Propagation of Subspaces in Primitives with Monomial Sboxes

Applications to Rescue and Variants of the AES

Aurélien Boeuf[1], Anne Canteaut[1], Léo Perrin[1]

[1]Inria Paris

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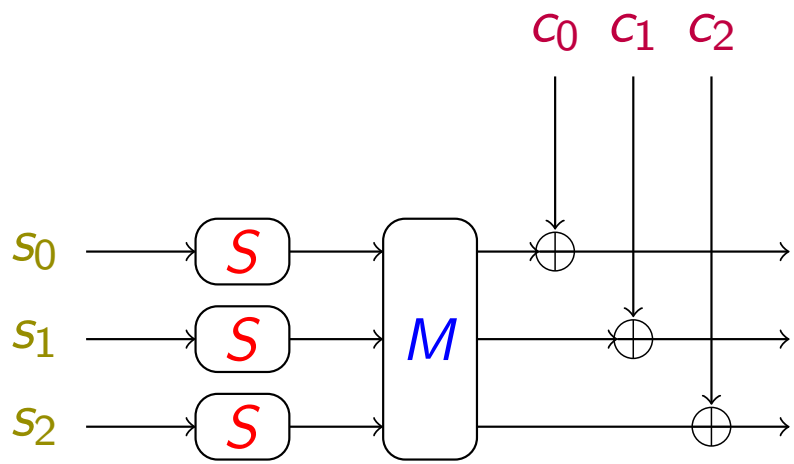
Which Round Functions?

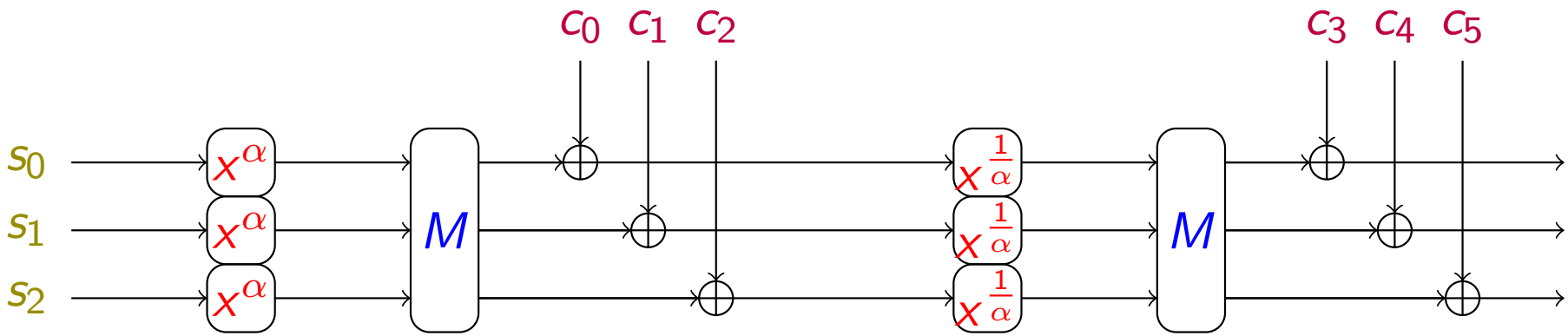
- $s_i \in \mathbb{F}_q$ (finite field of size q).



The round function of an SPN (Substitution-Permutation Network) Block Cipher. Design basis for the AES, very popular.

Rescue [AABDS'20]

- Defined in $\mathbb{F}_p$ with p prime $\simeq 2^{64}$ (unusually big!).



2 rounds of Rescue (repeated $N \approx 10$ times).

- Defined for any MDS matrix M and round constants c_i.

