PERK: Compact Signature Scheme Based on a New Variant of the Permuted Kernel Problem

Slim Bettaieb[2], Loïc Bidoux[2], **Victor Dyseryn**[1], Andre Esser[2],
Philippe Gaborit[1], Mukul Kulkarni[2], Marco Palumbi[2]

[1]XLIM, Université de Limoges, France
[2]Technology Innovation Institute, UAE

Journées C2 - October 16, 2023

In this talk

- What is this Permuted Kernel Problem (PKP)?
- Why is it so hard?
- What can we do with it?
- Why studying variants of PKP?

Permuted Kernel Problem

Definition (IPKP [Sha90])

Let $m < n$ be positive integers, Given

- $\boldsymbol{H} \in \mathbb{F}_q^{m \times n}$;
- $\boldsymbol{x} \in \mathbb{F}_q^{n}$;
- $\boldsymbol{y} \in \mathbb{F}_q^{m}$,

the Inhomogeneous Permuted Kernel Problem $\mathrm{IPKP}_{q,m,n}$ asks to find a permutation $\pi \in \mathcal{S}_n$ such that

$$\boldsymbol{H}\pi[\boldsymbol{x}] = \boldsymbol{y}.$$

Permuted Kernel Problem

Example

$\pi = id$

$$\begin{array}{cccc} & \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} & & \\ \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 0 \end{pmatrix} & \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{-1} \end{pmatrix} & \overset{?}{=} & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

Permuted Kernel Problem

Example

$\pi = (1, 2)$

$$\begin{array}{cc} & \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 0 \end{pmatrix} & \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

Permuted Kernel Problem

Example

$\pi = (1,2) \circ (2,3)$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Comparison with Syndrome Decoding

Definition (Syndrome Decoding $\mathtt{SD}(n, k, w)$)

Given:

- $\boldsymbol{H} \in \mathbb{F}_q^{(n-k)\times n}$ a parity check matrix;
- $\boldsymbol{s} \in \mathbb{F}_q^{n-k}$ a syndrome,

the Syndrome Decoding Problem asks to find an error $\boldsymbol{e}$ of Hamming weight $w_h(\boldsymbol{e}) = w$, such that

$$\boldsymbol{s} = \boldsymbol{He}.$$

Permuted Kernel	Syndrome Decoding
$\lvert\mathcal{S}_n\rvert = n!$	$\lvert\mathbb{F}_q^n\rvert = q^n$

Number of solutions

Proposition

The average number of solutions for a random $\mathrm{IPKP}_{q,m,n}$ *instance is*

$$\frac{n!}{q^m}.$$

Since all existing attacks on IPKP and variants are combinatorial, they benefit from a speedup equal to $\max(1, \frac{n!}{q^m})$.

Coding theory equivalent: Gilbert-Varshamov bound

Systematic form

$$H\pi[x] = y$$

Systematic form

$$\underbrace{PH}_{H'}\,\pi[x] = \underbrace{Py}_{y'}$$

Georgiades algorithm [Geo92]

For $\pi[\boldsymbol{x}] = (\boldsymbol{x}_1, \boldsymbol{x}_2)$,

$$\left(\begin{array}{c|c} \boldsymbol{I}_m & \boldsymbol{H}' \end{array}\right) \left(\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_2 \end{array}\right) = \boldsymbol{y}$$

Georgiades algorithm [Geo92]

For $\pi[\boldsymbol{x}] = (\boldsymbol{x}_1, \boldsymbol{x}_2)$,

$$\boldsymbol{x}_1 = \boldsymbol{y} - \boldsymbol{H}'\boldsymbol{x}_2$$

$\Rightarrow$ enumerate $\boldsymbol{x}_2$ as every subpermutation of $\boldsymbol{x}$ of size $n - m$.

