Lattices in cryptography: cryptanalysis, constructions and reductions

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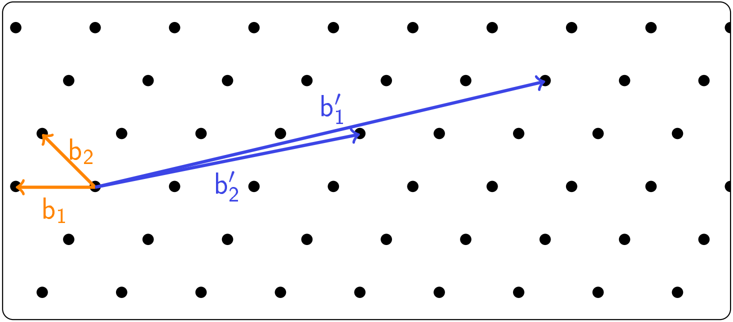
CNRS and Université de Bordeaux

Journées C2, 2023

Najac

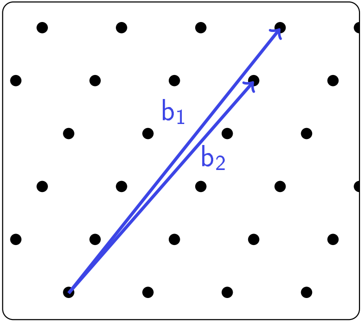
Lattices



- $\mathcal{L} = \{\sum_{i=1}^{n} x_i \mathrm{b}_i \mid \forall i,\, x_i \in \mathbb{Z}\}$ is a lattice
- $(\mathrm{b}_1, \ldots, \mathrm{b}_n) =: B \in \mathrm{GL}_n(\mathbb{R})$ is a basis (not unique)

Short basis problem

Input:



$\longrightarrow$

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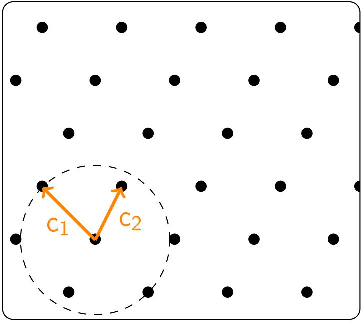

Shortest basis problem

$$\max_i \|c_i\| \leq \min_{B' \text{ basis of } L} \left(\max_i \|b'_i\| \right)$$

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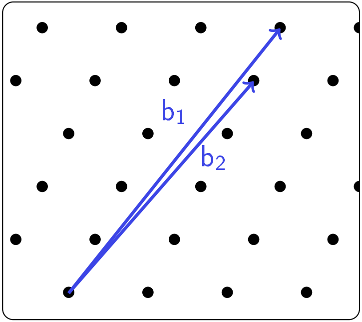

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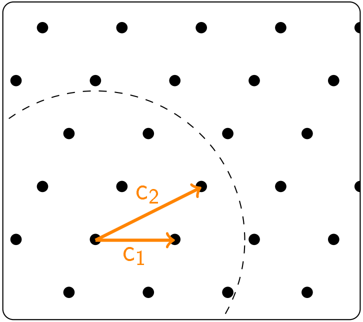
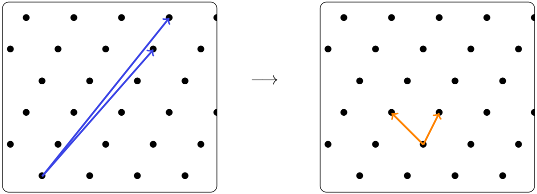


Approximate short basis problem

$$\max_i \|c_i\| \leq \gamma \cdot \min_{B' \text{ basis of } L} \left(\max_i \|b'_i\| \right)$$

Lattice reduction algorithms



Dimension 2: Lagrange-Gauss algorithm

video

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video

Theorem: The algorithm

- finds a shortest basis
- runs in polynomial time

The LLL algorithm [LLL82]

Input: basis $B = (b_1, \ldots, b_n)$

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Main idea: improve the basis locally on blocks of dimension 2
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Algorithm:

- while there exists i such that (b_i, b_{i+1}) is not a shortest basis of L_i
  (L_i is roughly the lattice spanned by (b_i, b_{i+1}))
    - run Lagrange-Gauss on L_i

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- finds an approximate short basis with $\gamma = 2^n$

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- finds an approximate short basis with $\gamma = 2^n$
- does not run in polynomial time

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Algorithm:

- while there exists i such that (b_i, b_{i+1}) is not a γ'-short basis of L_i with $\gamma' = 4/3$
  (L_i is roughly the lattice spanned by (b_i, b_{i+1}))
  - run Lagrange-Gauss on L_i

This algorithm

- finds an approximate short basis with $\gamma = 2^n$
- runs in polynomial time

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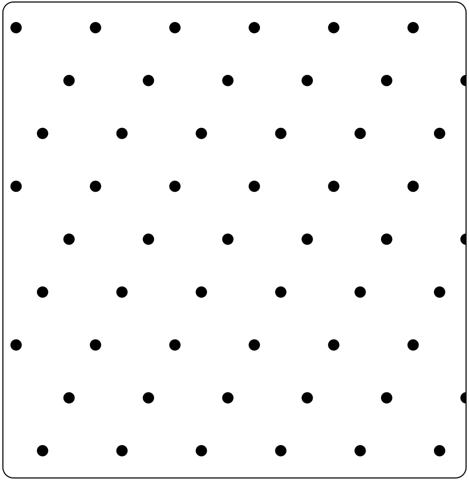
Sieving algorithm [AKS01]



Sieving:

[AKS01] Ajtai, Kumar, and Sivakumar. A sieve algorithm for the shortest lattice vector problem. STOC

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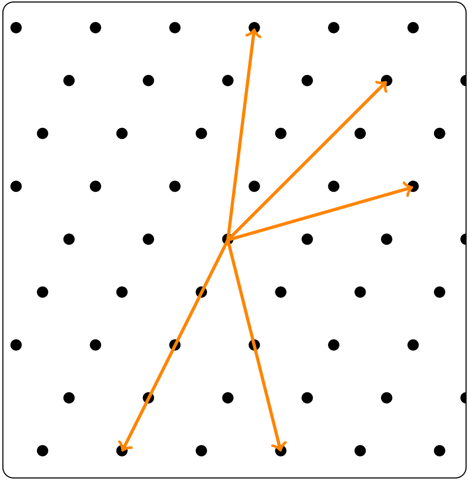

Sieving:

