Shooting for the Stars! The May-Ozerov Algorithm for Syndrome Decoding is “Galactic”

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Algorithms for Nearest Neighbor Problem and application to cryptanalysis of McEliece cryptosystem

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The Nearest Neighbor Problem and application to cryptanalysis

- Boolean context
- Decoding random linear codes ($DRLC$) :
  Find x such that $Hx = s$ with $|x| \leq w$ (NP-hard)
- Improve $DRLC$
  $\implies$ improve McEliece cryptosystem's cryptanalysis
- The best-known algorithms use in a crucial way a subroutine that solve NNP

Nearest Neighbor Problem in $\mathbb{F}_2^m$

L

Bit n°	0	1	2	3	4	5	6
Line 0	1	1	1	1	1	0	0
Line 1	1	0	0	0	0	0	1
Line 2	0	1	1	1	1	0	0
Line 3	1	0	0	1	0	0	0
Line 4	0	1	1	0	1	0	1
Line 5	0	0	0	1	0	0	0
Line 6	1	0	0	0	1	0	0
Line 7	0	0	0	0	0	1	1

C

R

Bit n°	0	1	2	3	4	5	6
Line 0	1	1	1	1	0	1	0
Line 1	1	1	1	0	1	0	1
Line 2	1	1	1	0	0	0	0
Line 3	1	0	0	1	1	1	1
Line 4	1	0	0	0	1	1	1
Line 5	0	1	1	1	0	1	0
Line 6	0	1	1	0	0	1	1
Line 7	0	0	0	1	1	1	1

$N=2^{\lambda m}$

m

Goal: Find $C = (x, y) \in L \times R$ such that $|x + y| \leq \gamma m$

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Line 3	1	0	0	1	0	0	0
Line 4	0	1	1	0	1	0	1
Line 5	0	0	0	1	0	0	0
Line 6	1	0	0	0	1	0	0
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Line 0	1	1	1	1	0	1	0
Line 1	1	1	1	0	1	0	1
Line 2	1	1	1	0	0	0	0
Line 3	1	0	0	1	1	1	1
Line 4	1	0	0	0	1	1	1
Line 5	0	1	1	1	0	1	0
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Goal: Find $C = (x, y) \in L \times R$ such that $|x + y| \leq \gamma m$

Here $\gamma m = 1 \implies \gamma = \frac{1}{m}$

The projection method

Probability that 2 bit strings we search coincide on k columns: $\frac{\binom{m}{k}}{\binom{m(1-\gamma)}{k}}$

The algorithm

- Pick k columns randomly
- Sort the 2 lists in lexicographic order according to the selected columns
- Compare all pairs of bit strings that coincide on the k columns
- Repeat $\simeq \frac{\binom{m}{k}}{\binom{m(1-\gamma)}{k}}$ times

$k = 2$, drawn column numbers $= \{0, 2\}$

L sorted

Bit n°	2	0	1	3	4	5	6
Line 0	0	0	0	1	0	0	0
Line 1	0	0	0	0	0	1	1
Line 2	0	1	0	0	0	0	1
Line 3	0	1	0	1	0	0	0
Line 4	0	1	0	0	1	0	0
Line 5	1	0	1	1	1	0	0
Line 6	1	0	1	0	1	0	1
Line 7	1	1	1	1	1	0	0

C

R sorted

Bit n°	2	0	1	3	4	5	6
Line 0	0	0	0	1	1	1	1
Line 1	0	1	0	1	1	1	1
Line 2	0	1	0	0	1	1	1
Line 3	1	0	1	1	0	1	0
Line 4	1	0	1	0	0	1	1
Line 5	1	1	1	1	0	1	0
Line 6	1	1	1	0	1	0	1
Line 7	1	1	1	0	0	0	0

$k = 2$, drawn column numbers $= \{1, 4\}$

L sorted

Bit n°	4	1	0	2	3	5	6
Line 0	0	0	1	0	0	0	1
Line 1	0	0	1	0	1	0	0
Line 2	0	0	0	0	0	1	1
Line 3	0	0	0	0	1	0	0
Line 4	1	0	1	0	0	0	0
Line 5	1	1	1	1	1	0	0
Line 6	1	1	0	1	1	0	0
Line 7	1	1	0	1	0	0	1

C

R sorted

Bit n°	4	1	0	2	3	5	6
Line 0	0	1	1	1	1	1	0
Line 1	0	1	1	1	0	0	1
Line 2	0	1	0	1	1	1	0
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Complexity

$$C_{Proj} = O\left(\left(N + \frac{N^2}{2^k}\right)\frac{\binom{m}{k}}{\binom{m-l}{k}}\right)$$

A well know complexity tradeoff is $k = \lambda m$ then

$$C_{Proj} = O\left(2^{m(\lambda + h(\lambda,\gamma))}\right)$$

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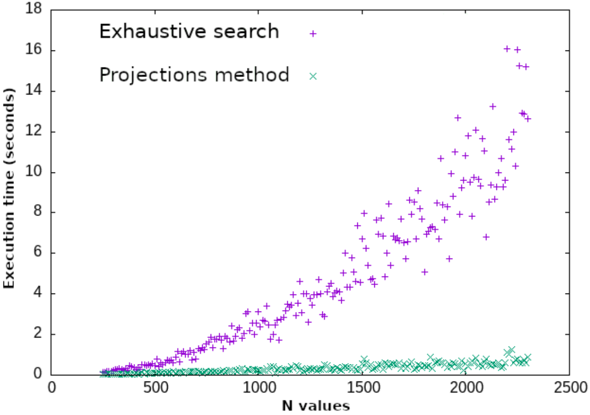
where $h(\lambda, \gamma) = H(\lambda) - (1-\gamma)H\left(\frac{\lambda}{1-\gamma}\right)$

