# We Are on the Same Side
# Alternative Sieving Strategies for the Number Field Sieve

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# RSA Cryptosystem

## Private key

- Used for decryption
- Generated from two random prime numbers $p$ and $q$

## Public key

- Used for encryption
- Generated from product $N = pq$

## Factorization

- RSA security is linked to the hardness of integer factorization
- Finding $p$ and $q$ from $N$ breaks RSA

# Factoring a large number

Shor's algorithm!

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Classically?

# Fermat's method

- Try many $x$'s
- Is $x^2 - N$ a square?

Then...

- $N = x^2 - y^2$
- $N = (x + y)(x - y)$
- $gcd(x \pm y, N)$ gives a factor of $N$

Smarter way than trying $x$'s until randomly getting a square?

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# Quadratic Sieve

## *Build* a square

- Generate many $y_i = x_i^2 \mod N$
- Build $Y^2 \mod N$ as a product of $y_i$'s

## Building $Y^2$

- Factor entirely many $y_i$'s (a *relation*)
- Linear algebra
  - Write each relation as a list of exponents of prime factors
  - Combine to get even exponents
  - It's a square!

From factoring a large number...
...to factoring many small numbers

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# NFS : Overview

State-of-the-art algorithm

## General idea

- $x^2 \equiv y^2 \pmod N$
- $x \pm y \neq 0 \pmod N$ ? ‘
- $gcd(x \pm y, N)$ gives a factor of N

**2 main parts**

1. Collection of relations
   - Find many relations
2. Linear algebra
   - Combine them

Very similar to the quadratic sieve (so far...)

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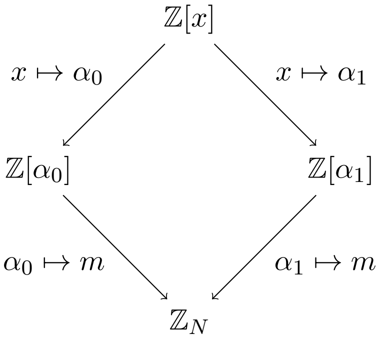
# NFS : Relations




Two sides in NFS

For each pair (a, b)

- Factor rational norm
- Factor algebraic norm

Small enough factors on both norms?

- Relation

# CADO-NFS

- Implementation of the NFS
- Open source : https://gitlab.inria.fr/cado-nfs/cado-nfs
- Can also compute discrete logarithms
- 2019 : Factorization record RSA-240 (240 digits)
- 2020 : Factorization record RSA-250 (current record)
- Computing time is dominated by the relation collection

# Relation collection in CADO-NFS

(a, b) pairs space is **large**

- No need to factor *all norms*

Objective

**Finding just enough relations in the shortest time**

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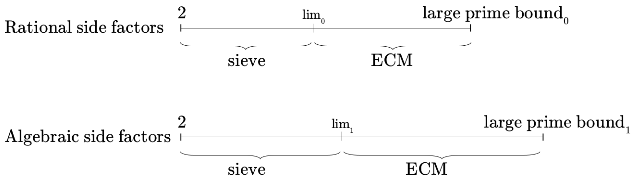
# Factoring norms

2 methods :

- Sieving to find small and medium factors
- Elliptic-curve factorization (ECM) to find large factors




- Step 1 : sieve all norms
- Step 2 : ECM on norms most likely to become relations

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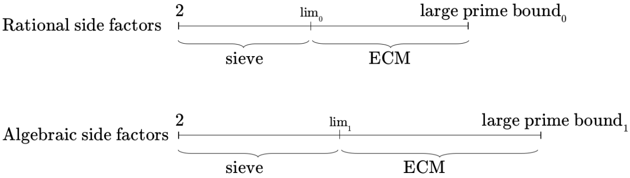


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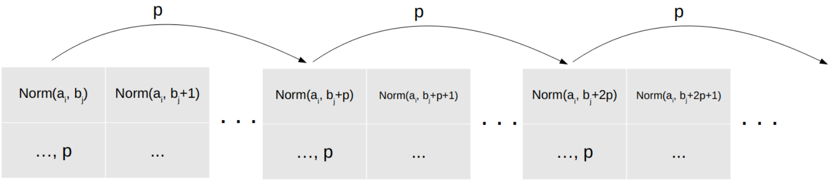
# Sieving process

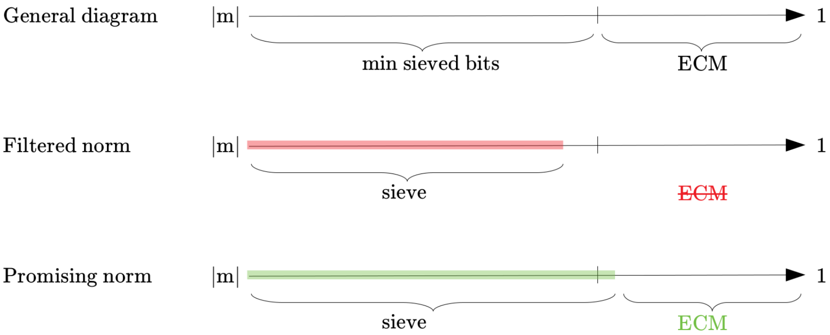
The structure of norms and $(a, b)$ pairs allows sieving on a side :

- Pick a side and a prime factor $p$
- Find and tick a pair $(a, b)$ whose norm it divides
- Tick the next $p$-th pair $(a + p, b)$
- Tick all $p$-th pairs



# Promising pairs

- Best candidates to give a relation
- Sieving factored enough for both norms
- Only promising pairs get to the ECM step



# Promising bound

If the bound deciding wether or not a pair is sent to ECM is...