- Create many large vectors

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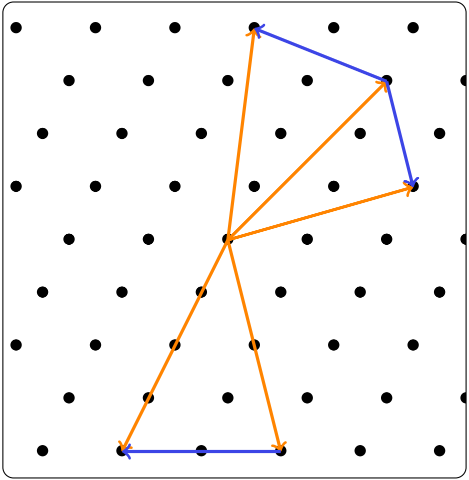

Sieving:

- Create many large vectors
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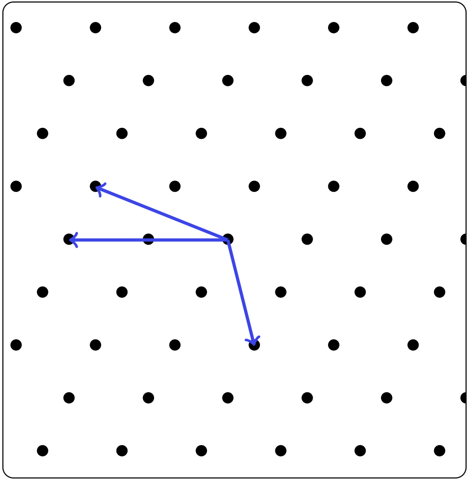

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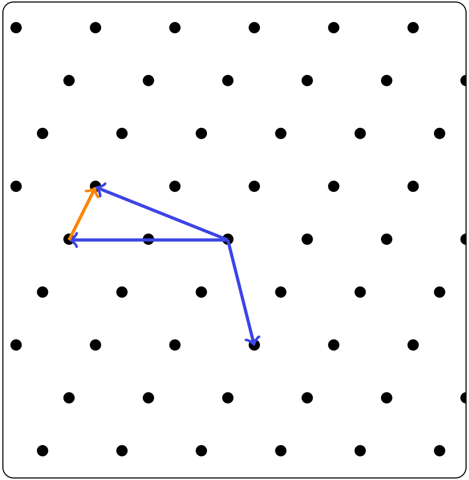

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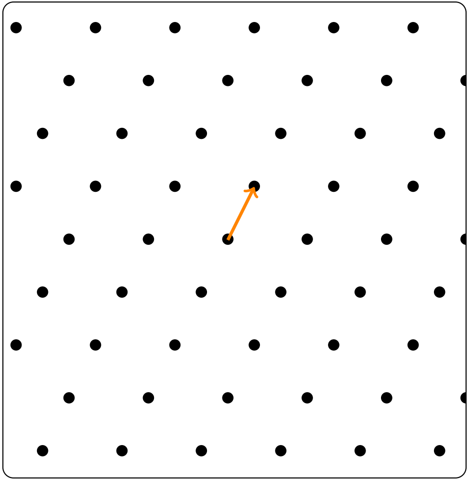

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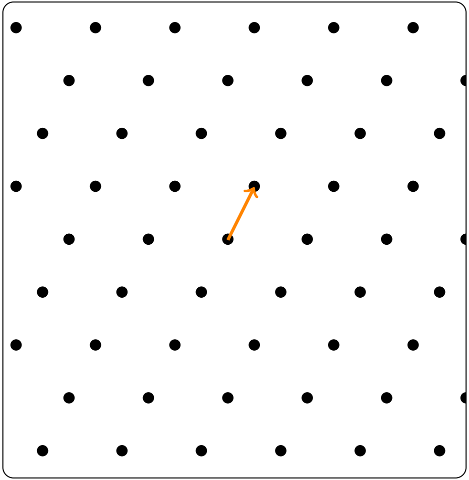

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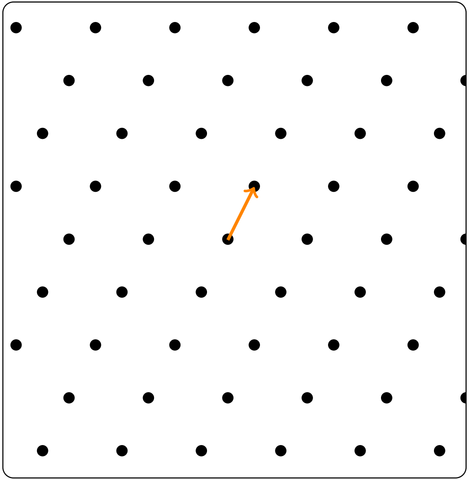

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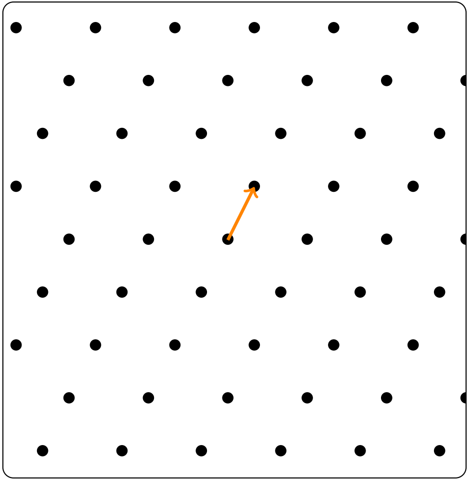

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- runs in time $2^{O(n)}$

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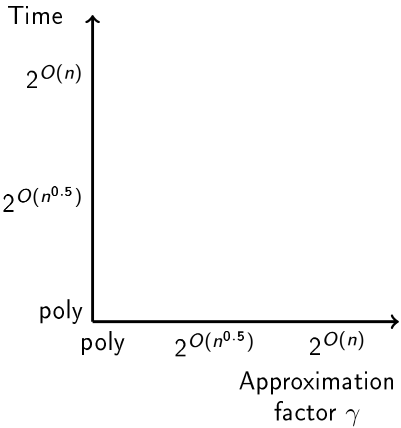
Summary and BKZ



