Ternary representation for large distance decoding in $\mathbb{F}_3$

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The Wave signature

1. Digital signature submitted to new NIST competition **[Debris,Sendrier,Tillich,Asiacrypt2019]**
2. Code based signature in $\mathbb{F}_3$

Wave security

Problem (Syndrom decoding problem)

Given

- $\boldsymbol{H} \in \mathbb{F}_3^{(n-k)\times n}$ *following a uniform distribution*
- $\mathbf{s} \in \mathbb{F}_3^{n-k}$ *following a uniform distribution*
- $t \in [\![0, n]\!]$

Find $\mathbf{e} \in \mathbb{F}_3^n$ *such that* $\boldsymbol{H}\mathbf{e}^T = \mathbf{s}$ *and* $\Delta(\mathbf{e}) = t$.

Security (see **[Sendrier,PQCrypto2023]**)

- Large weight decoding $\Rightarrow$ **Forgery Attack**
- Small weight decoding $\Rightarrow$ **Key Recovery Attack**

Best large/small weight decoding algorithms

ISD (Information Set Decoding introduced by **[Prange,1962]**).

Usual representation technique in Hamming weight

Definition (Representation in Hamming weight)

Let $\mathbf{z} \in \mathbb{F}_q^n$ *such that* $\Delta(\mathbf{z}) = t$.
A (u, v)-*representation of* $\mathbf{z}$ *is a pair* $(\mathbf{x}, \mathbf{y})$ *such that*

- $\Delta(\mathbf{x}) = u$
- $\Delta(\mathbf{y}) = v$
- $\mathbf{z} = \mathbf{x} + \mathbf{y}$

$$\underset{\mathbf{z}}{\boxed{\quad t \quad}} = \underset{\mathbf{x}}{\boxed{\quad u \quad}} + \underset{\mathbf{y}}{\boxed{\quad v \quad}}$$

A more precise operator than the Hamming weight

In $\mathbb{F}_3$, the Hamming weight gives us :

- number of 0
- The sum of the number of 1 and 2

We can be more precise

Definition : Symbol counter operator $\blacktriangle(.)$

Let $\mathbf{x} \in \mathbb{F}_q^n$, we define

$$\blacktriangle(\mathbf{x}) := (t_0, \cdots, t_{q-1}) \in [\![0, n]\!]^q$$

With $t_i := Card(\{j \in [\![1, n]\!] \ : \ \mathbf{x}(j) = i\})$

Idea to improve the state of the art of decoding in $\mathbb{F}_3$ with $\blacktriangle(.)$

$\blacktriangle(.)$ tool idea for large weight decodings

- **Small weight** : small quantity of 1 and 2 $\Rightarrow$ negligible precision gain
- **Large weight** : large quantity of 1 and 2 $\Rightarrow$ interesting precision gain

Idea to improve the state of the art of large weight decoding in $\mathbb{F}_3$

In $\mathbb{F}_2$, $\blacktriangle(.) \Leftrightarrow \Delta(.)$
$\Rightarrow$ Adapting the best current ISDs in $\mathbb{F}_2$ in Hamming weight **[Both,May,2018]** to $\blacktriangle(.)$ in $\mathbb{F}_3$

Adaptation of the concept of representation to $\blacktriangle(.)$

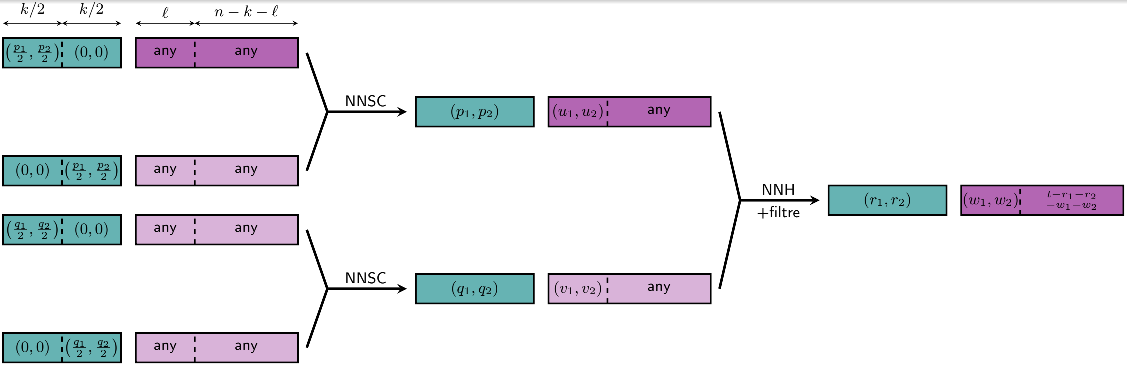
Definition (Representation with $\blacktriangle(.)$)