Georgiades algorithm [Geo92]

Proposition (Complexity)

$$\mathcal{T} = \mathcal{O}\left(\frac{n!}{(n-m)!}\right)$$

Coding theory equivalent: Prange algorithm

Time-memory trade-off

$$\left(\begin{array}{c|c} \boldsymbol{H}_1 & \boldsymbol{H}_2 \end{array}\right) \left(\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_2 \end{array}\right) = \boldsymbol{y}$$

$$\boldsymbol{H}_1\boldsymbol{x}_1 = \boldsymbol{y} - \boldsymbol{H}_2\boldsymbol{x}_2$$

Time-memory trade-off

$$L_1 = \{(\mathbf{x}_1, \mathbf{H}_1\mathbf{x}_1) \mid \mathbf{x}_1 \in \mathbb{F}_q^{n/2} \text{ sub-permutation of } \mathbf{x}\}$$

$$L_2 = \{(\mathbf{x}_2, \mathbf{y} - \mathbf{H}_2\mathbf{x}_2) \mid \mathbf{x}_2 \in \mathbb{F}_q^{n/2} \text{ sub-permutation of } \mathbf{x}\}$$

$$L_1 \bowtie L_2 = \{(\mathbf{x}_1, \mathbf{x}_2) \mid \exists \mathbf{z}, (\mathbf{x}_1, \mathbf{z}) \in L_1 \text{ and } (\mathbf{x}_2, \mathbf{z}) \in L_2\}$$

Time-memory trade-off

Proposition (Complexity)

$$\begin{aligned} \mathcal{T} &= \mathcal{O}\left(|L_1| + |L_2| + |L_1 \bowtie L_2|\right) \\ \mathcal{M} &= \mathcal{O}\left(|L_1| + |L_2|\right) \end{aligned}$$

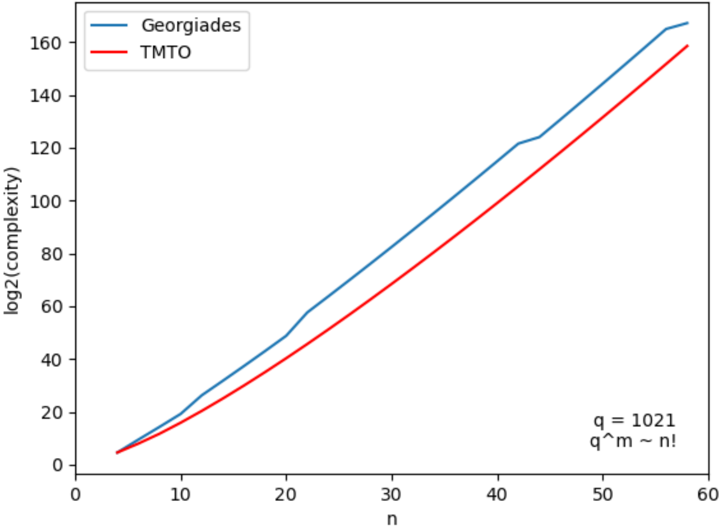
with

$$|L_1| = |L_2| = \frac{n!}{(n/2)!}$$

$$|L_1 \bowtie L_2| = \frac{|L_1| \times |L_2|}{q^m}$$

Coding theory equivalent: Birthday decoding

Comparison


Georgiades
TMTO
log2(complexity)
0
20
40
60
80
100
120
140
160
n
0
10
20
30
40
50
60
q = 1021
q^m ~ n!