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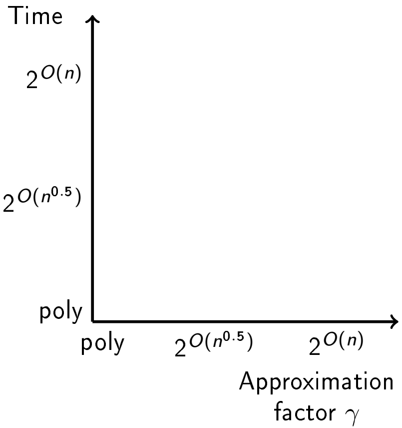
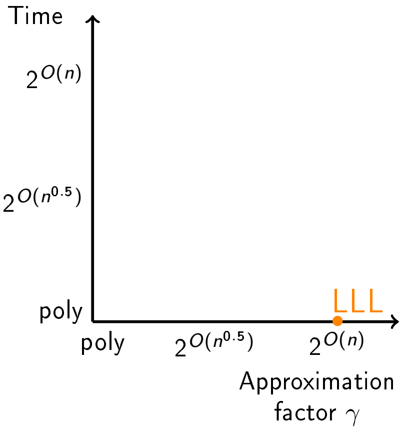


Lagrange-Gauss algorithm: dim 2

- shortest basis
- polynomial time



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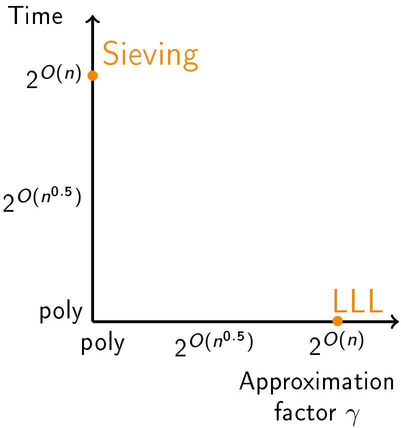

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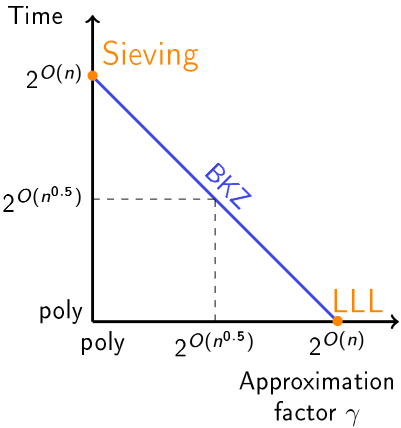
Summary and BKZ

BKZ trade-offs



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BKZ algorithm: combine LLL + Sieving $\Rightarrow$ various trade-offs

Some concrete numbers

Finding a shortest basis in practice:

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- up to $n = 180 \rightsquigarrow$ few days on big computers with good code [DSW21]

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- up to $n = 180 \rightsquigarrow$ few days on big computers with good code [DSW21]
- from $n = 500$ to $n = 1000 \rightsquigarrow$ cryptography

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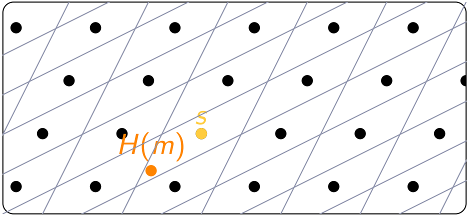
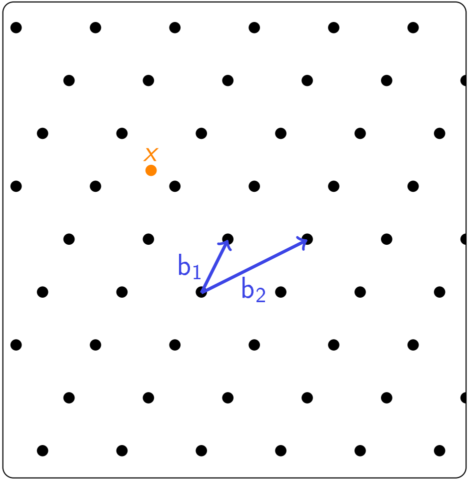
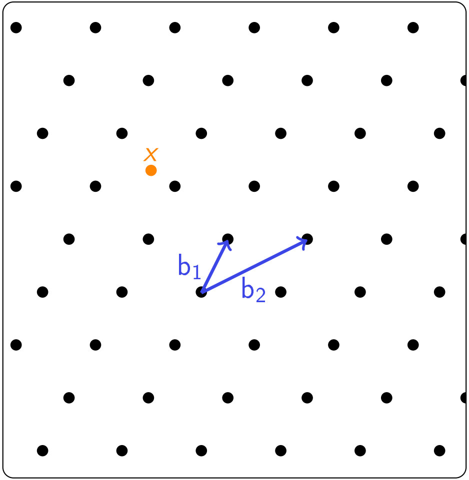
Hash-and-sign signature



Decoding in a lattice using a short basis



Input: $x = 3.7 \cdot \mathrm{b}_1 - 1.4 \cdot \mathrm{b}_2$



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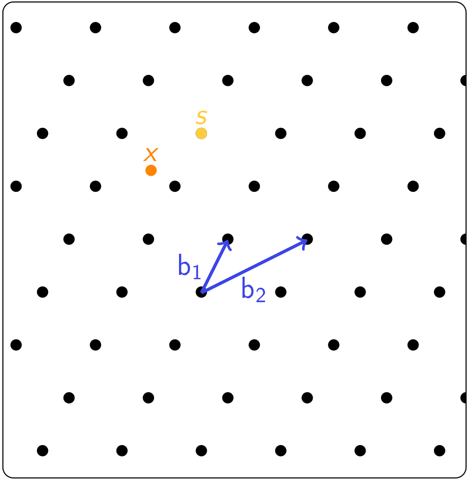

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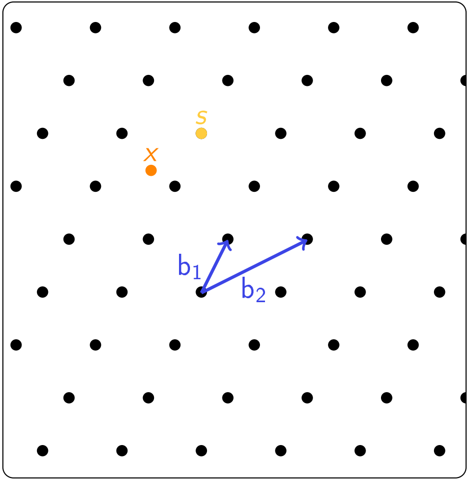