Let $\mathbf{z} \in \mathbb{F}_q^n$ *such that* $\blacktriangle(\mathbf{z}) = (t_0, t_1, t_2)$.
A (u_1, u_2, v_1, v_2)*-representation of* $\mathbf{z}$ *is a pair* $(\mathbf{x}, \mathbf{y})$ *such that*

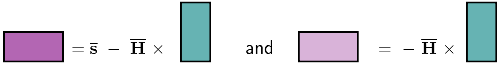
- $\blacktriangle(\mathbf{x}) = (u_0, u_1, u_2)$
- $\blacktriangle(\mathbf{y}) = (v_0, v_1, v_2)$
- $\mathbf{z} = \mathbf{x} + \mathbf{y}$

$$\underbrace{\boxed{(t_0, t_1, t_2)}}_{\mathbf{z}} = \underbrace{\boxed{(u_0, u_1, u_2)}}_{\mathbf{x}} + \underbrace{\boxed{(v_0, v_1, v_2)}}_{\mathbf{y}}$$

Two stages [Both,May,2018] with $\blacktriangle(.)$



Legend :



where $\overline{\mathbf{H}}|\mathbf{Id} = \mathbf{SHP}$ and $\overline{\mathbf{s}} = \mathbf{S}\mathbf{s}^\top$

Adaptation of the definition

What we need ?

1. Adapt the **Nearest Neighbor** search to $\blacktriangle(.)$
2. Count the **number of representations**

NNH Problem (Nearest Neighbor with Hamming weight)

Let two lists $\mathcal{L}_1$ and $\mathcal{L}_2$ of random elements of $\mathbb{F}_3^n$, and $t \in \mathbb{N}$, find :

$$\{(\mathbf{x}, \mathbf{y}) \in \mathcal{L}_1 \times \mathcal{L}_2 \text{ such that } \Delta(\mathbf{x} + \mathbf{y}) = t\}$$

NNSC Problem (Nearest Neighbor with Symbol Counter $\blacktriangle(.)$)

Let two lists $\mathcal{L}_1$ and $\mathcal{L}_2$ of random elements in $\mathbb{F}_3^n$, and $t_0, t_1, t_2 \in \mathbb{N}$ such that $t_0 + t_1 + t_2 = n$, find :

$$\{(\mathbf{x}, \mathbf{y}) \in \mathcal{L}_1 \times \mathcal{L}_2 \text{ such that } \blacktriangle(\mathbf{x} + \mathbf{y}) = (t_0, t_1, t_2)\}$$

Solving NNSC with Hamming weight

Let (t_0, t_1, t_2) be fixed.
Three ways to get back to Hamming's weight problem :

1. $\mathcal{L}_1 \times \mathcal{L}_2 \rightarrow (t_0, t_1, t_2) \rightarrow$ **Hamming weight** $t_1 + t_2$
2. $\mathcal{L}_1 \times (\mathcal{L}_2 + \mathbf{1}) \rightarrow (t_2, t_0, t_1) \rightarrow$ **Hamming weight** $t_0 + t_1$
3. $\mathcal{L}_1 \times (\mathcal{L}_2 + \mathbf{2}) \rightarrow (t_1, t_2, t_0) \rightarrow$ **Hamming weight** $t_0 + t_2$

The algorithm

At (t_0, t_1, t_2) fixed :

- Solve the **Nearest Neighbor in Hamming weight** which is the most efficient among the 3 previous ones using **[Carrier,2020]** optimal method.
- Filter the solutions $(\mathbf{x}, \mathbf{y}) \in \mathcal{L}_1 \times \mathcal{L}_2$ such that $\blacktriangle(\mathbf{x} + \mathbf{y}) = (t_0, t_1, t_2)$

Solving NNSC with $\blacktriangle(.)$

Adaptation of the Nearest Neighbor algorithm with random codes of **[Carrier,2020]** to $\blacktriangle(.)$

Principle

- We use hash functions that are the decoder of a random code $\mathcal{C}$.
- We use a hash table T indexed by $\mathcal{C}$.

