Rank Metric Trapdoor Functions with Homogeneous Errors

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Introduction

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- Code based post-quantum cryptography (NIST call)

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Main result:

- Security relying *on classical problems*
- For some parameters, public key *statistically indistinguishable* from a random matrix

Rank metric

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- The **rank** of $\mathbf{x}$ is the dimension of its support

Generic problem

Search Rank decoding

- $\mathbf{H} \in \mathbb{F}_{q^m}^{(n-k)\times n}$ a random matrix
- an integer $t > 0$
- $\mathbf{e} \in \mathbb{F}_{q^m}^{n}$ a random vector of rank t called *error vector*

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Assumption

Decision version of rank decoding in as hard as search version

Rank decoding's hardness

Proposition

There is a probabilistic reduction from decoding in Hamming metric to rank decoding.[1]

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[1] Gaborit, Zemor. *On the hardness of the decoding and the minimum distance problems for rank codes, ISIT 2016.*

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Combinatorial attacks[2]	Algebraic attacks [3]
$O\left((n-k)^3 m^3 q^{w\frac{(k+1)m}{n}-m}\right)$	Exponential

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Encryption schemes relying on rank decoding

	Transformed code	Hidden structure	Ciphertext in two parts
Description	$\mathbf{G} \rightarrow \mathbf{SGT}$	$\mathbf{G} \rightarrow \mathbf{SG}$	(C_1, C_2) : $C_2 - C_1\mathbf{V} = \mathbf{mG} + \mathbf{e}$
Used code	Gabidulin	Ideal LRPC	Gabidulin
Schemes	1991 GPT[4] 2017 Loidreau	2019 ROLLO	2020 RQC
Security problems	RD IfRD	IRD IfRD	IRD

RD :Rank decoding, IRD: Ideal rank decoding, IfRC: Indistinguishability from a random code

[4] Gabidulin, Paramonov, Tretjakov

New generic problem

Rank support learning (RSL)

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Assumption

Rank support learning is as hard as rank decoding if $N < kt$.

Attacks on rank support learning

	Nature	Complexity	Condition
2017[5]	Combinatorial	Poly	$N \geq nt$

[5] Gaborit, Hauteville, Phan, Tillich. *Identity-based encryption from rank metric, CRYPTO2017*

[6] Debris-Alazard, Tillich. *Two attacks on rank metric code-based schemes: Ranksign and an identity-based encryption scheme ASIACRYPT 2018*

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Used code	Gabidulin	LRPC	Gabidulin
Schemes	2022 LowMS	2022[9]	2019 Li-Ping Wang
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RSL: Rank syndrome learning,
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[9] Aguilar-Melchor, Aragon, Dyseryn, Gaborit, and Zémor. *LRPC codes with multiple syndromes: near ideal-size KEMs without ideals*

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Uses a generalisation of LRPC codes (that can only decode multiple syndromes) with semi-homogeneous matrices.

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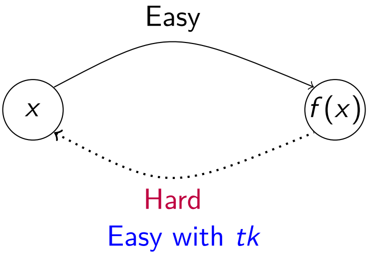
- Use of a transformed code
- Security relying on Rank decoding and RSL only

Construction of trapdoor function



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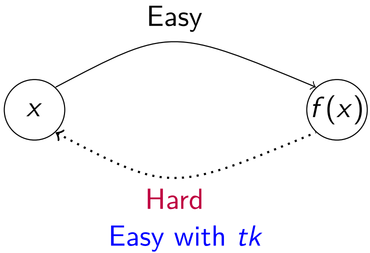

Three polynomial-time algorithms : $(Gen, Eval, Invert)$

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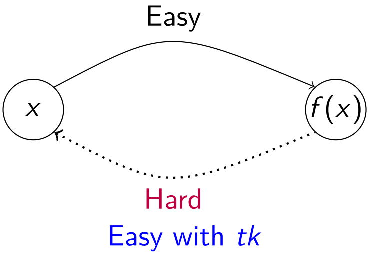

