Signing with higher dimensional isogenies

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Joint work with Antonin Leroux, Damien Robert and Benjamin Wesolowski
Acknowledgements to Luca De Feo

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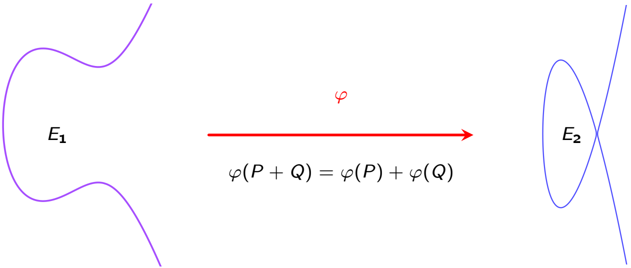
CANARI

The Deuring correspondence

Isogenies



The Deuring correspondence

Supersingular elliptic curves	Quaternions
$j(E)$ or $j(E)^p$ supersingular	$\mathcal{O} \cong \mathrm{End}(E)$ maximal order in $\mathcal{B}_{p,\infty}$
$\varphi : E \longrightarrow E'$	left $\mathcal{O}$-ideal and right $\mathcal{O}'$-ideal I_φ
$\varphi, \psi : E \longrightarrow E'$	$I_\varphi \sim I_\psi$ $(I_\psi = I_\varphi \alpha)$
$\widehat{\varphi}$	$\overline{I_\varphi}$
$\varphi \circ \psi$	$I_\psi \cdot I_\varphi$
$\theta \in \mathrm{End}(E)$	Principal ideal $\mathcal{O}\theta$
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi)$

Computing isogenies via the Deuring correspondence

- Let E_1 and E_2 of known endomorphism rings $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$.
- Compute a connecting ideal I between $\mathcal{O}_1$ and $\mathcal{O}_2$.
- Compute $J \sim I$ of smooth norm via [KLPT14].
- Translate J into an isogeny $\varphi_J : E_1 \longrightarrow E_2$.

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✗ Slow because of the red steps.

Effective Deuring correspondence and higher dimensional isogenies

Kani's embedding lemma

Theorem (Robert, 2022)

Let $\sigma : E_1 \longrightarrow E_2$ such that $\deg(\sigma) + a_1^2 + a_2^2 = \ell^e$. Then:

- *$\sigma : E_1 \longrightarrow E_2$ can be represented in dimension 4 by the ℓ^e-isogeny:*

$$F := \begin{pmatrix} a_1 & a_2 & \widehat{\sigma} & 0 \\ -a_2 & a_1 & 0 & \widehat{\sigma} \\ -\sigma & 0 & a_1 & -a_2 \\ 0 & -\sigma & a_2 & a_1 \end{pmatrix} \in \mathrm{End}(E_1^2 \times E_2^2).$$

- *F can be computed by evaluating σ on $E_1[\ell^e]$.*

Kani's embedding lemma

Corollary (Robert, 2022)

Let $\sigma : E_1 \longrightarrow E_2$ of degree $q < \ell^e$ such that $\ell^e - q$ is a prime $\equiv 1 \mod 4$. There exists a polynomial time algorithm with:

- ***Input:*** $(\sigma(P_1), \sigma(P_2))$, *where* (P_1, P_2) *is a basis of* $E_1[\ell^e]$ *and* $Q \in E_1(\mathbb{F}_{p^2})$.
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Context: This idea comes from the attacks against SIDH [CD23; MM22; Rob23].

A new algorithm for effective Deuring correspondence

Problem: Given $\phi : E_1 \longrightarrow E_2$, I_ϕ, $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$ (secret), find another path $\sigma : E_1 \longrightarrow E_2$.

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- Compute $I \sim I_\phi$ random of smooth norm $\simeq p^{15/4}$ via [KLPT14].