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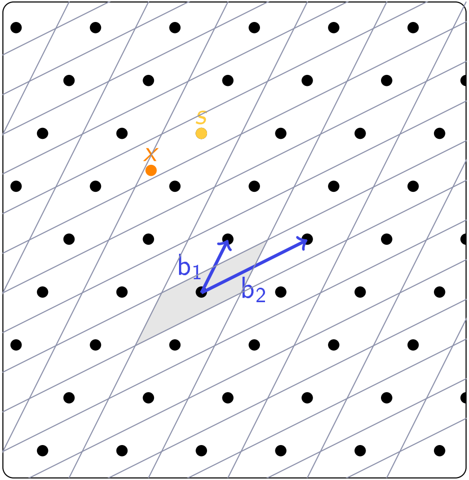

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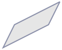
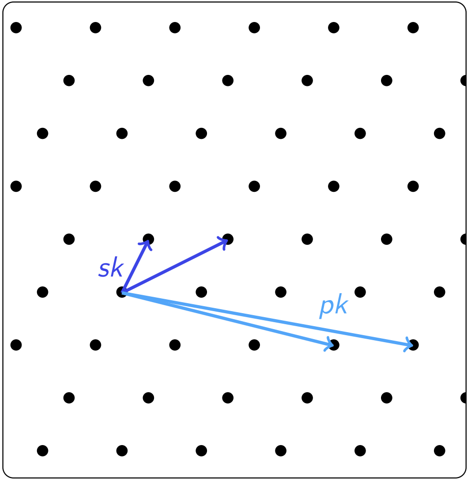
 $= \left\{ x_1 b_1 + x_2 b_2 \,\middle|\, |x_i| \leq \frac{1}{2} \right\}$

Hash-and-sign: first idea [GGH97]



KeyGen:

- pk = bad basis of $\mathcal{L}$
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[GGH97] Goldreich, Goldwasser, and Halevi. Public-key cryptosystems from lattice reduction problems. CRYPTO

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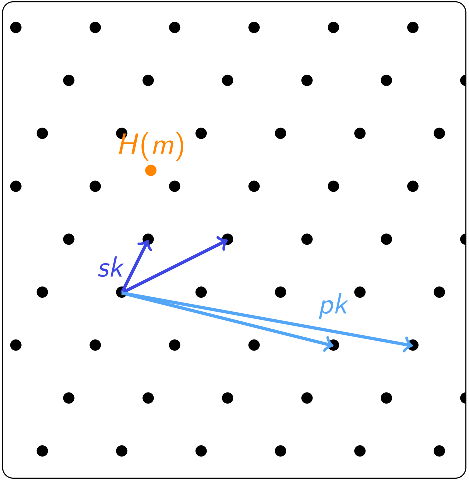

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Sign(m, sk):

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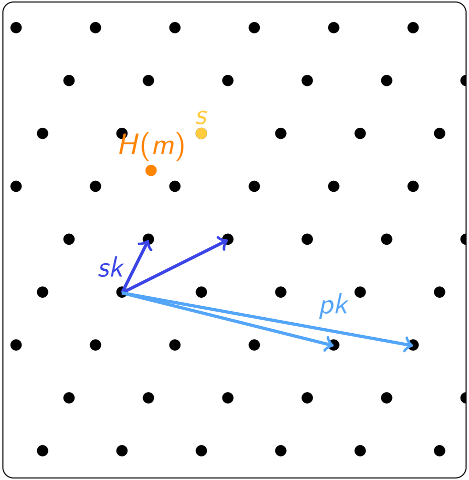

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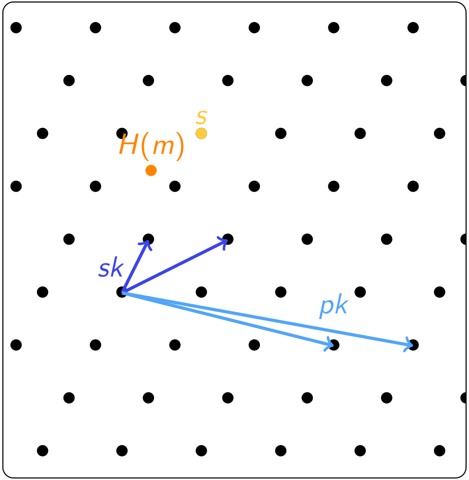

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Verify(s, pk):

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

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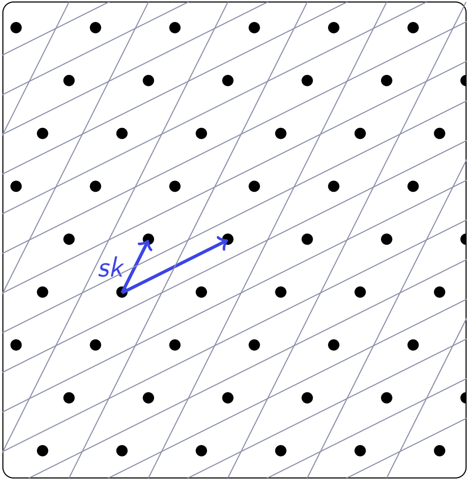
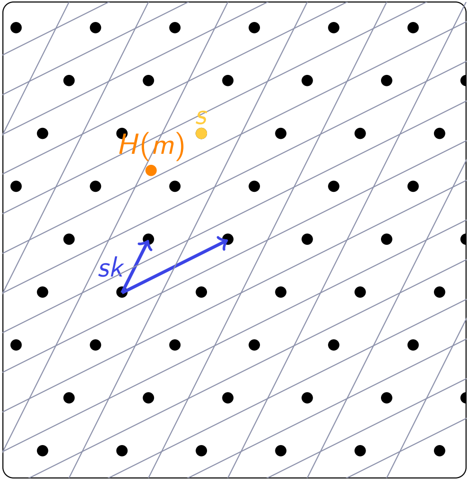


Parallelepiped attack:

[NR06] Nguyen and Regev. Learning a parallelepiped: Cryptanalysis of GGH and NTRU signatures. J. Cryptology

Attack on this first idea [NR06]



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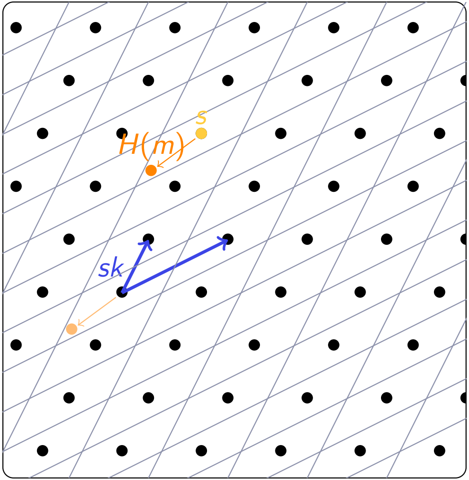

