The endomorphism ring problem given an endomorphism

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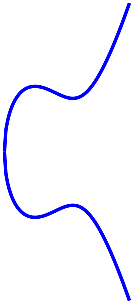
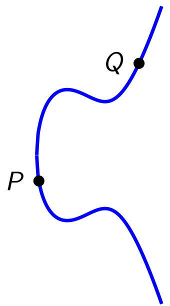


$E : y^2 = x^3 + Ax + B$

- **Elliptic curve:** smooth projective curve given by an affine model such as above.



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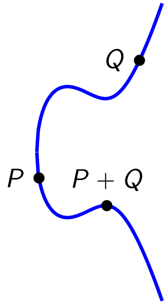

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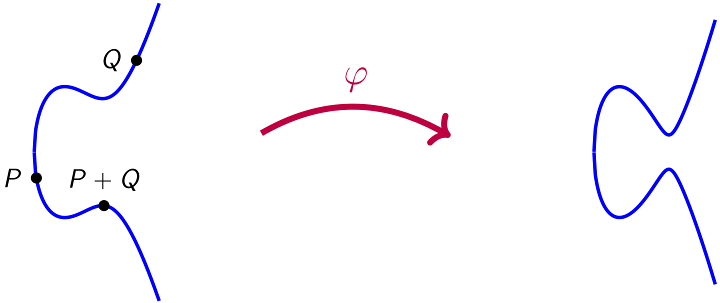

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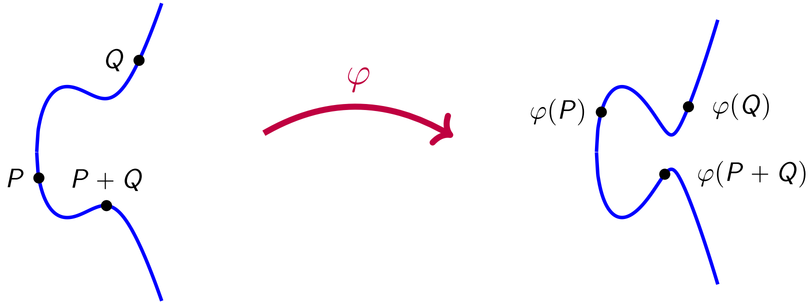

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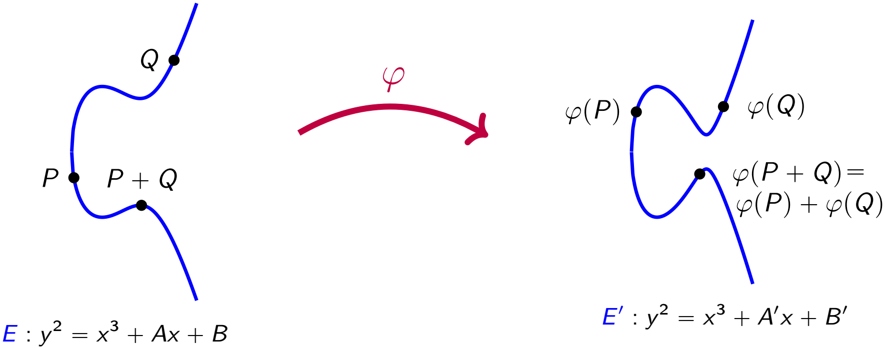
Elliptic curves and isogenies



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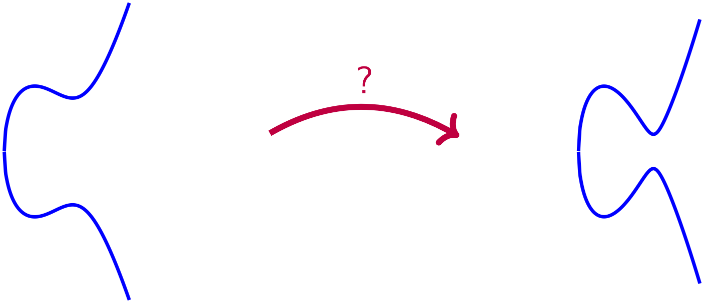
Isogeny hard problem



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The Isogeny Problem

Given two elliptic curves E and E', find an isogeny between them.

Endomorphism ring

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We denote $\mathrm{End}(E) := \{\varphi : E \to E\} \cup \{0\}$.

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- $(\mathrm{End}(E), +, \circ)$ is the **endomorphism ring** of E, where for every $P \in E$:

$$(\varphi + \psi)(P) = \varphi(P) + \psi(P) \quad \text{and} \quad (\varphi \circ \psi)(P) = \varphi(\psi(P)).$$

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- $\mathbb{Z} \hookrightarrow \mathrm{End}(E)$ as **subring**. For every $n \in \mathbb{Z}$, we have the endomorphism

$$\begin{aligned} [n] : E &\to E \\ P &\mapsto [n]P := \underbrace{P + \cdots + P}_{n \text{ times}}. \end{aligned}$$

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- $(\mathrm{End}(E), +)$ is a lattice of dimension
  - **2** then $\mathrm{End}(E) \simeq \mathbb{Z} \oplus \alpha\mathbb{Z}$,
    or
  - **4** then $\mathrm{End}(E) \simeq \mathbb{Z} \oplus \alpha\mathbb{Z} \oplus \beta\mathbb{Z} \oplus \gamma\mathbb{Z}$.