Differential Uniformity

Definition

Differential uniformity of a function F:

$$\delta(F) = \max_{\sigma \neq 0, \beta} |\{F(x+\sigma) - F(x) = \beta \text{ s.t. } x \in (\mathbb{F}_p)^m\}|$$

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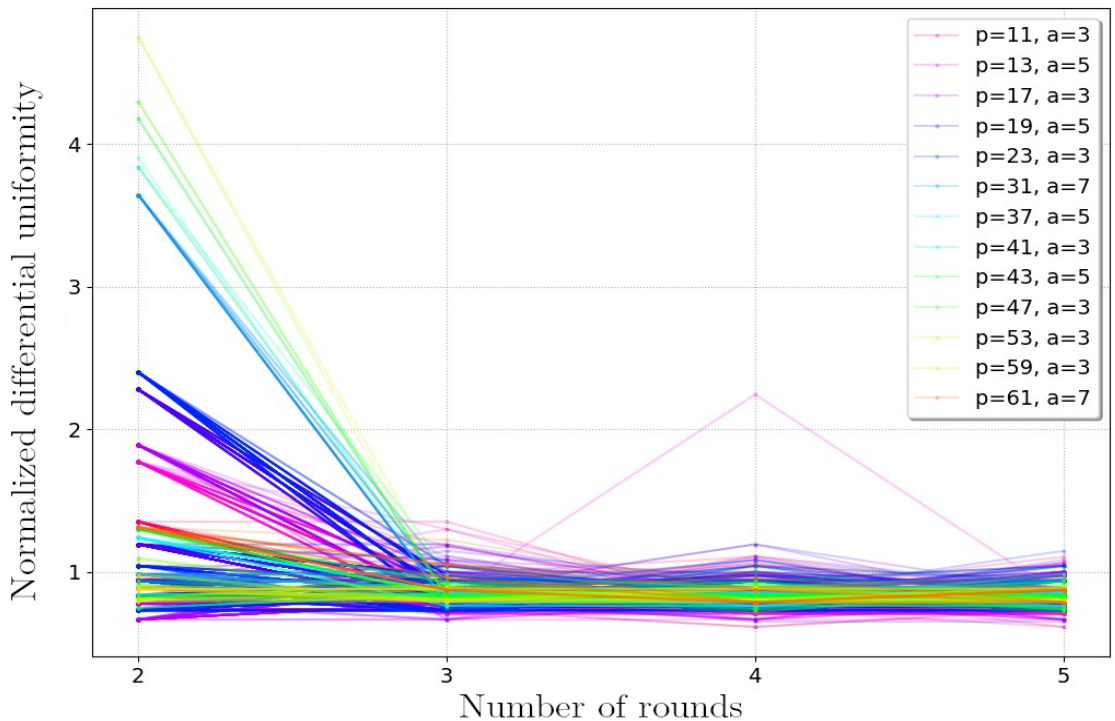
$\rightarrow$ This quantity must be minimized.

High Differential Uniformities in Rescue



Graph taken from `eprint.iacr.org/2020/820`.

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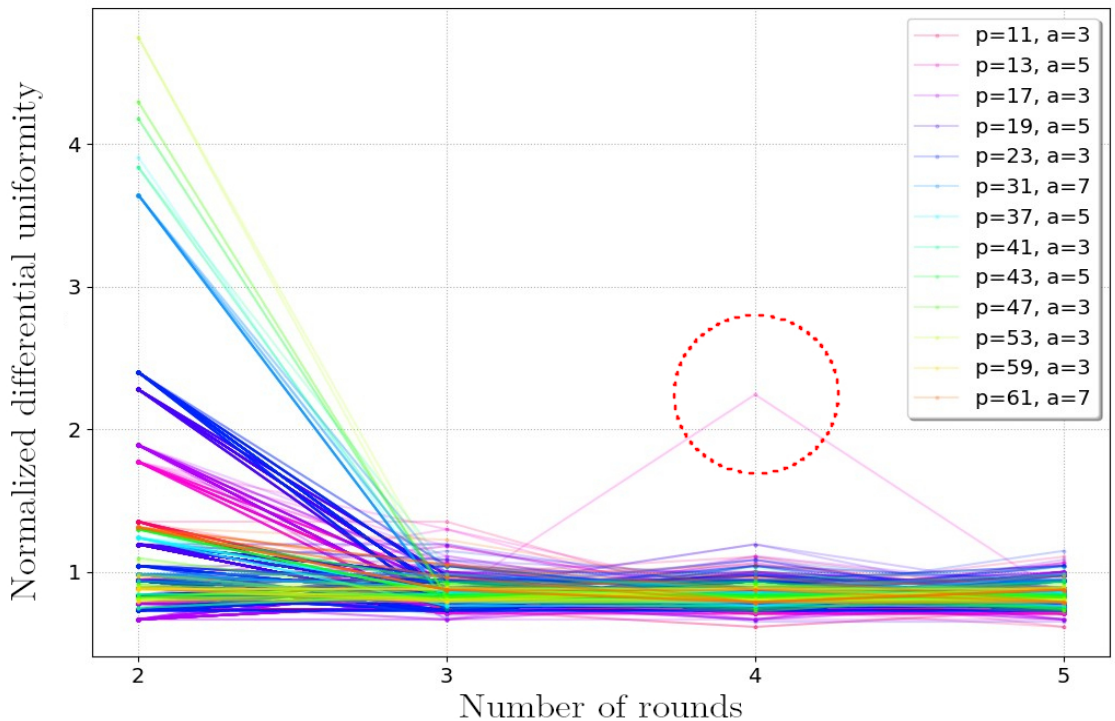

