Lattices in cryptography: cryptanalysis, constructions and reductions

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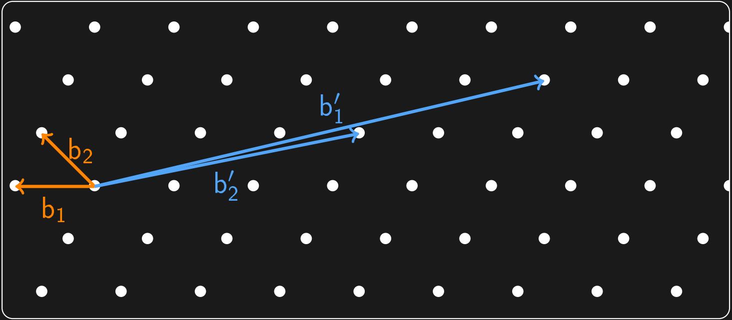
CNRS and Université de Bordeaux

Journées C2, 2023

Najac

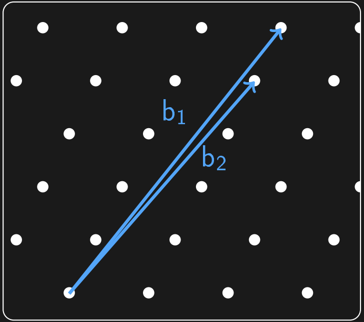
Lattices



- $\mathcal{L} = \{\sum_{i=1}^{n} x_i \mathrm{b}_i \mid \forall i,\ x_i \in \mathbb{Z}\}$ is a lattice
- $(\mathrm{b}_1, \ldots, \mathrm{b}_n) =: B \in \mathrm{GL}_n(\mathbb{R})$ is a basis (not unique)

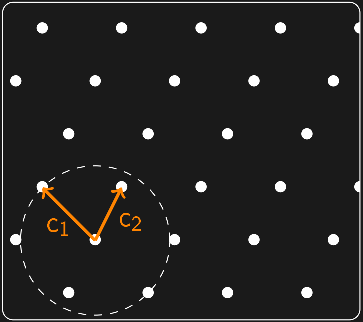
Short basis problem

Input:



$\longrightarrow$

Output:



Shortest basis problem

$$\max_i \|c_i\| \leq \min_{B' \text{ basis of } L} \left(\max_i \|b'_i\| \right)$$

Short basis problem

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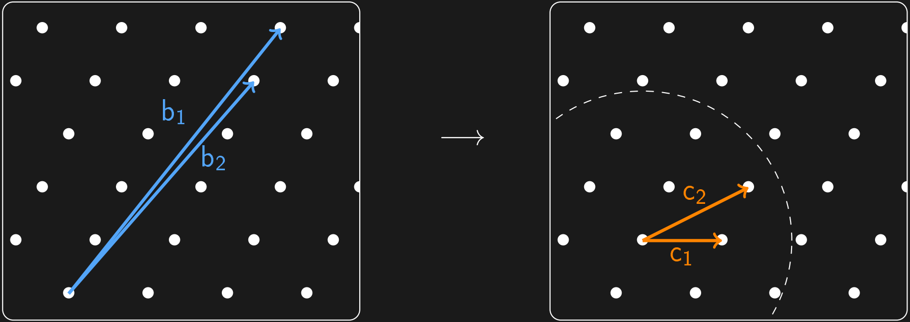
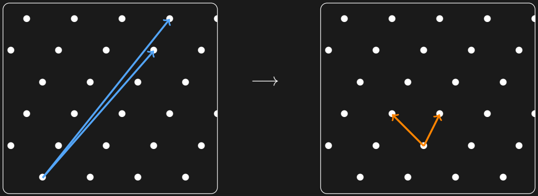


Approximate short basis problem

$$\max_i \|c_i\| \leq \gamma \cdot \min_{B' \text{ basis of } L} \left(\max_i \|b'_i\| \right)$$

Lattice reduction algorithms



Dimension 2: Lagrange-Gauss algorithm

video

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video

Theorem: The algorithm

- finds a shortest basis
- runs in polynomial time

The LLL algorithm [LLL82]

Input: basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$

[LLL82] Lenstra, Lenstra, and Lovász. Factoring polynomials with rational coefficients. Mathematische annalen.