Parallelepiped attack:

- ask for a signature s on m
- plot $H(m) - s$

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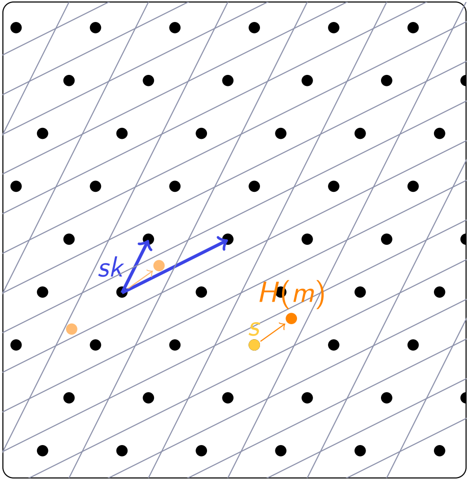

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- plot $H(m) - s$
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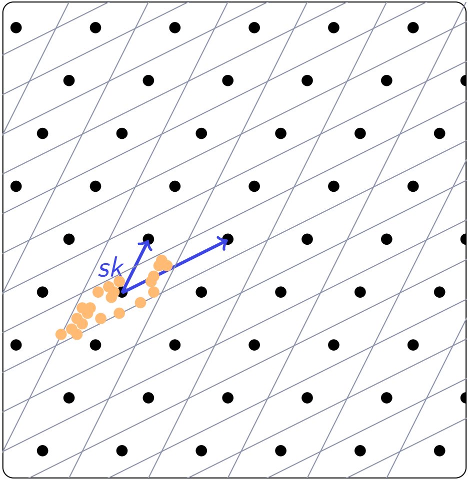

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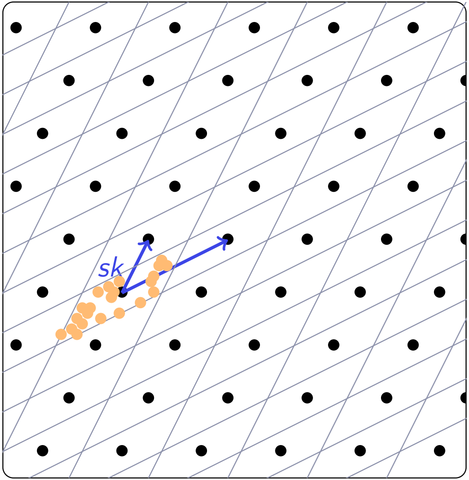

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From the shape of the parallelepiped, one can recover the short basis

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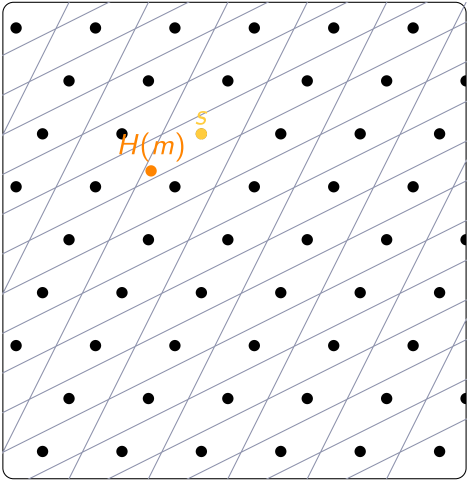
Preventing the attack [GPV08]



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[GPV08] Gentry, Peikert, and Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC.

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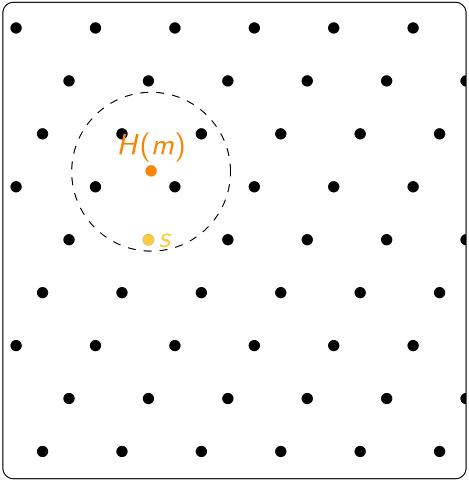

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Sign(m, sk):

- $x = H(m)$ (hash the message)
- sample $s \in \mathcal{L} \cap \mathcal{B}_r(x)$ (small radius r)

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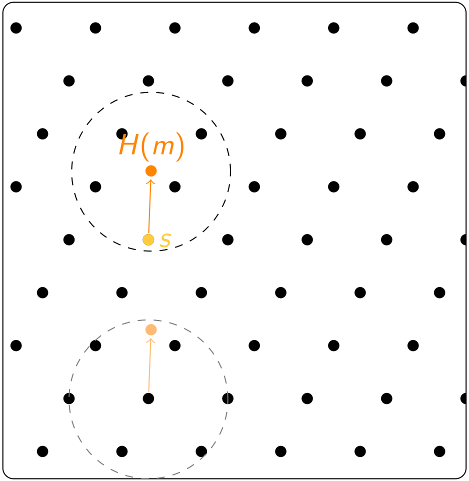

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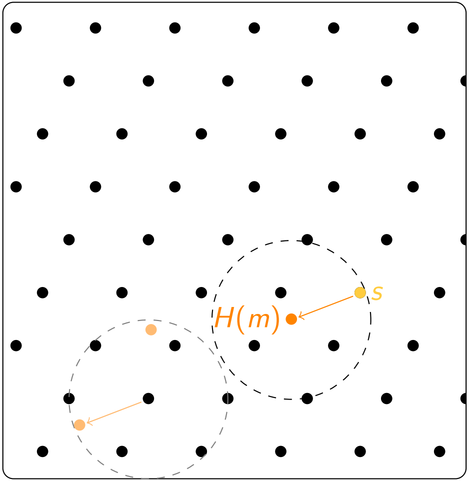

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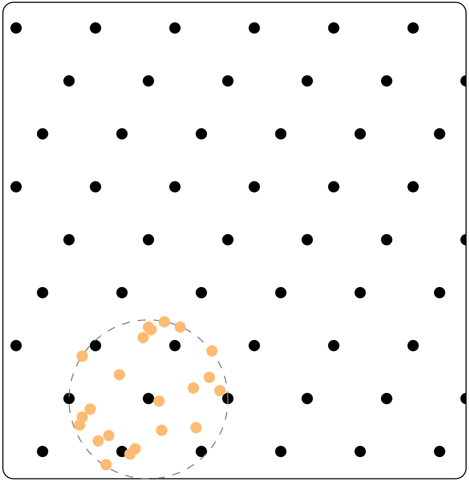