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- $(\mathrm{End}(E), +)$ is a lattice of dimension
  - **2** then $\mathrm{End}(E) \simeq \mathbb{Z} \oplus \alpha\mathbb{Z}$, $\}$ E is **ordinary**.

    or
  - **4** then $\mathrm{End}(E) \simeq \mathbb{Z} \oplus \alpha\mathbb{Z} \oplus \beta\mathbb{Z} \oplus \gamma\mathbb{Z}$. $\}$ E is **supersingular**.

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The Endomorphism Ring Problem given one Endomorphism :

Given a supersingular elliptic curve $\boldsymbol{E}$ and an endomorphism $\boldsymbol{\theta} \in \mathrm{End}(E) \setminus \mathbb{Z}$, find a basis of its endomorphism ring $\mathbf{End}(\boldsymbol{E})$.

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	`EndRing`	`EndRing given one endomorphism` θ
Classic	$p^{1/2}$	$(\deg\theta)^{1/4}$
Quantum	$p^{1/4}$	subexponential in $\log\deg\theta$

Complexity of `EndRing` and its variant for an elliptic curve defined over $\mathbb{F}_{p^2}$, with p a prime.

Orientations [CK20],[Onu20]

Let $\theta \in \mathrm{End}(E) \backslash \mathbb{Z}$.

- $\mathbb{Z}[\theta] \simeq \mathbb{Z}[X] / < X^2 + aX + b >$ for some $a, b \in \mathbb{Z}$, i.e. $\mathbb{Z}[\theta]$ is a **quadratic ring**.
- $\mathbb{Z}[\theta] \hookrightarrow \mathrm{End}(E)$.

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Let $\mathfrak{O}$ be a quadratic ring.

- An embedding $\iota : \mathfrak{O} \hookrightarrow \mathrm{End}(E)$ is called an $\mathfrak{O}$-**orientation**.
- It is a **primitive** $\mathfrak{O}$-orientation if for any quadratic ring $\mathfrak{O}' \supseteq \mathfrak{O}$, it is impossible to extend ι to $\mathfrak{O}'$ such that $\iota : \mathfrak{O}' \hookrightarrow \mathrm{End}(E)$.
- The ideal class group $\mathcal{C}l(\mathfrak{O})$ **acts** on elliptic curves endowed with an $\mathfrak{O}$-orientation.

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The $\mathfrak{O}$-oriented Endomorphism Ring Problem ($\mathfrak{O}$-EndRing):

Given a supersingular elliptic curve $\boldsymbol{E}$ and a **primitive orientation** $\boldsymbol{\iota} : \mathfrak{O} \hookrightarrow \mathrm{End}(E)$, find a basis of its endomorphism ring $\mathbf{End}\,(\boldsymbol{E})$.

`EndRing` given an endomorphism

$\boxed{E \text{ and } \theta \in \mathrm{End}(E)\backslash\mathbb{Z}}$

$\boxed{\text{A basis of } \mathrm{End}(E)}$

EndRing given an endomorphism

$$\begin{tikzcd} \boxed{E \text{ and } \theta \in \mathrm{End}(E) \backslash \mathbb{Z}} \arrow[d, "1"'] \\ \boxed{E \text{ and } \iota : \mathbb{Z}[\theta] \hookrightarrow \mathrm{End}(E)} \end{tikzcd}$$

$$\boxed{\text{A basis of } \mathrm{End}(E)}$$

① Immediate.

EndRing given an endomorphism

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\boxed{E \text{ and } \iota : \mathbb{Z}[\theta] \hookrightarrow \mathrm{End}(E) \textbf{ primitive}} \\
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② Hard problem with a subexponential quantum complexity. [Arp+23]

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Polynomial time given the factorisation of the discriminant of $\mathbb{Z}[\theta]$.

EndRing given an endomorphism

E and $\theta \in \mathrm{End}(E) \backslash \mathbb{Z}$

$\downarrow$ 1

E and $\iota : \mathbb{Z}[\theta] \hookrightarrow \mathrm{End}(E)$

$\downarrow$ 2 `Primitivisation`

E and $\iota : \mathbb{Z}[\theta] \hookrightarrow \mathrm{End}(E)$ **primitive**

$\downarrow$ 3 $\mathfrak{O}$-`EndRing`

A basis of $\mathrm{End}(E)$

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EndRing given an endomorphism

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\boxed{E \text{ and } \iota : \mathbb{Z}[\theta] \hookrightarrow \mathrm{End}(E) \ \textbf{primitive}} \arrow[d, "3\ \ \mathfrak{O}\text{-}\texttt{EndRing}"'] \\
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Rigorous complexity analysis.

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Thanks for your attention!

https://eprint.iacr.org/2023/1448

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