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Input: basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$

Main idea: improve the basis locally on blocks of dimension 2
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Algorithm:

- while there exists i such that (b_i, b_{i+1}) is not a shortest basis of L_i
  (L_i is roughly the lattice spanned by (b_i, b_{i+1}))
  - run Lagrange-Gauss on L_i

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This algorithm

- finds an approximate short basis with $\gamma = 2^n$

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Algorithm:

- while there exists i such that (b_i, b_{i+1}) is not a γ'-short basis of L_i with $\gamma' = 4/3$
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  - run Lagrange-Gauss on L_i

This algorithm

- finds an approximate short basis with $\gamma = 2^n$
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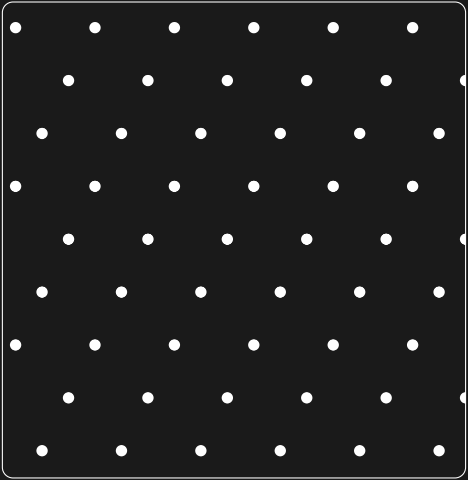
Sieving algorithm [AKS01]



Sieving:

[AKS01] Ajtai, Kumar, and Sivakumar. A sieve algorithm for the shortest lattice vector problem. STOC

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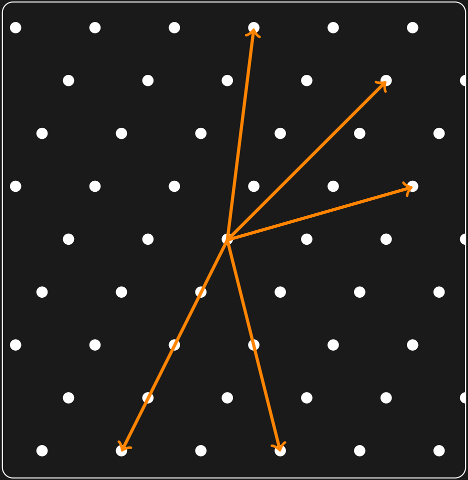

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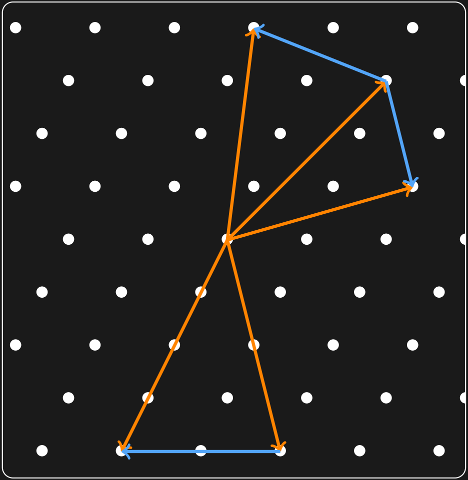

Sieving:

- Create many large vectors
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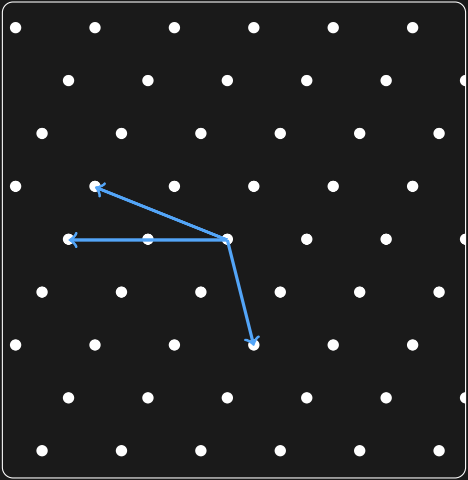

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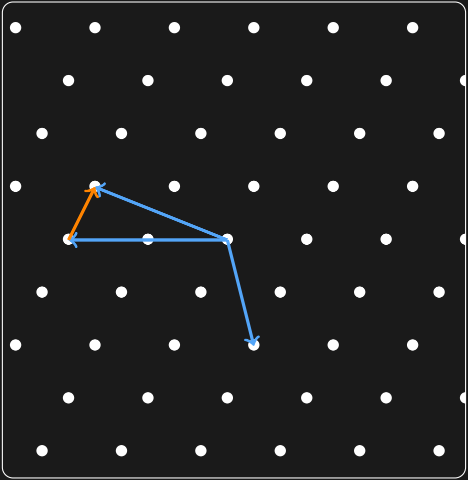

