Scattering on self-dual Black holes

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Quantum & classical fields interacting with geometry,
IHP, 3/26/2024.

Work with: Tim Adamo, Giuseppe Bogna & Atul Sharma
2309.03834 building on 2003.13501, 2103.16984, 2110.06066, 2203.02238 . . .

First steps to extend twistor integrability for gravity scattering amplitude formulae on SD backgrounds to SD black holes.

Amplitudes on nontrivial backgrounds

Promise: Encode fully nonlinear effects, exact to all-orders!

Challenges:

- Construct momentum eigenstate analogues.
- Construct exact propagators on background.
- Perform space-time perturbation theory.

What has been done?

- 3-4 points on generic plane waves general: [2018 Adamo, Casali, M.,Nekovar], BMN: [Constable-et. al., Spradlin-Volovich].
- AdS/dS correlators: 5 pt [Goncalves, Perreira, Zhou], n-pt MUV [Green-Wen].
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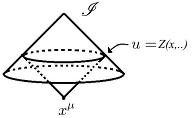
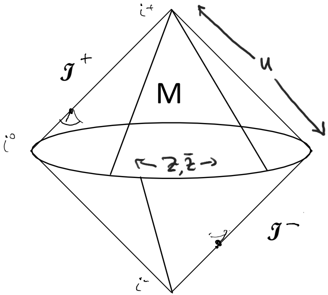
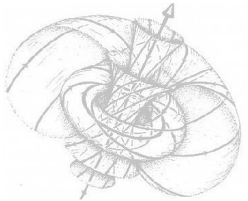
Twistors at null infinity, and integrability

- Newman's good cuts attempt to rebuild space-time from $\mathscr{I}$ data.





- Yields instead $\mathcal{H}$-*space*: a complex *self-dual* space-time.
- We use Penrose's asymptotic twistor space at $\mathscr{I}$, reformulating Newman's good cuts.
- Penrose's construction embodies integrability of self-dual sector.
- Use amplitudes to give perturbations of $\mathcal{H}$-space approximating real space-time.



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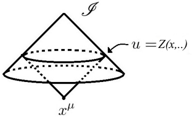

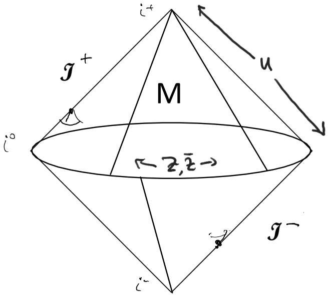

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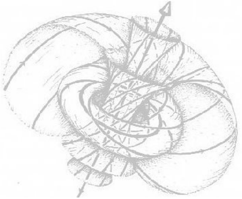

Gravity amplitudes at MHV: $- - + \dots +$ helicity.

Scatter n gravitons with momenta k_i, $i = 1, \dots n$.

- 2-component spinors express

$$x^{\dot\alpha\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} t-z & x+iy \\ x-iy & t+z \end{pmatrix} \rightsquigarrow t^2 - x^2 - y^2 - z^2 = \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} x^{\dot\alpha\alpha} x^{\dot\beta\beta}$$

- So null momenta factorize: $k_{i\alpha\dot\alpha} = \kappa_{i\alpha}\kappa_{i\dot\alpha}$.
- Spinor helicity: $\langle 1\,2\rangle := \kappa_{1\alpha}\kappa_2^{\alpha}$, $[1\,2] := \kappa_{1\dot\alpha}\kappa_2^{\dot\alpha}$,
- MHV formula: [Hodges 2012] define $n \times n$ matrix:

$$\mathbb{H}_{ij} = \begin{cases} \frac{[ij]}{\langle ij\rangle} & i \neq j \\ -\sum_k \frac{[ik]}{\langle ik\rangle} & i = j\,. \end{cases}$$

- Then: $\mathcal{M}(1, \dots, n) = \langle 12\rangle^6 \det{}'\mathbb{H}\, \delta^4(\sum_i k_i)$.
- $\mathbb{H}$ is Laplace matrix for matrix-tree theorem $\overset{\text{[Feng,He 2012]}}{\rightsquigarrow}$
- Sum of tree diagrams with propagators $\frac{[ij]}{\langle ij\rangle}$ [Bern, et. al. '98]

For what theory???