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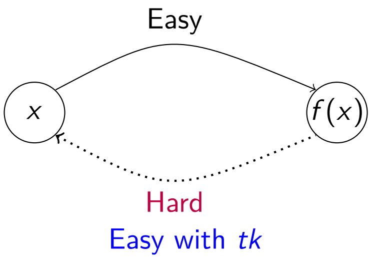

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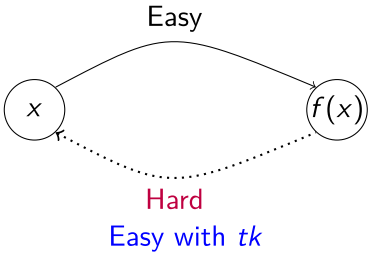

Three polynomial-time algorithms : (*Gen*,*Eval*,*Invert*)

1. pk,tk $\leftarrow$ *Gen*$(\mathbb{1}^{\lambda})$
2. *Eval*(pk,$\mathbf{x}$) will evaluate with public key pk in $\mathbf{x}$
3. Invert(tk,*Eval*(pk,$\mathbf{x}$)) returns $\mathbf{x}$ with overwhelming probability

Gen

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Remark : $\mathbf{G}(\mathbf{W}, \mathbf{I}_n)^\top = 0$

Eval

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1. $\mathbf{X} \in \mathbb{F}_{q^m}^{N \times k}$: *input*
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2. $\mathbf{E} \xleftarrow{\$} \mathbb{F}_{q^m}^{N \times (n+L)}$ *homogeneous of weight* t
3. Compute and return the output

$$\mathbf{C} = \mathbf{X}\mathbf{G} + \mathbf{E}$$

Invert

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2. Compute $\mathbf{C}(\mathbf{W}, \mathbf{I}_n)^\mathsf{T} = (\mathbf{X}\mathbf{G} + \mathbf{E})(\mathbf{W}, \mathbf{I}_n)^\mathsf{T}$

$$= \mathbf{X}\mathbf{G}(\mathbf{W}, \mathbf{I}_n)^\mathsf{T} + \mathbf{E}(\mathbf{W}, \mathbf{I}_n)^\mathsf{T}$$

$$= \mathbf{E}(\mathbf{W}, \mathbf{I}_n)^\mathsf{T}$$

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3. Recover $\mathbf{E}$ with *Homogeneous error decoding*

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$$\begin{aligned} \mathbf{C}(\mathbf{W}, \mathbf{I}_n)^\intercal &= (\mathbf{X}\mathbf{G} + \mathbf{E})(\mathbf{W}, \mathbf{I}_n)^\intercal \\ &= \mathbf{X}\mathbf{G}(\mathbf{W}, \mathbf{I}_n)^\intercal + \mathbf{E}(\mathbf{W}, \mathbf{I}_n)^\intercal \\ &= \mathbf{E}(\mathbf{W}, \mathbf{I}_n)^\intercal \end{aligned}$$

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4. Compute $\mathbf{C} - \mathbf{E} = \mathbf{X}\mathbf{G}$ and recover $\mathbf{X}$ with linear algebra

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4. Compute $\mathbf{C} - \mathbf{E} = \mathbf{XG}$ and recover $\mathbf{X}$ with linear algebra
5. Return $(\mathbf{X}, \mathbf{E})$

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Theorem (Burle, Gaborit, Hartri, Otmani)

If $N \geq wt$ and $\ell w \geq n$, there is a polynomial time algorithm that recovers $\mathbf{E}$ with a failure probability upper bounded by

$$\left(1 - \prod_{i=0}^{tw-1}(1 - q^{i-N}) + \frac{q^{2(w-1)t}}{q^m - q^{t-1}}\right)^\ell + 1 - \left(1 - \frac{q^{tw}}{q^m - q^{t-1}}\right)^\ell$$

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Asymptotically equivalent to ℓq^{tw-m}

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   - $\mathbf{s}_i \in \mathbb{F}_{q^m}^N$ seen as a sample of N elements that generates $W_i \cdot \mathcal{E}$ (with $E \cdot F := < ef | e \in E, f \in F >_{\mathbb{F}_q}$)

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   - W_i : support of $\mathbf{h}_i$
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   - $\mathbf{s}_i \in \mathbb{F}_{q^m}^N$ seen as a sample of N elements that generates $W_i \cdot \mathcal{E}$ (with $E \cdot F := < ef | e \in E, f \in F >_{\mathbb{F}_q}$) $\rightarrow N \geq tw$