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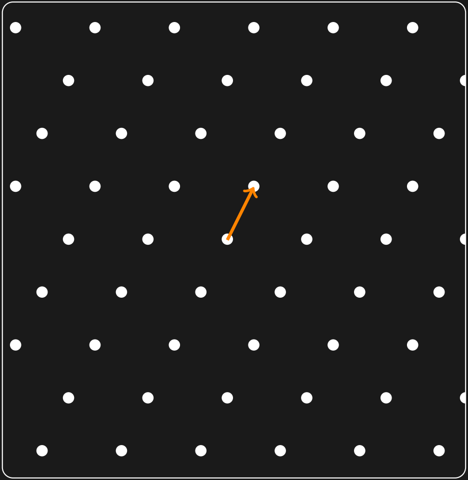

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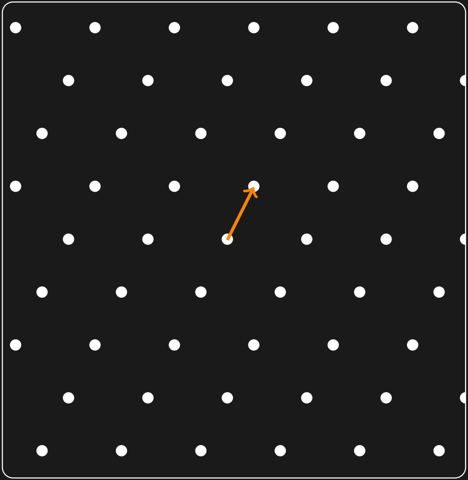

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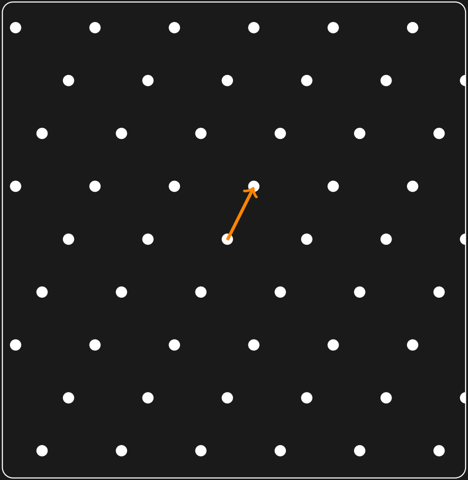

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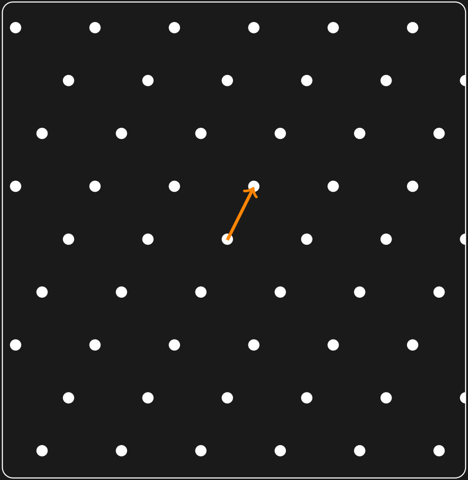

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- runs in time $2^{O(n)}$

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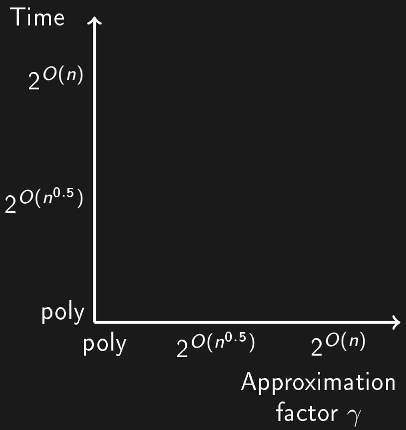
Summary and BKZ



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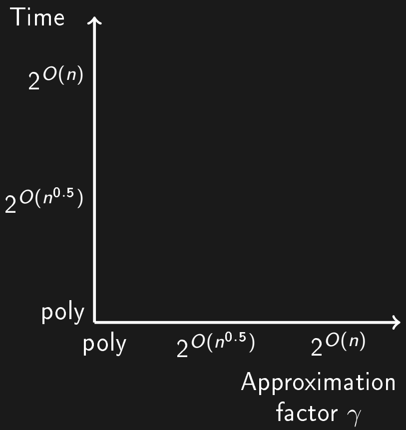