KMP algorithm [KMP19]

Meet in the middle approach between Georgiades and TMTO

$$\left(\begin{array}{ccc} \boldsymbol{I}_{m-u} & \multicolumn{2}{c}{\boldsymbol{H}'} \\ \mathbf{0} & \boldsymbol{H}_2 & \boldsymbol{H}_3 \end{array}\right) \left(\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_2 \\ \hline \boldsymbol{x}_3 \end{array}\right) = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix}$$

$$\boldsymbol{x}_1 = \boldsymbol{y}_1 - \boldsymbol{H}'(\boldsymbol{x}_2, \boldsymbol{x}_3)$$

$$\boldsymbol{H}_2\boldsymbol{x}_2 + \boldsymbol{H}_3\boldsymbol{x}_3 = \boldsymbol{y}_2$$

KMP algorithm [KMP19]

Proposition (Complexity)

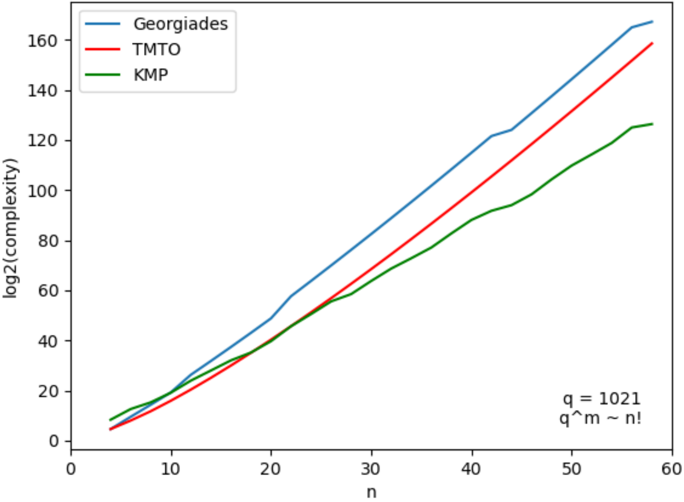
$$\mathcal{T} = \mathcal{O}\left(|L_1| + |L_2| + |L_1 \bowtie L_2|\right)$$

with

$$|L_1| = |L_2| = \binom{n}{(n-m+u)/2}\left((n-m+u)/2\right)!$$

$$|L_1 \bowtie L_2| = \frac{|L_1| \times |L_2|}{q^u}$$

Comparison



*KMP/SBC cost estimation courtesy of `https://github.com/Crypto-TII/CryptographicEstimators`

Other attacks on IPKP

- [BCCG93]
- [PC94]
- Joux-Jaulmes attack [JJ01]

Applications of PKP

Encryption	✗
Hash-and-sign	✗
Proof of Knowledge	✓

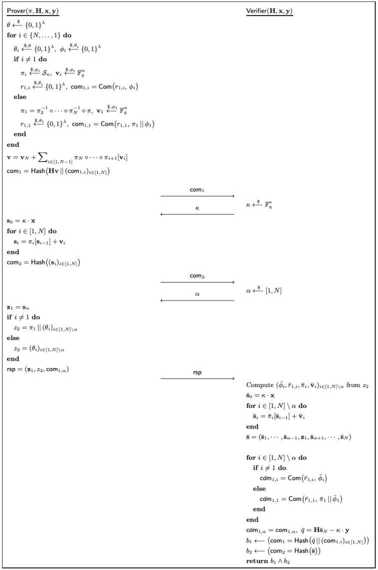
PKP-based proof of knowledge

Algorithm 2 The original 5-pass PKP identification protocol

```
$\mathcal{P}(\mathsf{sk}, \mathsf{pk})$                                      $\mathcal{V}(\mathsf{pk})$

$\sigma \xleftarrow{\$} S_n$
$\mathbf{r} \xleftarrow{\$} \mathbb{F}_p^n$
$\mathsf{C}_0 \leftarrow \mathrm{Com}(\sigma, \mathbf{Ar})$
$\mathsf{C}_1 \leftarrow \mathrm{Com}(\pi\sigma, \mathbf{r}_\sigma)$
                      $\xrightarrow{\mathsf{C}_0, \mathsf{C}_1}$
                                            $c \xleftarrow{\$} \mathbb{F}_p$
                      $\xleftarrow{c}$
$\mathbf{z} \leftarrow \mathbf{r}_\sigma + c\mathbf{v}_{\pi\sigma}$
                      $\xrightarrow{\mathbf{z}}$
                                            $b \xleftarrow{\$} \{0, 1\}$
                      $\xleftarrow{b}$
if $b = 0$ then
    $\mathsf{rsp} \leftarrow \sigma$
else
    $\mathsf{rsp} \leftarrow \pi\sigma$
end if
                      $\xrightarrow{\mathsf{rsp}}$
                                            if $b = 0$ then
                                                accept if $\mathsf{C}_0 = \mathrm{Com}(\sigma, A\mathbf{z}_{\sigma^{-1}})$
                                            else
                                                accept if $\mathsf{C}_1 = \mathrm{Com}(\pi\sigma, \mathbf{z} - c\mathbf{v}_{\pi\sigma})$
                                            end if
```