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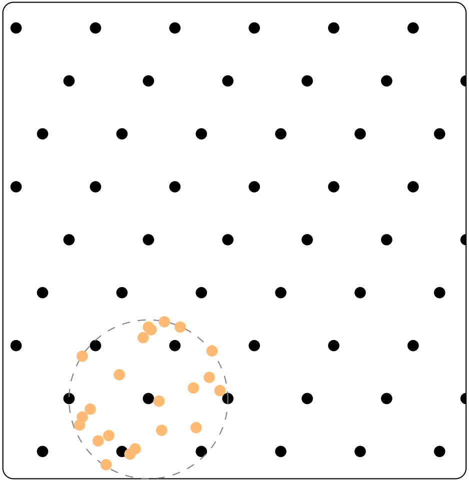

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Lemma: if an adversary can forge signatures, then she can recover a short basis of $\mathcal{L}$ using only pk (in the ROM)

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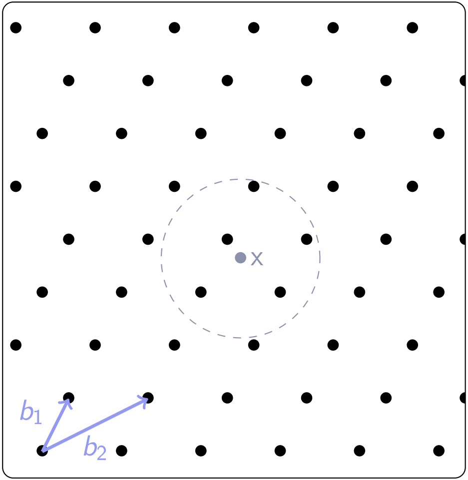
Sampling uniformly in a ball [PP21]

Input: center x, radius r

(and a short basis $(b_1, \ldots, b_n)$)

Output: $s \leftarrow \mathcal{U}\left(\mathcal{L} \cap \mathcal{B}_r(x)\right)$



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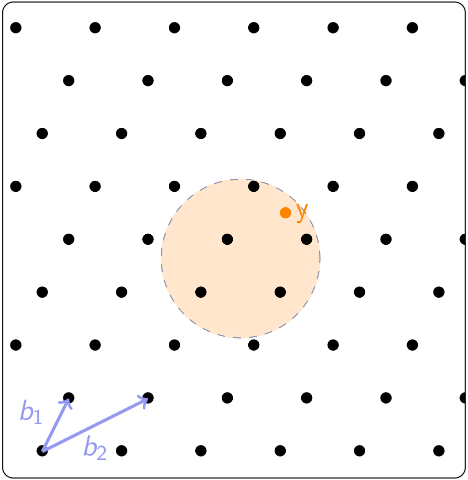

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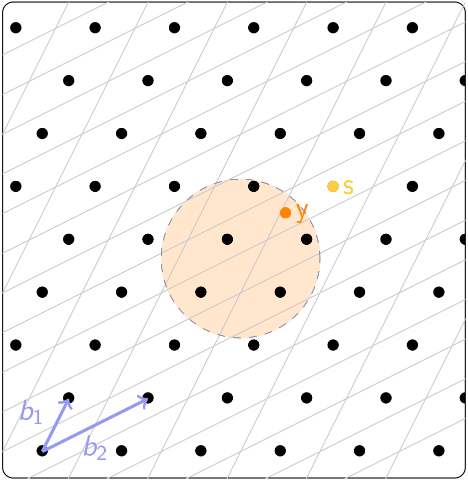

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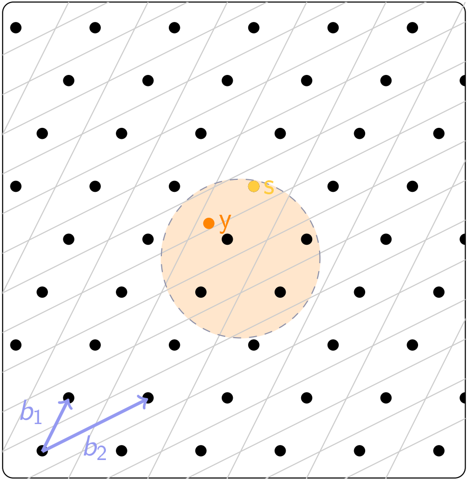

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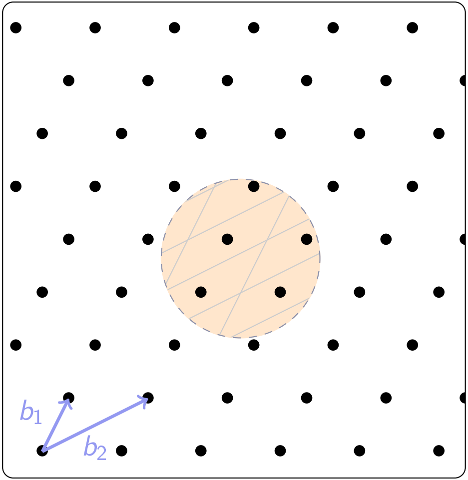

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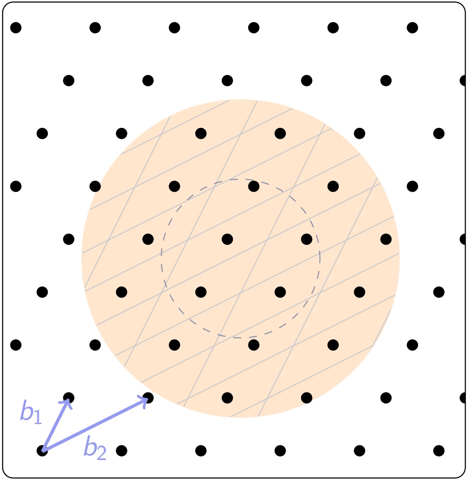