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$\Rightarrow$ $\mathbf{G}$ *computationally indistinguishable* from a uniform matrix

Parameters

λ	m	L	k	n	w	t	N	pk	ct
80	179	37	16	163	6	14	84	64	367
128	293	43	20	261	8	19	153	203	1,664
192	443	59	27	391	9	26	237	618	5,694
256	409	200	33	521	4	32	128	1,134	4,608

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Security	pkSize (KB)	ctSize (KB)
128	1.90	2.04
192	2.29	2.41
256	2.50	2.63

Table: ROLLO encryption parameters

Other property on $\mathbf{G}$

$$\mathbf{G} = (\mathbf{R} | -\mathbf{R}\mathbf{W}^{\intercal})$$

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Theorem (Burle, Gaborit, Hartri, Otmani)

The statistical distance between $\mathbf{G}$ and a uniformly random matrix in $\mathbb{F}_{q^m}^{k\times(n+L)}$ is $\leq \frac{n}{2}\sqrt{\frac{q^{mk}}{\left|S_w\left(\mathbb{F}_{q^m}^{L}\right)\right|}}$

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New condition :

Choose parameters in order to have this distance $< 2^{-\lambda}$:
$\mathbf{G}$ statistically indistinguishable from uniform

$S_w\left(\mathbb{F}_{q^m}^{L}\right)$: set of vectors of length L and rank w

Statistically indistinguishable parameters

λ	m	L	k	n	w	t	pk	ct
80	499	59	17	163	16	13	212	2,813
128	907	130	21	261	19	20	860	16,450
192	1657	234	29	391	26	28	3,496	92,033
256	2707	129	36	521	35	35	7,304	263,116

Table: $q = 2$, sizes of public key and ciphertext are in KB, probability of error $< 2^{-\lambda}$

Conclusion

First rank metric trapdoor function with a public key **statistically indistinguishable** from uniform

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Thank you for your attention !

Probability of error

1. For recovering $\mathcal{E}$, one of those two events occur :
   - $\langle \mathbf{s}_i \rangle_{\mathbb{F}_q} \neq \mathcal{E} \cdot W_i$
   - $\langle \mathbf{s}_i \rangle_{\mathbb{F}_q} = \mathcal{E} \cdot W_i$ but recovering $\mathcal{E}$ fails

   Probability $\leq 1 - \prod_{i=0}^{tw-1}(1-q^{i-N}) + \frac{q^{2(w-1)t}}{q^m - q^{t-1}}$

   ℓ rows for $\mathbf{H} \to \ell$ attempts:

   $$\leq \left(1 - \prod_{i=0}^{tw-1}(1-q^{i-N}) + \frac{q^{2(w-1)t}}{q^m - q^{t-1}}\right)^\ell$$

2. For recovering $\mathbf{E}$, not possible if $\dim(W_i \cdot \mathcal{E}) < \dim W_i \dim \mathcal{E}$

   Probability $\leq \frac{q^{tw}}{q^m - q^{t-1}}$

   At least one of the ℓ spaces $\to \leq 1 - \left(1 - \frac{q^{tw}}{q^m - q^{t-1}}\right)^\ell$

Probability of error upper bounded by :

$$\left(1 - \prod_{i=0}^{tw-1}(1-q^{i-N}) + \frac{q^{2(w-1)t}}{q^m - q^{t-1}}\right)^\ell + 1 - \left(1 - \frac{q^{tw}}{q^m - q^{t-1}}\right)^\ell$$

Rank metric

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_{q^m}^n$$

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{F}_{q^m}^m$ be a basis of $\mathbb{F}_{q^m}$. For all $i \in \{1 \ldots n\}$ we have

$$x_i = \sum_{j=1}^{m} x_{i,j}\alpha_j$$

So if we consider the matrix

$$\mathbf{M} \triangleq \begin{pmatrix} x_{1,1} & \cdots & x_{n,1} \\ \vdots & \vdots & \vdots \\ x_{1,m} & \cdots & x_{n,m} \end{pmatrix} \in \mathbb{F}_q^{m \times n}$$

Then $\mathbf{x} = \boldsymbol{\alpha}\mathbf{M}$ and $|\mathbf{x}| = \mathrm{Rank}(\mathbf{M})$.