PKP-based proof of knowledge



```
Prover$(\pi, \mathbf{H}, \mathbf{x}, \mathbf{y})$                                Verifier$(\mathbf{H}, \mathbf{x}, \mathbf{y})$
$\theta \xleftarrow{\$} \{0,1\}^\lambda$
for $i \in \{N, \dots, 1\}$ do
    $\theta_i \xleftarrow{\$,\theta} \{0,1\}^\lambda,\ \phi_i \xleftarrow{\$,\theta_i} \{0,1\}^\lambda$
    if $i \neq 1$ do
        $\pi_i \xleftarrow{\$,\phi_i} \mathcal{S}_n,\ \mathbf{v}_i \xleftarrow{\$,\phi_i} \mathbb{F}_q^n$
        $r_{1,i} \xleftarrow{\$,\theta_i} \{0,1\}^\lambda,\ \mathsf{com}_{1,i} = \mathsf{Com}(r_{1,i},\ \phi_i)$
    else
        $\pi_1 = \pi_2^{-1} \circ \cdots \circ \pi_N^{-1} \circ \pi,\ \mathbf{v}_1 \xleftarrow{\$,\phi_1} \mathbb{F}_q^n$
        $r_{1,1} \xleftarrow{\$,\theta_1} \{0,1\}^\lambda,\ \mathsf{com}_{1,1} = \mathsf{Com}(r_{1,1},\ \pi_1 \,||\, \phi_1)$
    end
end
$\mathbf{v} = \mathbf{v}_N + \sum_{i \in [1,N-1]} \pi_N \circ \cdots \circ \pi_{i+1}[\mathbf{v}_i]$
$\mathsf{com}_1 = \mathsf{Hash}(\mathbf{H}\mathbf{v} \,||\, (\mathsf{com}_{1,i})_{i \in [1,N]})$

                            $\xrightarrow{\mathsf{com}_1}$
                            $\xleftarrow{\kappa}$                $\kappa \xleftarrow{\$} \mathbb{F}_q^*$

$\mathbf{s}_0 = \kappa \cdot \mathbf{x}$
for $i \in [1, N]$ do
    $\mathbf{s}_i = \pi_i[\mathbf{s}_{i-1}] + \mathbf{v}_i$
end
$\mathsf{com}_2 = \mathsf{Hash}((\mathbf{s}_i)_{i \in [1,N]})$

                            $\xrightarrow{\mathsf{com}_2}$
                            $\xleftarrow{\alpha}$                $\alpha \xleftarrow{\$} [1, N]$

$\mathbf{z}_1 = \mathbf{s}_\alpha$
if $i \neq 1$ do
    $z_2 = \pi_1 \,||\, (\theta_i)_{i \in [1,N] \setminus \alpha}$
else
    $z_2 = (\theta_i)_{i \in [1,N] \setminus \alpha}$
end
$\mathsf{rsp} = (\mathbf{z}_1, z_2, \mathsf{com}_{1,\alpha})$

                            $\xrightarrow{\mathsf{rsp}}$

                                 Compute $(\bar{\phi}_i, \bar{r}_{1,i}, \bar{\pi}_i, \bar{\mathbf{v}}_i)_{i \in [1,N] \setminus \alpha}$ from $z_2$
                                 $\bar{\mathbf{s}}_0 = \kappa \cdot \mathbf{x}$
                                 for $i \in [1, N] \setminus \alpha$ do
                                     $\bar{\mathbf{s}}_i = \bar{\pi}_i[\bar{\mathbf{s}}_{i-1}] + \bar{\mathbf{v}}_i$
                                 end
                                 $\bar{\mathbf{s}} = (\bar{\mathbf{s}}_1, \cdots, \bar{\mathbf{s}}_{\alpha-1}, \mathbf{z}_1, \bar{\mathbf{s}}_{\alpha+1}, \cdots, \bar{\mathbf{s}}_N)$

                                 for $i \in [1, N] \setminus \alpha$ do
                                     if $i \neq 1$ do
                                         $\bar{\mathsf{com}}_{1,i} = \mathsf{Com}(\bar{r}_{1,i},\ \bar{\phi}_i)$
                                     else
                                         $\bar{\mathsf{com}}_{1,1} = \mathsf{Com}(\bar{r}_{1,1},\ \bar{\pi}_1 \,||\, \bar{\phi}_1)$
                                     end
                                 end
                                 $\bar{\mathsf{com}}_{1,\alpha} = \mathsf{com}_{1,\alpha},\ \bar{q} = \mathbf{H}\bar{\mathbf{s}}_N - \kappa \cdot \mathbf{y}$
                                 $b_1 \longleftarrow (\mathsf{com}_1 = \mathsf{Hash}(\bar{q} \,||\, (\mathsf{com}_{1,i})_{i \in [1,N]}))$
                                 $b_2 \longleftarrow (\mathsf{com}_2 = \mathsf{Hash}(\bar{\mathbf{s}}))$
                                 return $b_1 \wedge b_2$
```