Answer: [Adamo, M., Sharma, 2103.16984] $\mathcal{M} = \langle V_1 \dots V_{n-2}\rangle_{\text{Tree}}$

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- The $\kappa_{i\alpha}$ survive as contants.
- We can 'dress' the $\kappa_{i\dot\alpha}$ and send $[ij] \to [\![ij]\!]$ in formulæ.
- Can define (x-dependent) $\mathbb{H}$

But:

- there are, say, t interactions with background, $t < n-2$,
- and fields generate tails after hitting background...
- So define $(n+t) \times (n+t)$ generating matrix for t interactions with background

$$\mathcal{H} := \begin{pmatrix} \mathbb{H} & \mathfrak{h} \\ \mathfrak{h}^T & \mathbb{T} \end{pmatrix}$$

- Gives contribution $\int_M d^4x \, \prod_{m=1}^{t} \partial_{\epsilon_m}^{p_m} \det{}' \mathcal{H}|_{\epsilon_m=0} \times \ldots$

But, radiative $\leftrightarrow$ defined from radiation data at $\mathscr{I}$:
Stationary black holes are trivial at $\mathscr{I}$, so doesnt apply.

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Outline

Extension to self-dual (SD) black hole backgrounds:

1. Self dual Kerr $=$ SD Taub-NUT with single copy SD dyon.
2. Introduce charged Killing spinors for self-dual dyons $\rightsquigarrow$
3. momentum eigenstates on SD dyon background.
4. Solutions lift directly to SD Taub-NUT.
5. Find anomalous spin weights and fall-off from nontrivial topology of SD dyons and Taub-NUTs.
6. Two point functions for helicity-flip through backgrounds obtained by direct integration.
7. Spin is incorporated by exponential factor.
8. MHV amplitudes at n-points via twistor theory.

Black holes and valence-2 Killing spinors

Vacuum black-holes, Kerr etc. have Petrov type D Weyl-spinor

$$\Psi_{\alpha\beta\gamma\delta} = \Psi_2 o_{(\alpha} o_\beta \iota_\gamma \iota_{\delta)} \,.$$

Proposition (Penrose, Walker, Hughston, Sommers, 1972,3)

For any vacuum space-time of Petrov type D

- $\exists$ *Killing spinor* $\chi^{\alpha\beta}$: $\qquad \nabla^{\dot\alpha(\alpha}\chi^{\beta\gamma)} = 0\,.$
- *related to ASD Weyl spinor & ASD Maxwell field by*

$$\Psi_{\alpha\beta\gamma\delta} = \frac{\chi_{(\alpha\beta}\chi_{\gamma\delta)}}{\chi^5}\,, \quad \phi_{\alpha\beta} := \frac{\chi_{\alpha\beta}}{\chi^3}\,, \quad \chi^{\alpha\beta} = \chi o^{(\alpha}\iota^{\beta)}\,.$$

- *SD parts are the complex conjugates* $\leftrightarrow \bar\chi^{\dot\alpha\dot\beta}$.

In Kerr's original rectangular coords: $\chi_a^{\alpha\beta} = \begin{pmatrix} x+iy & z-ia \\ z-ia & x-iy \end{pmatrix}$.

a =Kerr parameter.

Newman-Janis shift: $\quad z \to z - ia$ sends $\chi_0^{\alpha\beta} \to \chi_a^{\alpha\beta}$.