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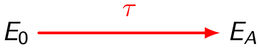
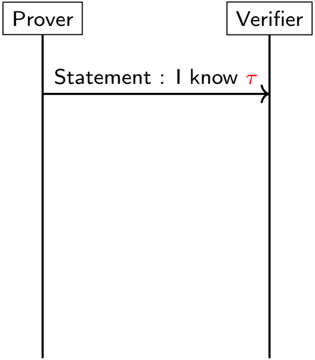
- Evaluate $\sigma : E_1 \longrightarrow E_2$ associated to I on $E_1[\ell^e]$, using ϕ.
- $(q, \sigma(E_1[\ell^e]))$, is sufficient to represent σ.
- We can then compute $F \in \mathrm{End}(E_1^2 \times E_2^2)$ embedding σ.

SQIsignHD

The SQIsignHD identification scheme [DLRW23]





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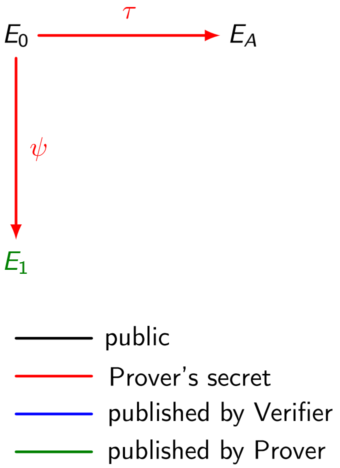

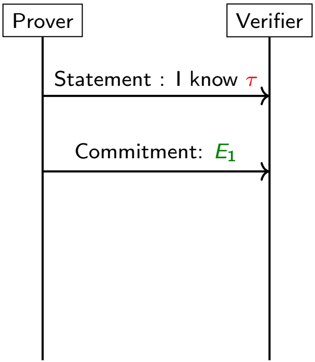

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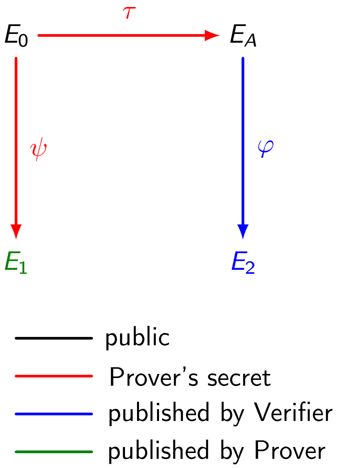

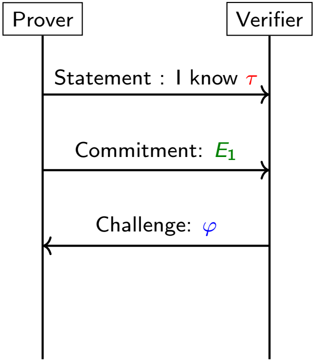

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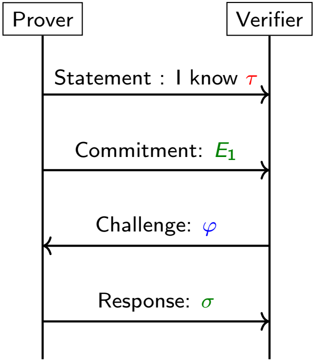

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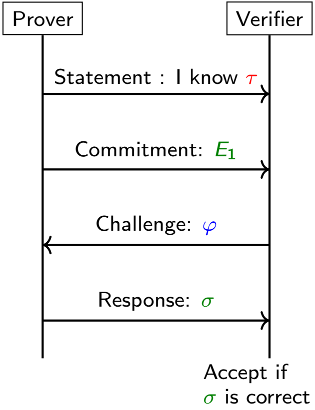

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Proof of concept.
850 ms in `sagemath`.

Comparison of SQIsignHD with SQIsign

	SQIsign	SQIsignHD
Security	✗ Ad-hoc heuristic: • Distribution of σ.	✓ Simpler heuristics: • Oracle (RUGDIO); • Distribution of E_1.
Signing time	✗ 400 ms for NIST-1	✓ 28 ms for NIST-1
Signature size	✓ 204 bytes for NIST-1	✓ 109 bytes for NIST-1
Verification	✓ Fast (6 ms for NIST-1)	✗ 850 ms for NIST-1 in `sagemath`

Thank you for listening.

Find our pre-print here: https://eprint.iacr.org/2023/436