(+)}(x,y) = (\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\alpha),$$
$$G_\alpha^{(-)}(x,y) = (\Omega_\alpha|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\alpha).$$

are the Klein-Gordon kernels of $\Pi_\alpha^{(+)}$ and $\Pi_\alpha^{(-)}$.

Let

$$\mathcal{W}_{\mathrm{KG}} = \mathcal{Z}_\beta^{(+)} \oplus \mathcal{Z}_\beta^{(-)} \tag{8}$$

be another orthogonal decomposition of the Krein space $\mathcal{W}_{\mathrm{KG}}$ into a maximal positive and maximal negative subspace, defining the vacuum Ω_β. One can show that we have a (non-orthogonal) direct sum

$$\mathcal{W}_{\mathrm{KG}} = \mathcal{Z}_\beta^{(+)} \oplus \mathcal{Z}_\alpha^{(-)} \tag{9}$$

Let $\Pi_{\beta\alpha}^{(+)}$, $\Pi_{\alpha\beta}^{(-)}$ be the corresponding projections. Then

$$G_{\alpha\beta}^{(+)}(x,y) = \frac{(\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)}, \tag{10}$$

$$G_{\beta\alpha}^{(-)}(x,y) = \frac{(\Omega_\beta|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\alpha)}{(\Omega_\beta|\Omega_\alpha)} \tag{11}$$

are the Klein-Gordon kernels of $\Pi_{\beta\alpha}^{(+)}$ and $\Pi_{\alpha\beta}^{(-)}$.

Green functions of 1-dimensional Schrödinger operators.

Suppose ψ_1, ψ_2 solve

$$(H + k^2)\psi_i(t) = 0, \qquad i = 1, 2.$$

Then their *Wronskian*

$$\mathcal{W}(\psi_1, \psi_2) := \psi_1(t)\psi_2'(t) - \psi_1'(t)\psi_2(t)$$

does not depend on t.

The function

$$G^{\leftrightarrow}(-k^2; t, s) := \frac{1}{\mathcal{W}(\psi_1, \psi_2)}\big(\psi_1(t)\psi_2(s) - \psi_2(t)\psi_1(s)\big)$$

does not depend on the choice of ψ_1, ψ_2 and defines the so-called *canonical bisolution*, the analog of the Pauli Jordan propagator. From $G^{\leftrightarrow}$ we can define the *forward* and *backward Green functions*:

$$G^{\rightarrow}(-k^2; t, s) := G^{\leftrightarrow}(-k^2, t, s)\theta(t - s),$$
$$G^{\leftarrow}(-k^2; t, s) := -G^{\leftrightarrow}(-k^2, t, s)\theta(s - t).$$

For $\operatorname{Re} k > 0$ we define the left and right *Jost solutions* to be the unique solutions of

$$(H + k^2)\psi_\pm(t,k) = 0, \quad \psi_\pm(t,k) \sim \mathrm{e}^{\mp tk}, \quad \pm t \to \infty.$$

We also introduce the *Jost function*

$$\mathcal{W}(k) := \mathcal{W}\big(\psi_+(\cdot,k), \psi_-(\cdot,k)\big).$$

The *resolvent* of H, denoted $G(-k^2) := (H + k^2)^{-1}$ has the integral kernel

$$G(-k^2;t,s) = \frac{1}{\mathcal{W}(k)}\big(\theta(t-s)\psi_+(t,k)\psi_-(s,k) + \theta(s-t)\psi_-(t,k)\psi_+(s,k)\big).$$

We say that H is *reflectionless* if there exist functions $T(\pm\mathrm{i}p)$ such that

$$\psi_+(\pm\mathrm{i}p) = T(\pm\mathrm{i}p)\psi_-(\mp\mathrm{i}p).$$