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High Differential Uniformities in Rescue

The cause? Affine spaces of dimension 1 nicely mapping from one to another.

$$\begin{pmatrix} z \\ X \end{pmatrix} \xrightarrow{\text{2 rounds}} \begin{pmatrix} aX + b \\ cX + d \end{pmatrix} \xrightarrow{\text{2 rounds}} \begin{pmatrix} eX + f \\ gX + h \end{pmatrix}$$

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- 1 round or 3 rounds: the function is not affine.
- Because p is big ($\geq 2^{64}$), affine spaces of dim 1 are also big.

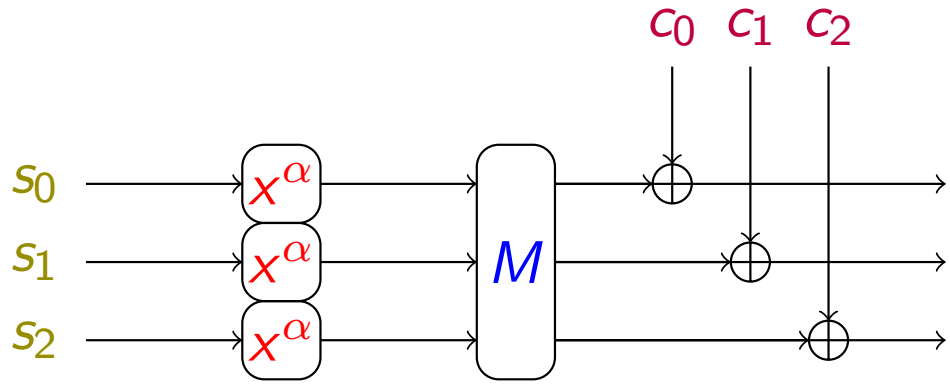
Structure of our work

Affine Space Chains

Note $\boldsymbol{a} + \left\langle \boldsymbol{v} \right\rangle := \{\boldsymbol{a} + X\boldsymbol{v} \text{ such that } X \in \mathbb{F}_p\}$.

$$\boldsymbol{a}_0 + \langle \boldsymbol{v}_0 \rangle \xrightarrow{f} \boldsymbol{a}_1 + \langle \boldsymbol{v}_1 \rangle \xrightarrow{f} \ldots \xrightarrow{f} \boldsymbol{a}_N + \langle \boldsymbol{v}_N \rangle$$

Main Observation



Rescue round.

Write elements of $\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ v \\ 0 \end{pmatrix} \right\rangle$ as $\begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} X \\ vX \\ a \end{pmatrix}$.