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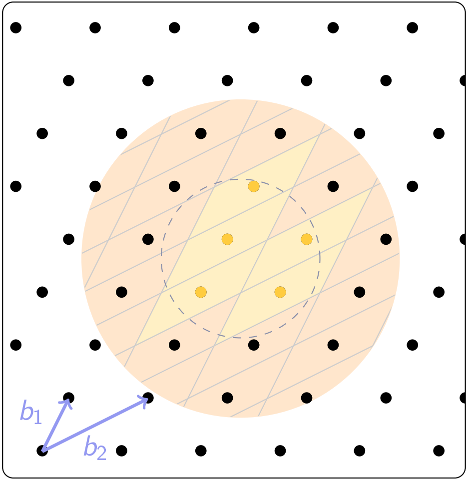

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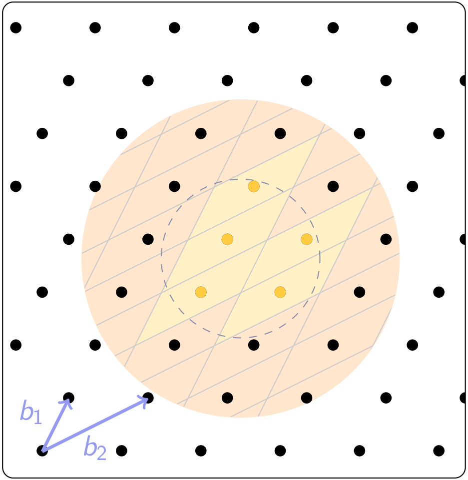
polynomial time if

$r \geq 2n^2 \cdot \max_i \|b_i\|$



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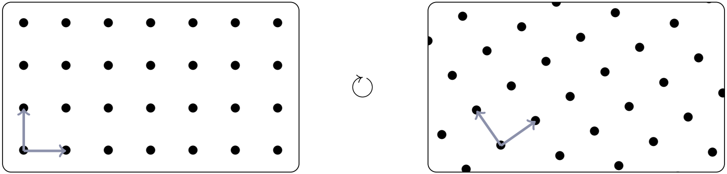
Summary

Hash-and-sign signature scheme:

- requires a lattice $\mathcal{L}$ + a short basis B_s + a bad basis B_p;
- provably secure if recovering a short basis from B_p is hard.

How to generate a hard lattice?



Objective

What we want: An algorithm `KeyGen` such that

- `KeyGen` computes
  - a random lattice $\mathcal{L}$
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  - a bad basis B_p of $\mathcal{L}$ (pk)
- computing a short basis of $\mathcal{L}$ from B_p is hard with overwhelming probability

Worst-possible basis

There is a basis B_0 of $\mathcal{L}$ that can be computed in poly time from any other basis B

$\Rightarrow$ B_0 is a worst possible basis

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Input: any basis B of $\mathcal{L}$

Algorithm:

- Compute LLL-reduced basis $C = (c_1, \ldots, c_n)$
  - poly time
  - $\max_i \|c_i\| \leq 2^n \cdot \min_{C'} \max_i \|c'_i\|$ (C' ranging over all bases of $\mathcal{L}$)

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- extract a basis B_0 from the v_j's
  - linear algebra $\Rightarrow$ poly time

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There is a random basis B_0 of $\mathcal{L}$ that can be computed in poly time from any other basis B

$\Rightarrow$ B_0 is a worst possible distribution over bases

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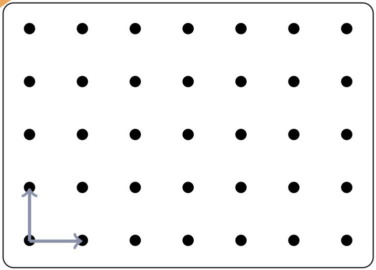
Using lattice isomorphism [DW22,BGPS23]



$\mathcal{L}_0 = \mathbb{Z}^n$

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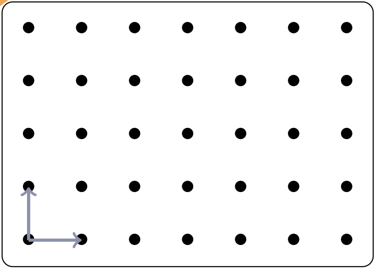

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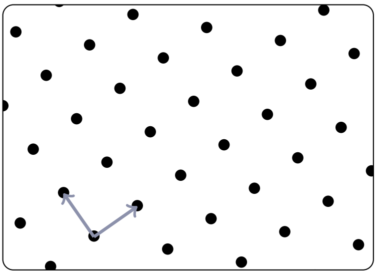
rotate
$\longrightarrow$
(choose O orthogonal matrix)



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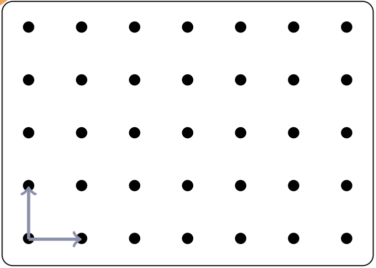

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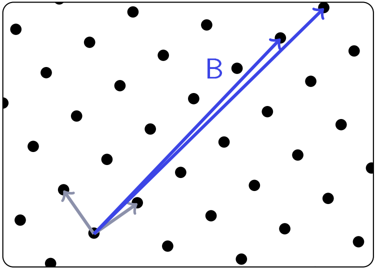

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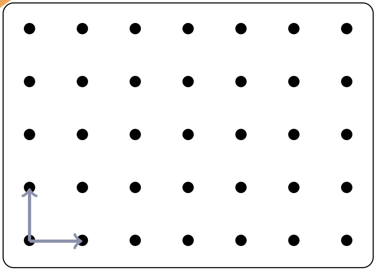

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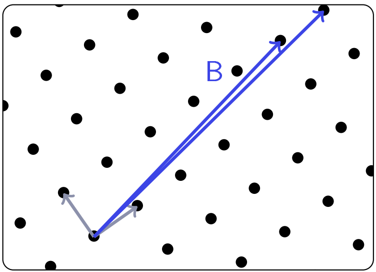

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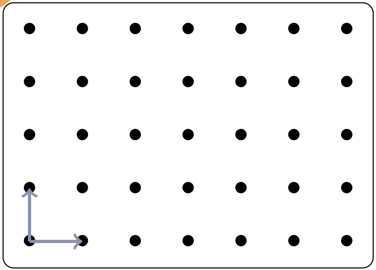

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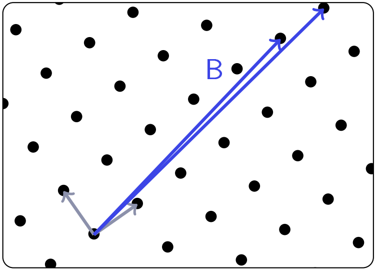