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Self-dual Taub-NUT as SD part of Kerr

Euclidean SD Taub-NUT is

$$ds^2 = \frac{1}{V}(dt - 2m\,\mathbf{a})^2 + V d\mathbf{x}\cdot d\mathbf{x}\,,$$

here bold $\leftrightarrow$ 3-vector $\mathbf{x} = (x, y, z)$, $r = |\mathbf{x}|$, and

$$V = 1 + \frac{2m}{r}, \qquad \nabla \wedge \mathbf{a} = \nabla V\,, \qquad \mathbf{a} = (1 - \cos\theta)d\phi$$

- The ASD Weyl spinor $= 0$ and the SD Weyl spinor is

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The self-dual dyon as single copy of SD Taub-NUT

Minimal ($e = g = 1$) SD Euclidean Maxwell dyon potential:

$$A = \frac{1}{2}\left(\frac{dt}{r} - \mathbf{a}\right) \qquad \rightsquigarrow \quad \text{field} \quad F_{\dot\alpha\dot\beta} = \frac{\tilde\chi_{\dot\alpha\dot\beta}}{2\tilde\chi^3}\,, \quad F_{\alpha\beta} = \mathbf{E} - \mathbf{B} = 0.$$

- 'single-copy' of SD Taub-NUT: Coulomb + monopole.
- $\frac{1}{2}\mathbf{a} = \frac{1}{2}(1 - \cos\theta)d\phi$ has 'wire singularity at $\theta = \pi$.
- Gauge transformation $\mathrm{e}^{-i\phi}$ gives wire singularity at $\theta = 0$

$$\mathbf{a}' = -\frac{1}{2}(1 + \cos\theta)d\phi\,.$$

- So A is topologically non-trivial $U(1)$ connection on Line bundle L with Chern class 1.
- Defines line bundles $L^e \to \mathbb{M} - \{r = 0\}$, $c_1(L^e) = e$, connection ieA.

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$$\chi^{\alpha\beta} = \chi_+^{(\alpha}\chi_-^{\beta)}\,, \qquad \chi_-^{\alpha} = \sqrt{2}\, t^{\alpha}_{\dot\alpha}\bar\chi_+^{\dot\alpha}\,,$$

- Using $\zeta = \frac{x+iy}{r+z}$, stereographic coord, find

$$\chi_+^{\alpha} = \sqrt{\frac{r}{1+|\zeta|^2}}\,(\bar\zeta, -1)\,, \qquad \chi_-^{\alpha} = \sqrt{\frac{r}{1+|\zeta|^2}}\,(1, \zeta)$$

- $\chi_\pm^\alpha$ are globally defined as sections of $L^{\pm 1}$.
- They satisfy the charged Killing spinor equation

$$\partial_{\dot\alpha}^{(\alpha}\chi_\pm^{\beta)} \pm \mathrm{i}\, A_{\dot\alpha}^{(\alpha}\chi_\pm^{\beta)} = 0\,,$$

- Let $\psi_{\alpha_1\ldots\alpha_{n+1}}$ be a massless field, $\partial_{\dot\alpha}^{\alpha}\psi_{\alpha_1\ldots\alpha_n\alpha} = 0$, then

$$\phi_{\alpha_1\ldots\alpha_n} = \psi_{\alpha_1\ldots\alpha_n\beta}\chi_\pm^{\beta}$$

is a charged massless field of charge ± 1.

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Constructing charged momentum eigenstates

Start with momentum eigenstate of helicity $-(n+e)/2$

$$\psi_{\alpha_1\dots\alpha_n} = \kappa_{\alpha_1}\dots\kappa_{\alpha_{n+e}}\mathrm{e}^{ik\cdot x}\,, \qquad k_{\dot\alpha\alpha} = \kappa_\alpha\tilde\kappa_{\dot\alpha}\,.$$

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Nontrivial topology of **a** in SD Taub-Nut

$$ds^2 = \frac{1}{V}(dt - 2m\mathbf{a})^2 + V d\mathbf{x} \cdot d\mathbf{x}$$

implies periodic time $t \sim t + 4m$, quantizes frequency $2m\omega \in \mathbb{Z}$.

Proposition

Charge e momentum eigenstates $\psi^{SDD}_{\alpha_1 \dots \alpha_n}$ on the SD dyon background naturally lift to momentum eigenstates $\psi^{SDTN}_{\alpha_1 \dots \alpha_n}$ on SD Taub-NUT with energy ω quantized by $2m\omega = e$.