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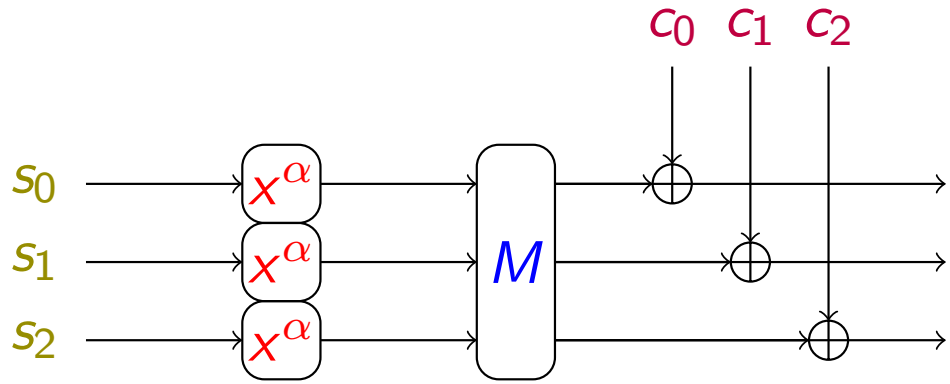

Rescue round.

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} X \\ vX \\ a \end{pmatrix} \longrightarrow \begin{pmatrix} X^\alpha \\ v^\alpha X^\alpha \\ a^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a^\alpha \end{pmatrix} + X^\alpha \begin{pmatrix} 1 \\ v^\alpha \\ 0 \end{pmatrix}$$

This is the most important part! It only relies on the fact that the Sbox is a monomial.

Separable Affine Spaces

Definition

An affine space of dimension 1 is separable if and only if there exists a representation of it denoted $\boldsymbol{a} + \langle \boldsymbol{v} \rangle$ such that:

$$\forall\, 1 \leq i \leq m\,,\ \ a_i \cdot v_i = 0\ .$$

or, equivalently, $\mathrm{supp}(\boldsymbol{v}) \cap \mathrm{supp}(\boldsymbol{a}) = \emptyset$.

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- $\begin{pmatrix} a \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ b \end{pmatrix} \right\rangle$ is a separable affine space for all a and b.

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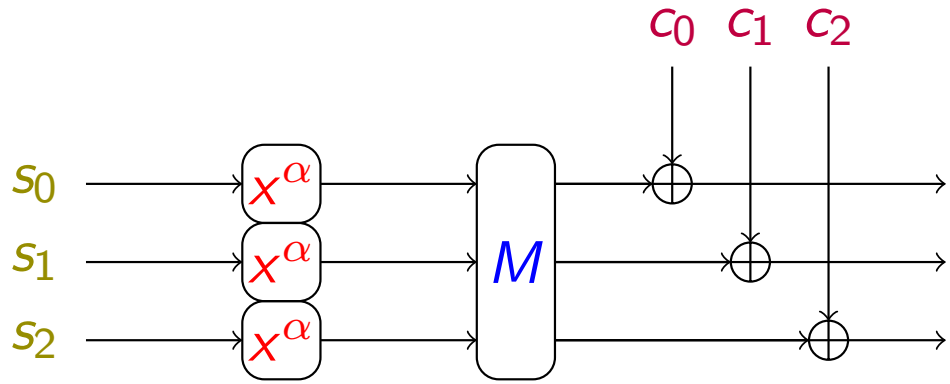
- $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$ is not.

Main Observation



Rescue round.

$$\begin{pmatrix} 0 \\ 0 \\ a^\alpha \end{pmatrix} + x^\alpha \begin{pmatrix} 1 \\ v^\alpha \\ 0 \end{pmatrix} \longrightarrow M \begin{pmatrix} 0 \\ 0 \\ a^\alpha \end{pmatrix} + x^\alpha M \begin{pmatrix} 1 \\ v^\alpha \\ 0 \end{pmatrix}$$

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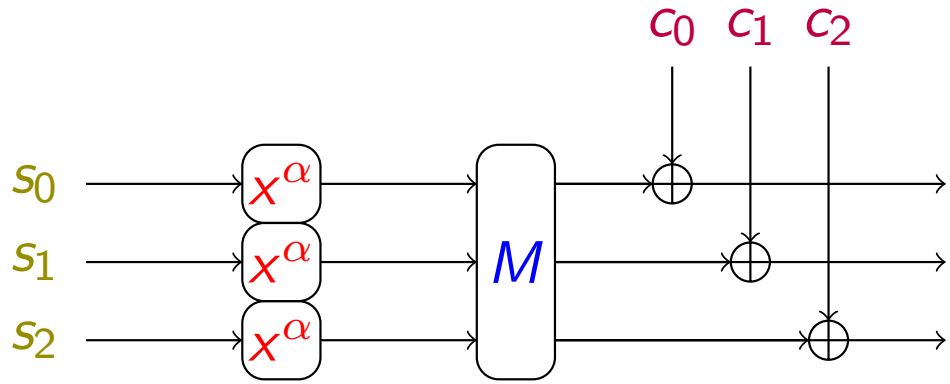