MPC-in-the-Head and PKP

Name	Type	σ size
Shamir [Sha90]	5-round	∼28 kB
PKP-DSS	5-round	∼21 kB
SUSHYFISH [Beu20]	5-round with helper	12-18 kB
[Fen22]	7-round	13-16 kB
[BG22]	5-round using structure	9-10 kB

Table: Comparison of recent digital signature schemes based on PKP assumptions for 128-bit security

Parameters in [BG22]

PKP parameters (q, n, m) ⟶ attacks on IPKP

MPC parameters (N, τ) ⟶ KZ attack on 5-round protocols [KZ20]

Increasing the challenge space leads to a decrease in τ.

	[BG22]	our work
Challenge space	$\mathbb{F}_q$	$\mathbb{F}_q^t$

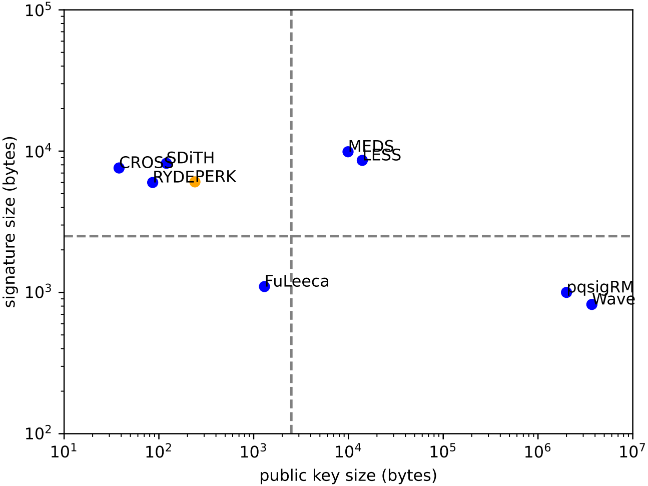
$t = 3 \longrightarrow$ 27% size decrease
$t = 5 \longrightarrow$ 33% size decrease

MPC-in-the-Head and PKP (with PERK)