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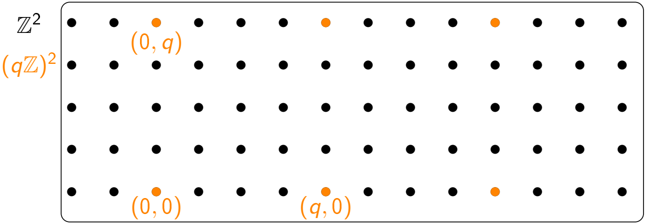
- Hawk: hash-and-sign + (module) LIP [DPPW23]

[DPPW23] Ducas, Postlethwaite, Pulles, van Woerden. Hawk: Module LIP makes lattice signatures [...] Asiacrypt

NTRU [HPS98]



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[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

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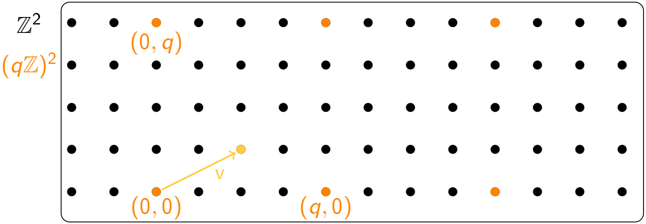

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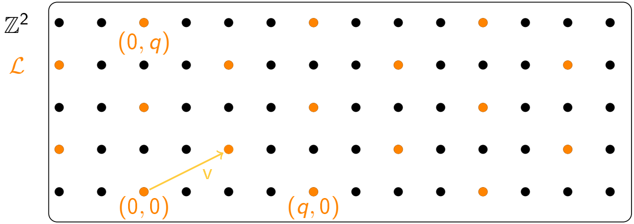

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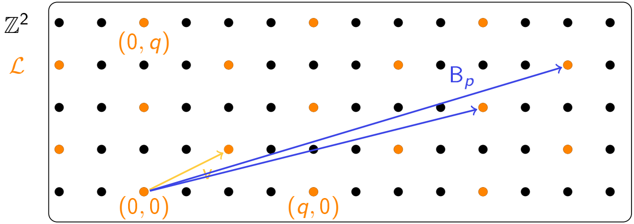

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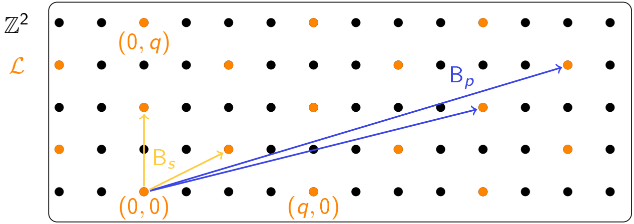

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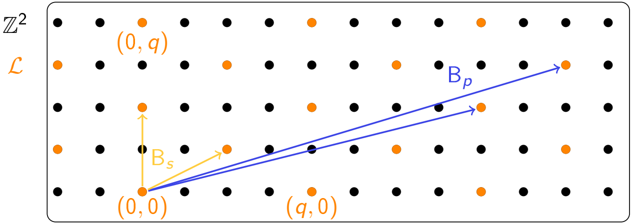

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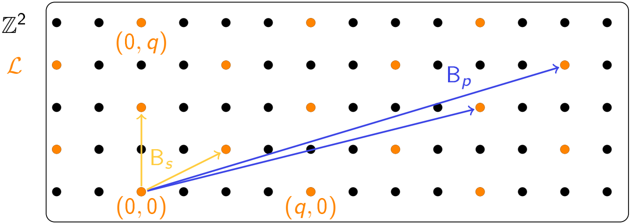

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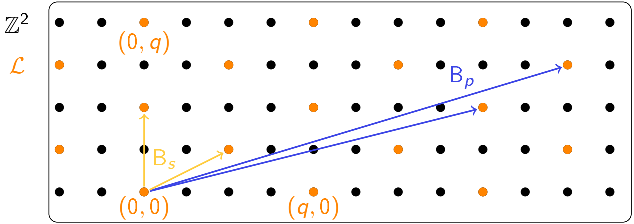

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- Falcon: hash-and-sign + NTRU

[Falcon] Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Prest, Ricosset, Seiler, Whyte, Zhang. NIST standard

Short Integer Solution [Ajt96]

Short Integer Solution (SIS) assumption

Let $A \leftarrow \mathcal{U}(\mathbb{Z}_q^{m \times n})$ $(m > n \log q)$ and

$$\mathcal{L}(A) := \{x \in \mathbb{Z}^m \,|\, xA = 0 \bmod q\}.$$

Finding a short basis of $\mathcal{L}(A)$ is hard with overwhelming probability.

[Ajt96] Ajtai. Generating hard instances of lattice problems. STOC.

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Lemma: if there exists one lattice for which the short basis problem is hard, then the SIS assumption holds. [Ajt96]

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Short Integer Solution (SIS) assumption

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[Ajt99] Ajtai. Generating hard instances of the short basis problem. ICALP.

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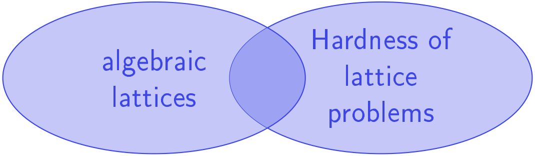
- GPV: hash-and-sign + SIS [GPV08]

[GPV08] Gentry, Peikert, Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC

Conclusion

Some questions that interest me



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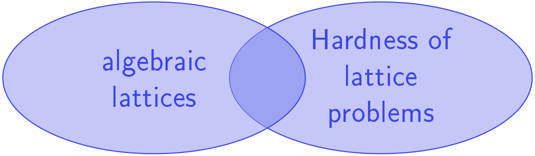

Some concrete questions: (come ask me if you want to know more)

- can we generate a random prime ideal $\mathfrak{p}$ in a number field K together with a short element in it?
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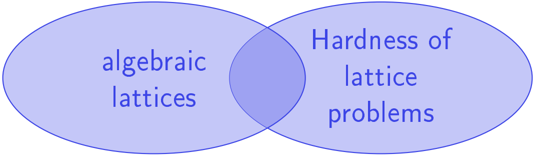

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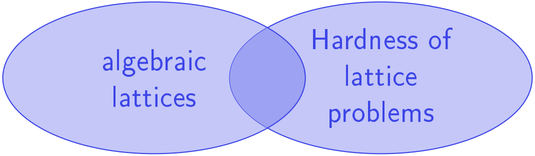

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Thank you