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Two point functions

Intuition: SD background $\leftrightarrow$ many $+$ particles.

On flat space, the all $+$ and one $-$ amplitudes vanish $\rightsquigarrow$ expect:

- $++$ and $+-$ 2-point functions $= 0$ on SD background.
- First non-trivial 2-point function is for $--$.
- SUSY $\rightsquigarrow$ expect vanishing $--$ for Fermion on SD dyon and helicity $-1/2, -1, -3/2$ on SD Taub-NUT.

Embed SD dyon into $SU(2)$ Yang-Mills via $\sigma_3 A$ and compute two-point function of charge $\pm e$ gluons as variation of action

$$\mathcal{A}^{SDD}(1-,2-) = \int_{\mathbb{M}} d^4x\, F_{1\alpha\beta} F_2^{\alpha\beta}\,.$$

For spin-2 similarly compute as 2^{nd} variation of Plebanski action

$$\mathcal{A}^{SDTN}(1-,2-) = \int_{M_{SDTN}} \Gamma_{1\alpha}^{\gamma} \wedge \Gamma_{2\beta\gamma} \wedge \theta^{\alpha}_{\dot\alpha} \wedge \theta^{\beta\dot\alpha}$$

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$(+,+),(+-)$ cases vanish as expected.
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This simplifies in SD sector:

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Twistors become essential Adamo, M., Sharma, 2203.02238.
For the SD dyon the output is Adamo, Bogna, M., Sharma, 240?.????.

$$\mathcal{A}_n^{\text{MHV}}(\{\kappa_a^\alpha, \tilde{\kappa}_a^{\dot\alpha}, e_a, m_a\}) = 2\pi g^{n-2}\delta(\omega)\,\delta(e)\,\frac{\langle r\, s\rangle^4}{\langle 1\,2\rangle \ldots \langle n\, 1\rangle}\times$$
$$\times \int \mathrm{d}^3\vec{x}\, \mathrm{e}^{i\vec{k}\cdot\vec{x}} \prod_{a=1}^{n} \left(\frac{r}{1+\zeta\bar{\zeta}}\right)^{m_a} (\bar{\zeta} z_a + 1)^{e_a+m_a}(\zeta - z_a)^{e_a - m_a}\,, \tag{1}$$

Whereas for gravity we have:

$$\mathcal{M}_n^1 = \frac{\tilde{\delta}\left(\sum_{i=1}^n \omega_i\right)\,\langle 1\,2\rangle^6}{\langle 1\, i\rangle^2\,\langle 2\, i\rangle^2} \sum_{\substack{p+q+r=t \\ t=0}}^{\infty} \int_{M\times(\mathbb{P}^1)^t} \mathrm{d}^4 x \prod_{a=1}^{p} \frac{\varpi_a\,[\tilde{\iota}\, F_a]}{24\, M}\, \frac{\partial^3}{\partial \varepsilon_a^{-\,2}\partial\varepsilon_a^{+}}$$
$$\prod_{b=1}^{q} \frac{\varpi_b\,[\tilde{o}\, F_b]}{24\, M}\, \frac{\partial^3}{\partial\varepsilon_b^{-}\partial\varepsilon_a^{+\,2}} \prod_{c=1}^{r} \varpi_c 8\, M\, \frac{\partial^4}{\partial\varepsilon_a^{-\,2}\partial\varepsilon_a^{+\,2}} \rho\left(|\mathcal{H}[t]_i^i|\right)\Big|_{\varepsilon=0} \prod_{i=1}^{n} \mathrm{e}^{i\,k_i\cdot x}\,.$$

Conclusions and further developments

In summary our explicit formulae

- describes helicity flip of ASD particle by SD background,
- are tractable from integrability of SD sector,
- resum infinitely many diagrams.

The two ASD particles shouldnt interact with ASD sector so:

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Thank You!