1. $\forall \mathbf{x} \in \mathcal{L}_1$, $\forall \mathbf{c} \in \mathcal{C}$ such that $\blacktriangle(\mathbf{x}+\mathbf{c}) = (u_0, u_1, u_2)$, append $\mathbf{x}$ to $T[c]$.
2. $\forall \mathbf{y} \in \mathcal{L}_2$, $\forall \mathbf{c} \in \mathcal{C}$, such that $\blacktriangle(\mathbf{y}-\mathbf{c}) = (v_0, v_1, v_2)$, test the candidate (pairs hashed to the same value).
3. Iterate with new code $\mathcal{C}$

About $u_i's$ and $v_i's$

Parameters that are chosen to optimize complexity

Algorithm complexity

Complexity

$$C = \min_{u_1,u_2,v_1,v_2} \left(\frac{C_{iter}}{P_{succ}}\right)$$

- C_{iter} **the complexity of an iteration** can be computed efficiently
- P_{succ} **the success probability** can be computed efficiently using Gröebner basis techniques

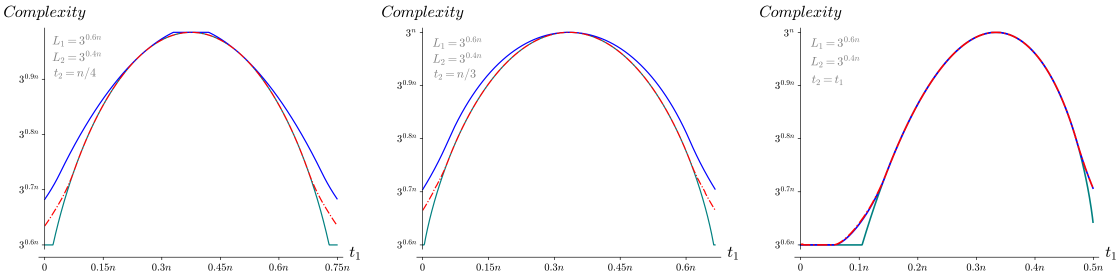
Calculation

We can compute efficiently the number of representation of a fixed vector (same technique with Gröebner basis)

One problem

To get C, we must optimize the parameters $(u_i, v_i) \Rightarrow$ numerical optimization $\Rightarrow$ expensive
(We start to better understand these parameters).

Comparison of complexities



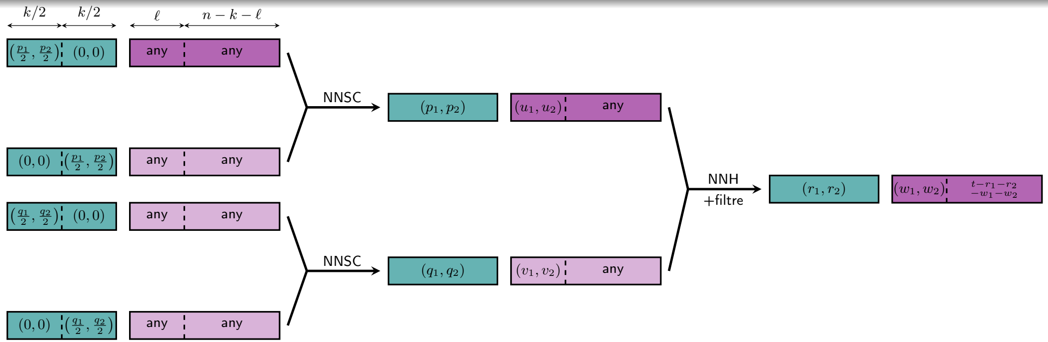
Red curve : complexity for solving NNSC problem using $\blacktriangle(.)$
Blue curve : complexity for solving NNSC problem using Hamming weight
Green curve : complexity lower bound

Result

Comparison of the complexity of several algorithms for ternary decoding with high weight $t = 0.98n$ and rate $k = n/2$.

Algorithm	Complexity
[Prange,1962]	0.12072
[Dumer,1991]	0.09091
[Stern,1988],[May,Ozerov,2015]	0.08491
[Bricout,Chailloux,Debris,Lequesne,2019]	$\mathbf{0.06535}$
This work	0.07239

Increase the depth of [Both,May,2018]



- **[Both,May,2018]** uses 4 stages. **[Bricout,Chailloux,Debris,Lequesne,2019]** uses 7 stages. **Our work** 2 stages.
- The number of parameters to optimize explode.
- We are better understanding the intern parameters of our Nearest Neighbor by getting inspiration from **[Carrier,2020]**.