Name	Type	σ size
Shamir [Sha90]	5-round	∼28 kB
PKP-DSS	5-round	∼21 kB
SUSHYFISH [Beu20]	5-round with helper	12-18 kB
[Fen22]	7-round	13-16 kB
[BG22]	5-round using structure	9-10 kB
PERK	5-round using structure	6-8 kB

Table: Comparison of recent digital signature schemes based on PKP assumptions for 128-bit security

Comparaison with NIST onramp code-based signatures



A variant of the Permuted Kernel Problem

Definition (r-IPKP)

Let $m < n$ and t be positive integers, Given

- $\boldsymbol{H} \in \mathbb{F}_q^{m \times n}$;
- $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) \in (\mathbb{F}_q^n)^t$;
- $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_t) \in (\mathbb{F}_q^m)^t$,

the Relaxed Inhomogeneous Permuted Kernel Problem r-IPKP$_{q,m,n,t}$ asks to find a permutation $\pi \in \mathcal{S}_n$ such that

$$\boldsymbol{H}\pi\Big[\sum_{i\in[1,t]} \kappa_i \boldsymbol{x}_i\Big] = \sum_{i\in[1,t]} \kappa_i \boldsymbol{y}_i$$

for some $(\kappa_1, \ldots, \kappa_t) \in (\mathbb{F}_q)^t \setminus \{(0, \ldots, 0)\}$.

Coding theory equivalent: (Rank) Support Learning

Number of solutions

Proposition

The average number of solutions for a random r-IPKP$_{q,m,n,t}$ *instance is*

$$\frac{n!}{q^m} \cdot \frac{q^t - 1}{q - 1}$$

Idea of our attack

- Take the smallest weight vector $\boldsymbol{x}$ in $\langle \boldsymbol{x}_1, \ldots, \boldsymbol{x}_t \rangle$,

  $$\boldsymbol{x} = \sum_{i \in [1,t]} \kappa_i \boldsymbol{x}_i$$

  of weight w.
- Define

  $$\boldsymbol{y} = \sum_{i \in [1,t]} \kappa_i \boldsymbol{y}_i$$

  and solve IPKP instance $\boldsymbol{H}\pi[\boldsymbol{x}] = \boldsymbol{y}$.
- Adapt KMP algorithm to take advantage of the $n - w$ zeros in $\boldsymbol{x}$.

KMP adaptation with zeros

$$\begin{array}{cl} \left(\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_2 \\ \hline \boldsymbol{x}_3 \end{array}\right) & \begin{array}{l} \leftarrow n - w - z \text{ zeros} \\ \leftarrow z/2 \text{ zeros} \\ \leftarrow z/2 \text{ zeros} \end{array} \end{array}$$

$$\begin{pmatrix} \boldsymbol{I}_{m-u} & \multicolumn{2}{c}{\boldsymbol{H}'} \\ \mathbf{0} & \boldsymbol{H}_2 & \boldsymbol{H}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix}$$

$$\boldsymbol{x}_1 = \boldsymbol{y}_1 - \boldsymbol{H}'(\boldsymbol{x}_2, \boldsymbol{x}_3)$$

$$\boldsymbol{H}_2\boldsymbol{x}_2 + \boldsymbol{H}_3\boldsymbol{x}_3 = \boldsymbol{y}_2$$

Our attack

Proposition (Complexity)

$$\mathcal{T} = \mathcal{O}\left(\mathcal{T}_{ISD} + \left(|L_1| + |L_2| + |L_1 \bowtie L_2|\right) P\right)$$

