On the module-Lattice Isomorphism Problem,
based on a joint work with A. Pellet-Mary, G. Pliatsok and A. Wallet.

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INRIA, Université de Bordeaux

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- Defining the module-Lattice Isomorphism Problem (module-LIP).

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- An (algebraic) attack on module-LIP in a special case.

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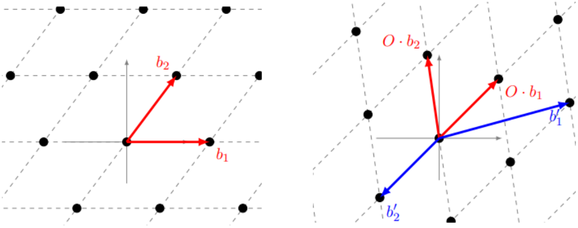

(a) $\mathcal{L}_1$ with known basis $B = (b_1, b_2)$. (b) $\mathcal{L}_2$ with known basis $B' = (b'_1, b'_2)$.

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- Use the algebraic structure to solve LIP more efficiently ?

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Modules and module-LIP

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- More generally (for non-free modules), one can use pseudo-bases and pseudo-Gram matrices.

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Writing elements as sums of two squares in K is equivalent to solve a norm equation (in the appropriate extension).

First attack on module-LIP

$\rightarrow$ Using arithmetic of ideals (factorization, splittings) and algorithmic tools (variants of Gentry Szydlo algorithm) we can solve module-LIP for rank two modules when K is totally real !

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Breaking module-LIP on $\mathcal{O}_K^2$.

Suppose K is totally real and $M = \mathcal{O}_K^2$. There exists a polynomial time algorithm that, given any G' congruent to I_2, returns **all** $U \in \mathrm{GL}_2(\mathcal{O}_K)$ such that $G' = U^* U$.

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We proved a more general statement : for K totally real and most of rank two modules $M \subset K^2$, there exists a probabilistic and heuristic polynomial time algorithm that solves module-LIP on M (G. M., A. Pellet-Mary, G. Pliatsok, A. Wallet).

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Thank you !

[1] On the Lattice Isomorphism Problem, I. Haviv, O. Regev, 2013.

[2] HAWK : Module LIP makes Lattice Signatures Fast, Compact and Simple, L. Ducas, E.W. Postlethwaite, L. N. Pulles. W. van Woerden, 2023.

[3] Just how hard are rotations of $\mathbb{Z}^n$? H. Bennett, A. Ganju, P. Peetathawatchai, N. Sthephens-Davidowitz, 2023.

[4] On the lattice isomorphism problem, quadratic forms, remarkable lattices and cryptography, L. Ducas, W. van Woerden.