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For this space to be separable, we need that there exists $\lambda \in \mathbb{F}_p$ such that

$$M \begin{pmatrix} 1 \\ v^\alpha \\ 0 \end{pmatrix} \text{ and } M \begin{pmatrix} 0 \\ 0 \\ a^\alpha \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} + \lambda M \begin{pmatrix} 1 \\ v^\alpha \\ 0 \end{pmatrix}$$

have disjoint supports. □

Main Result

Theorem

The image of a separable affine space $\boldsymbol{a} + \langle \boldsymbol{v} \rangle$ *by a round of a monomial SPN is an affine space. Also, the image is still separable if and only if there exists* λ *in* $\mathbb{F}_p$ *such that:*

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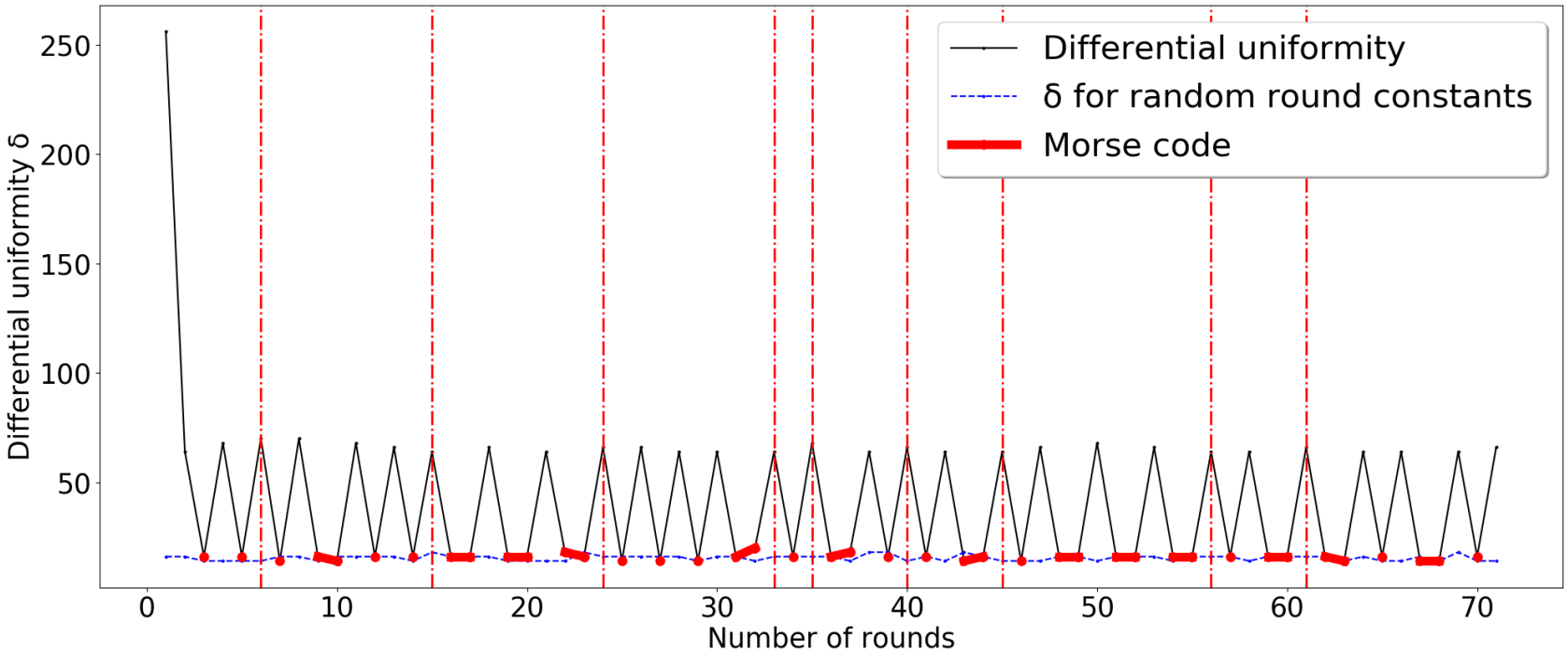
$$\forall i \in \mathrm{supp}(M \circ S)(\boldsymbol{v}),$$