In practice

Parameters

$n = 200,\ \gamma m = 60,\ N \in [250, 2300],\ step = 10$



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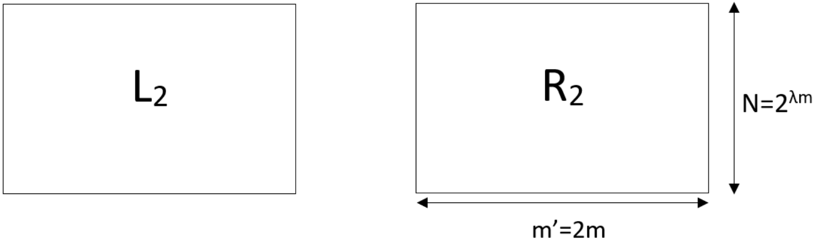
- $C_{MO} = \tilde{O}(2^{y(\lambda,\gamma)m})$ where $y(\lambda,\gamma) = (1-\gamma)\left(1 - \frac{H(H^{-1}(1-\lambda)-\gamma/2)}{1-\gamma}\right)$
- With $\lambda = 0.025$ and $\gamma = 0.1$ $C_{MO} \geq C_{Proj} \implies m \geq 256000 \implies |L| = |R| \geq 2^{8000}$

At a fixed list size N and a fixed γ, what happens if the vectors are twice as long?



$$2^{\lambda m} = 2^{\frac{\lambda}{2} 2m}$$

Goal: Find $C = (x', y') \in L_2 \times R_2$ such that $|x' + y'| \leq \gamma 2m$

Complexity

$$C_{Proj2} = O\left(\left(N + \frac{N^2}{2^k}\right)\frac{\binom{2m}{k}}{\binom{2m(1-\gamma)}{k}}\right)$$

If we choose $k = \lambda m$ then

$$C_{Proj2} = O\left(2^{m(\lambda+2h(\frac{\lambda}{2},\gamma))}\right) \text{ with } 2h\left(\lambda/2,\gamma\right) \leq h(\lambda,\gamma)$$

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And if we concatenate L with itself, the same for R ?

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That won't work: some of the k columns drawn can be identical

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$\Longrightarrow$ Filtering on less than k columns

But let see the function $f(t) = t.h(\frac{\lambda}{t}, \gamma)$

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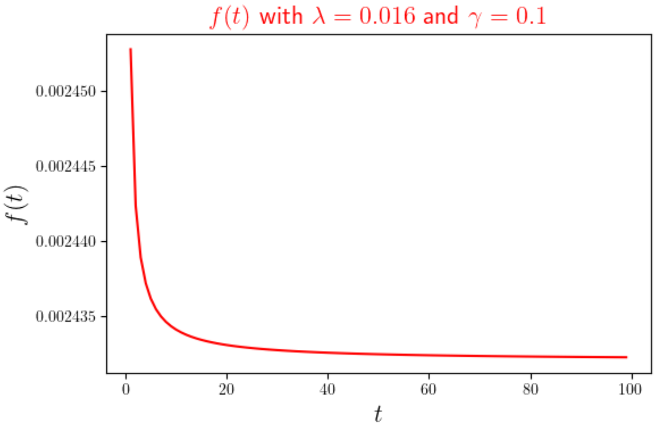

In fact, $f(t)$ is decreasing

At a fixed list size N and a fixed γ, what happens if the vectors are k times longer?

Complexity

$$C_{Projk} = O\left(\left(N + \frac{N^2}{2^k}\right)\frac{\binom{km}{k}}{\binom{km(1-\gamma)}{k}}\right)$$

If we choose $k = \lambda m$ then

$$C_{Projk} = O\left(2^{m(\lambda + k.h(\frac{\lambda}{k},\gamma))}\right)$$

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Concatenate L with itself k times, the same for R ?

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Concatenate L with itself k times, the same for R ?

Again columns drawn can be identical

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Concatenate L with itself k times, the same for R ?

Again columns drawn can be identical

It looks like drawing with replacement

The projection method drawing columns with replacement

Complexity

$$C_{ProjR} = O\left(\left(N + \frac{N^2}{2^{m(1-(1-\frac{1}{m})^k)}}\right)(1-\gamma)^{-k}\right)$$

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If we choose $k = \frac{\ln(1-\lambda)}{\ln(1-\frac{1}{m})}$ then

The projection method drawing columns with replacement

Complexity

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If we choose $k = \frac{\ln(1-\lambda)}{\ln(1-\frac{1}{m})}$ then

$$C_{ProjR} = O\left(N(1-\gamma)^{-k}\right) = O\left(2^{m(\lambda+\log_2(1-\gamma)\ln(1-\lambda))}\right)$$

The projection method drawing columns with replacement

Complexity

$$C_{ProjR} = O\left(\left(N + \frac{N^2}{2^{m(1-(1-\frac{1}{m})^k)}}\right)(1-\gamma)^{-k}\right)$$

If we choose $k = \frac{\ln(1-\lambda)}{\ln(1-\frac{1}{m})}$ then

$$C_{ProjR} = O\left(N(1-\gamma)^{-k}\right) = O\left(2^{m(\lambda + \log_2(1-\gamma)\ln(1-\lambda))}\right)$$

$$\log_2(1-\gamma)\ln(1-\lambda) \leq h(\lambda,\gamma)$$

Work in progress:

- Drawing columns with replacement in practice ?
- Concatenate lists seems to improve complexity of Esser, Kübler and Zweydinger algorithm

$$2y(\lambda/2, \gamma) \leq y(\lambda, \gamma)$$

Thank you