with

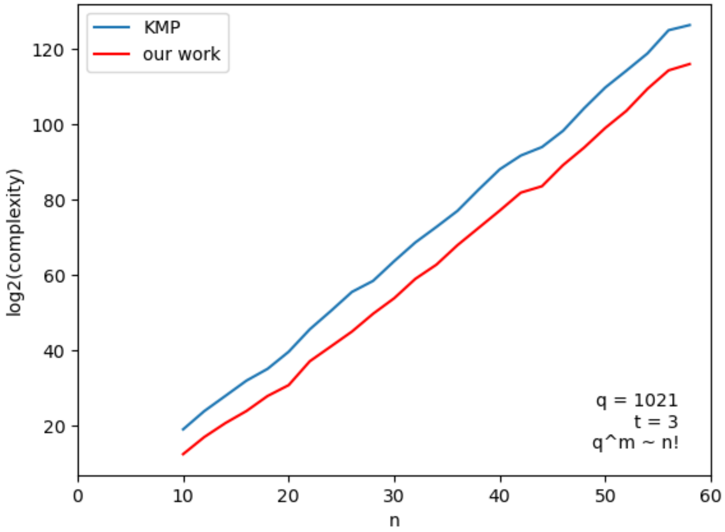
$$k = (n - m + u)/2\,,\; z \leq n - w$$

$$|L_1| = |L_2| = \binom{k}{z/2}\binom{n-z}{k-z/2}(k - z/2)!$$

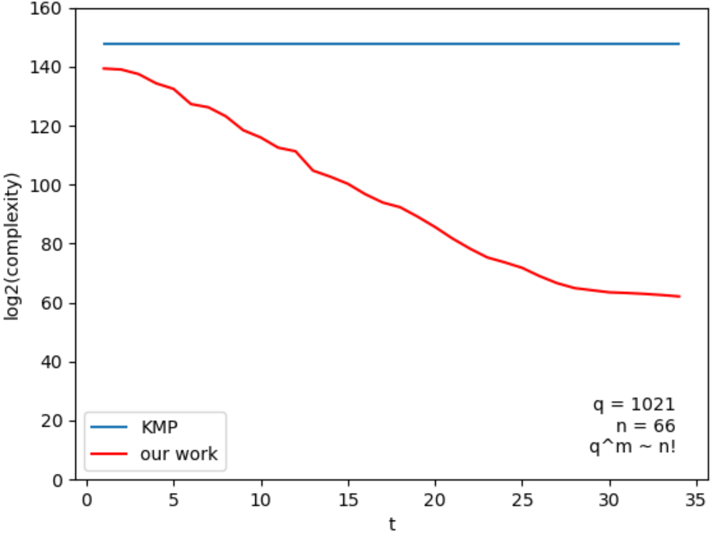
$$|L_1 \bowtie L_2| = \frac{|L_1| \times |L_2|}{q^u}$$

$$P = \frac{\binom{n}{n-w}}{\binom{n-2k}{n-w-z}\binom{k}{z/2}^2}$$

Comparison with KMP



Comparison with KMP for higher t



Conclusion

PERK was submitted to the NIST on-ramp call for digital signatures with the following augmented team:

Najwa Aaraj, Technology Innovation Institute, UAE

Slim Bettaieb, Technology Innovation Institute, UAE

Loïc Bidoux, Technology Innovation Institute, UAE

Alessandro Budroni, Technology Innovation Institute, UAE

Victor Dyseryn, XLIM, University of Limoges, France

Andre Esser, Technology Innovation Institute, UAE

Philippe Gaborit, XLIM, University of Limoges, France

Mukul Kulkarni, Technology Innovation Institute, UAE

Victor Mateu, Technology Innovation Institute, UAE

Marco Palumbi, Technology Innovation Institute, UAE

Lucas Perin, Technology Innovation Institute, UAE

Jean-Pierre Tillich, INRIA, Paris, France

Perspectives

- Combinatorial attacks
  - Refine our attack
  - Exploit the multiple instances directly in KMP?
- Algebraic attacks
  - Modelling of permutations in a PhD thesis [Sae17]
  - Polynomial attack when mt is sufficiently high (ongoing work)
  - No efficient attack derived so far in the typical regime
  - Work in progress

Questions?

Thank you for your attention !

https://pqc-perk.org

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