$$c_i = \lambda(M \circ S)(\boldsymbol{v})_i - (M \circ S)(\boldsymbol{a})_i$$

Morse Code with Differential Uniformity



This differential uniformity graph spells "`..  .-..  ---  ...-  .  -. .-  .---  .-  -.-.`" (ILOVENAJAC) over 71 rounds ($m = 2$, $\mathbb{F}_{2^6}$).

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- Bad choice of round constants may lead to affine space chains or high differential uniformities.
- It's possible to define "backdoored" primitives that enforce this kind of behaviour.
- Such weak designs satisfy state-of-the art security arguments (APN Sbox, MDS matrix, wide-trail strategy...). Usual security arguments are not sufficient in the AO context.
- Look out for similar algebraic patterns in AO primitives; they can improve algebraic attacks.

Thank you for listening!

Questions?

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$\rightarrow \delta(F) \geq p$

Arithmetization-Oriented Symmetric Primitives

- Term coined for the first time in a 2020 paper from Aly et al.
- Symmetric primitives with a "simple" arithmetic description.
- Minimize verification cost in Zero-Knowledge schemes and other advanced protocols.
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Example

Primitives using the nonlinear component $S : x \mapsto x^3$ (Mimc and variants, Rescue...).

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Main motivation: Are the usual security arguments sufficient?

Our Designs

- Stir, a weak instance of Rescue.

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- Stir, a weak instance of Rescue.
- ~~Snare, a tweakable cipher with a secret weak tweak. Directly based on the MALICIOUS framework[1].~~
- AES-like ciphers where we can introduce and control differential uniformity spikes.

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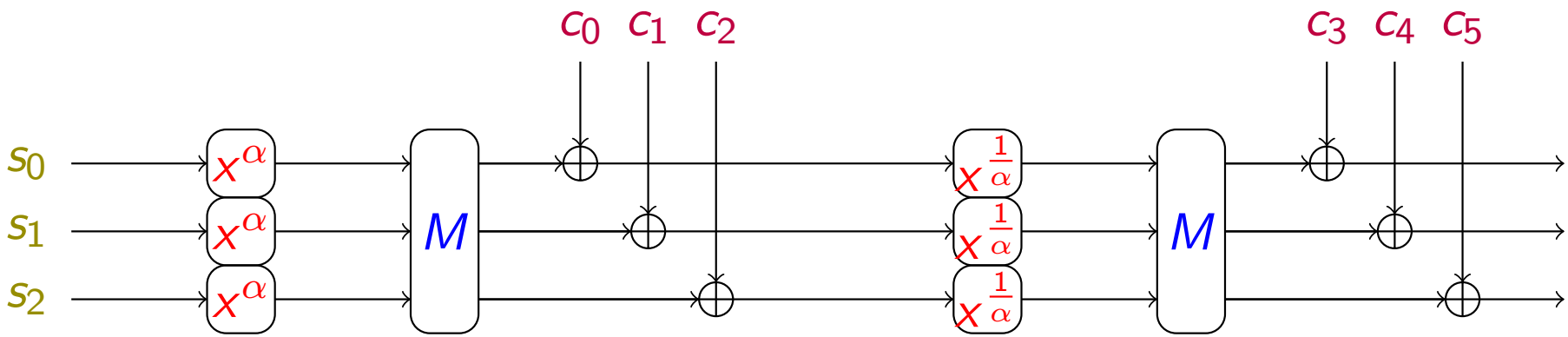
Stir

- Based on Rescue.
- MDS matrix M and round constants r are carefully chosen to impose one affine space chain over the whole permutation.