- Too high
  - Many pairs of low quality will take too much time in ECM
- Too low
  - Few pairs of high quality will give too few relations and additional sieving will be needed

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# Improving relation collection in CADO-NFS

**Goal : find *almost* as many promising pairs at a much lower cost**

Small sieve

Subroutine of CADO-NFS sieving finding small primes

- Small factors are worth few bits
- Not decisive on promising pairs

**Remove small sieve?**

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# Batch factoring

## How to find smooth parts of integers [Bernstein 2004]

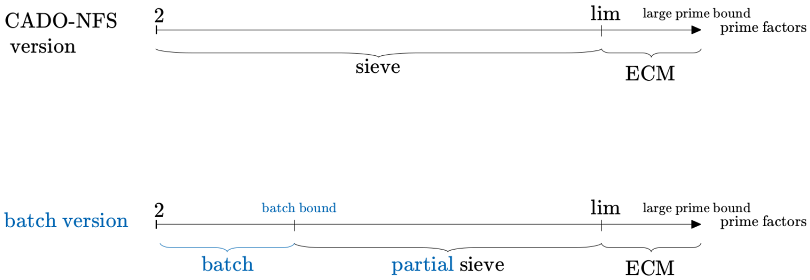
- Input : list of integers, factor base ($b$ bits)
- Output : list of smooth parts, meaning the product of factors from the base found in each integer
- $O(b(\lg b)^{2+o(1)})$

# Hybrid version

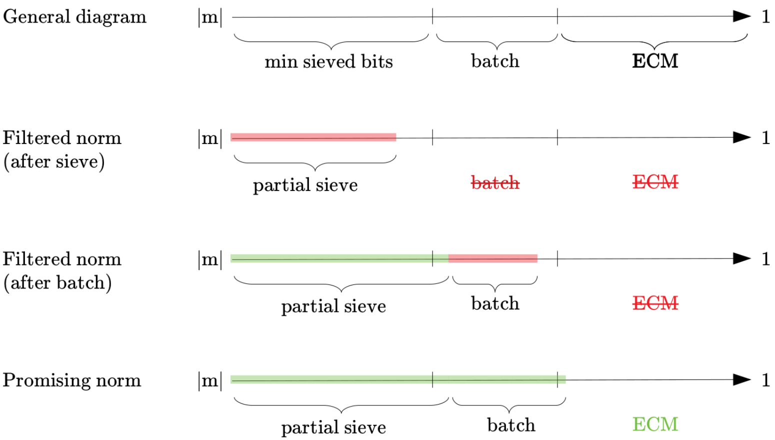
Pick an intermediate "batch promising" bound larger than the "ECM promising" bound, then :

1. Sieve only on medium primes
2. Remove non-batch promising pairs
3. Get small factors using batch factoring
4. Remove non-ECM promising pairs
5. Get large factors using ECM
6. Relations!

# Method for each prime factors interval



# Path to ECM


General diagram
|m|
1
min sieved bits
batch
ECM
Filtered norm
(after sieve)
|m|
1
partial sieve
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# Implementation in CADO-NFS

## RSA-250's relations

- Targeted number of relations
- Sets of parameters

## Results

- Fewer relations are found
- Speedup counteracts this
- Better efficiency
- Up to 1.1 overall speedup

## Benchmarks

- Sampled sieved regions
- Easy extrapolation

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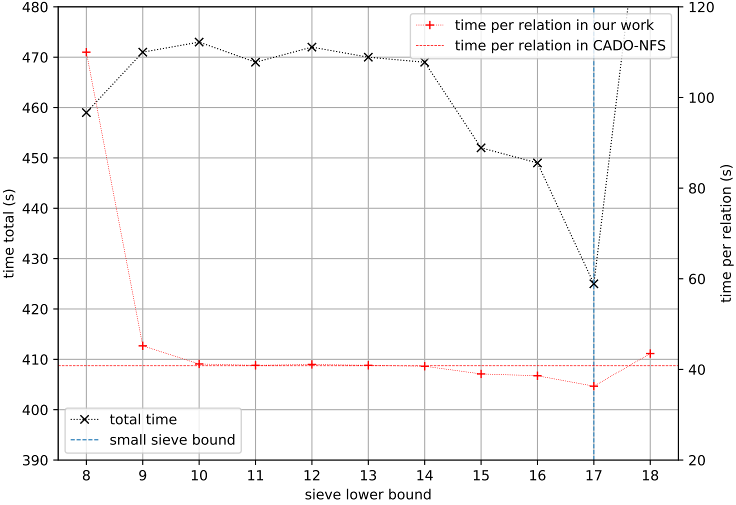
# Speedup

Target : 90% of relations



*Thank you!*