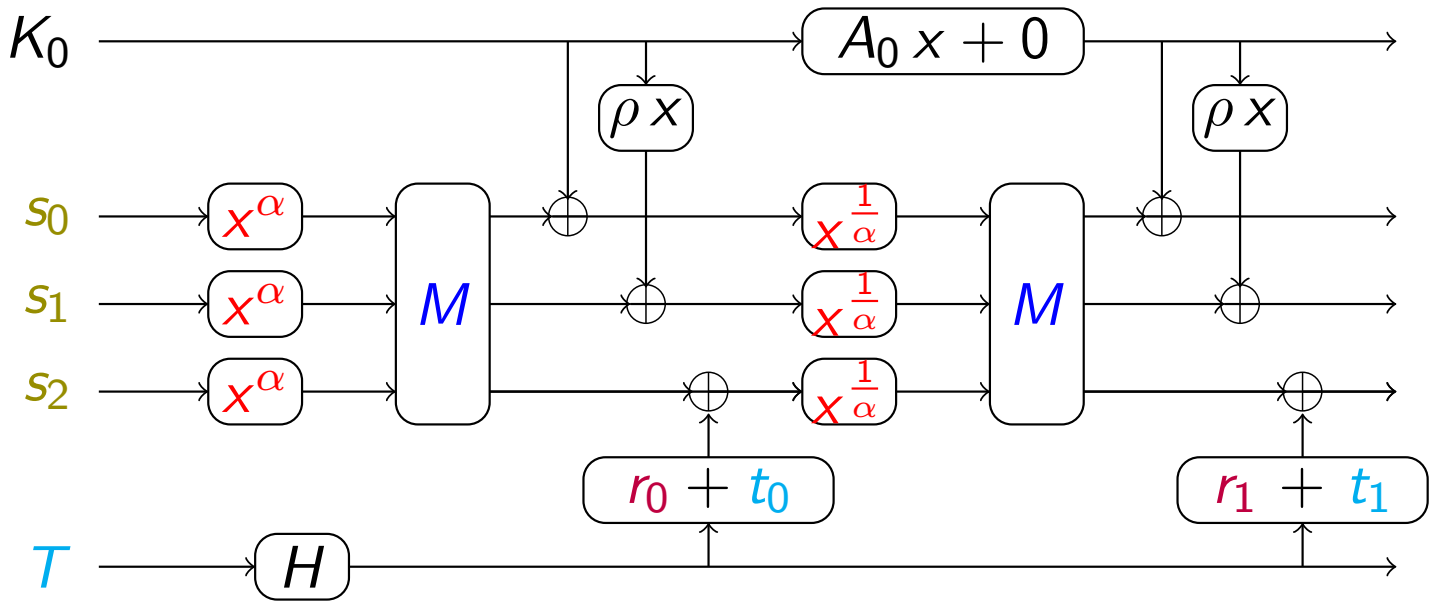


Stir

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} \right\rangle \longrightarrow \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix} + \left\langle \begin{pmatrix} v_1' \\ v_2' \\ 0 \end{pmatrix} \right\rangle \longrightarrow \ldots \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} v_1'' \\ v_2'' \\ 0 \end{pmatrix} \right\rangle$$

- Yields $p \approx 2^{64}$ solutions to the "CICO problem". This breaks security arguments in sponge constructions.

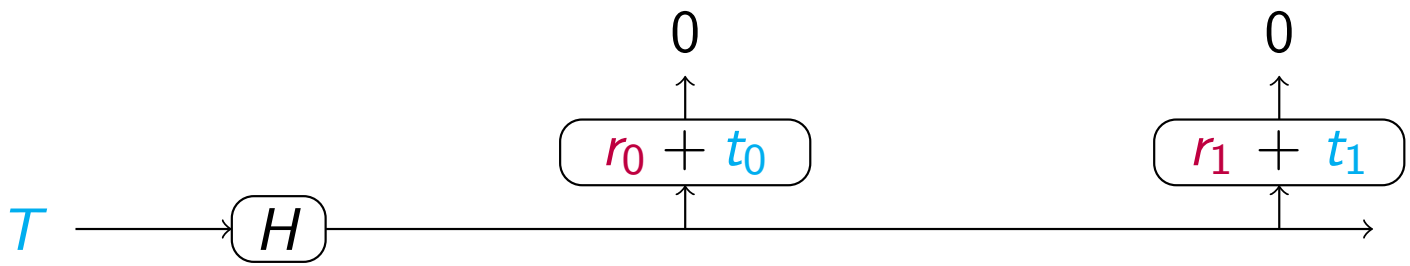
Snare



- H is some hash function, like SHAKE256.
- The t_i are the tweak hashes.

Snare

Idea: Choose $r_i = -H(T^*)_i$ for some secret tweak T^*.
$\rightarrow$ When $T = T^*$, r_i and t_i annihilate one another and an invariant vector space appears.



Snare

$$\left\langle \begin{pmatrix} 1 \\ \rho \\ 0 \end{pmatrix} \right\rangle \xrightarrow{\text{1 round}} \left\langle \begin{pmatrix} 1 \\ \rho \\ 0 \end{pmatrix} \right\rangle \longrightarrow \ldots \longrightarrow \left\langle \begin{pmatrix} 1 \\ \rho \\ 0 \end{pmatrix} \right\rangle$$

Snare

$$\begin{pmatrix} 1 \\ \rho \\ 0 \end{pmatrix} \xrightarrow{\text{1 round}} P_1(K_0) \begin{pmatrix} 1 \\ \rho \\ 0 \end{pmatrix} \longrightarrow \ldots \longrightarrow P_n(K_0) \begin{pmatrix} 1 \\ \rho \\ 0 \end{pmatrix}$$

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- Retrieve K_0 with multivariate polynomial solving (Gröbner bases), with m times less equations as the general case.

$\rightarrow$ Algebraic attack complexity put to the mth root!

Affine Space Chain vs Affine Function

- Last design is based on affine space chains.
- Having an affine space chain doesn't mean that the function itself is affine.
- In the beginning we measured high differential uniformities because the function itself is affine on these subspaces.
- Can we recreate that?

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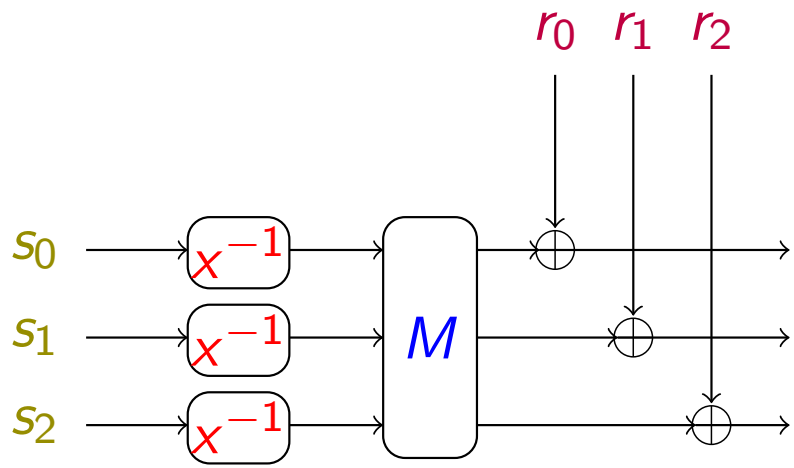
$$\boldsymbol{a}_1 + X\boldsymbol{v}_1 \longrightarrow \boldsymbol{a}_2 + (X^\alpha + \lambda)\boldsymbol{v}_2 \longrightarrow \boldsymbol{a}_3 + (X^\alpha + \lambda)^{\frac{1}{\alpha}}\boldsymbol{v}_3$$

Morse Code with Differential Uniformity

- Same thing as Snare, but with elements over $\mathbb{F}_{2^n}$ and the inverse function $x \mapsto x^{-1}$ as an Sbox.



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- For a 2-round delay, the coefficient X of the affine space basis verifies $X \longrightarrow X^{-1} \longrightarrow X$ (Case $\lambda = 0$).
- High differential uniformity every 2 or 3 rounds (controlled by our choices of r_i).