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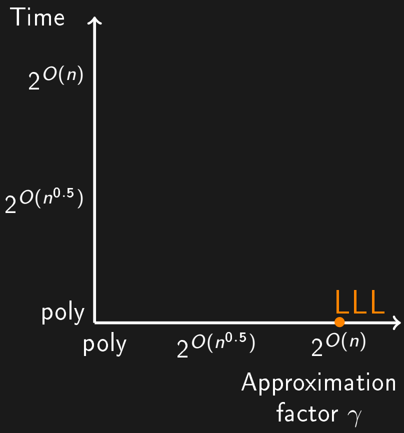

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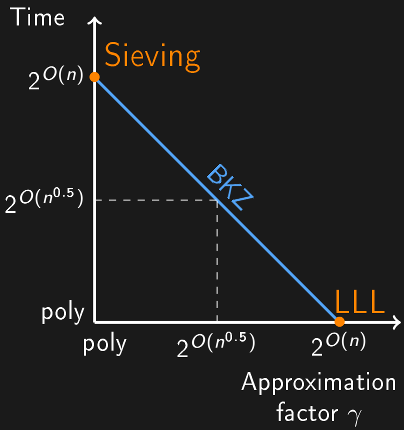
Summary and BKZ

BKZ trade-offs



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- shortest basis
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BKZ algorithm: combine LLL + Sieving $\Rightarrow$ various trade-offs

Some concrete numbers

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- from $n = 500$ to $n = 1000 \rightsquigarrow$ cryptography

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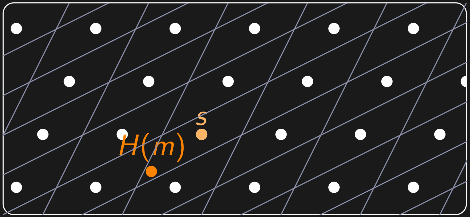
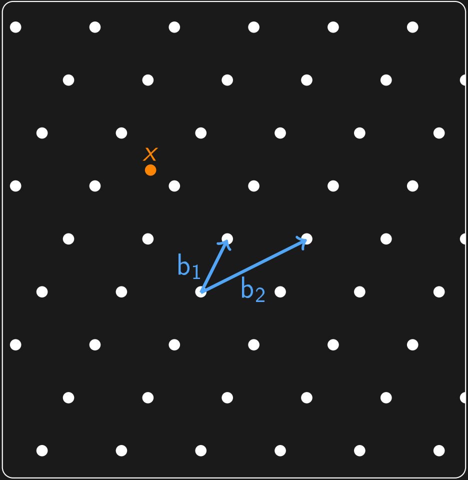
Hash-and-sign signature



Decoding in a lattice using a short basis



Input: $x = 3.7 \cdot b_1 - 1.4 \cdot b_2$

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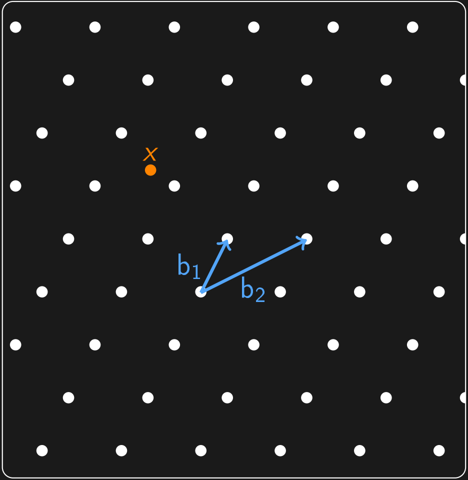

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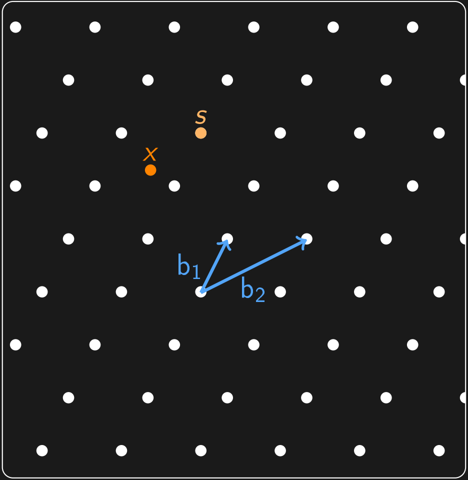

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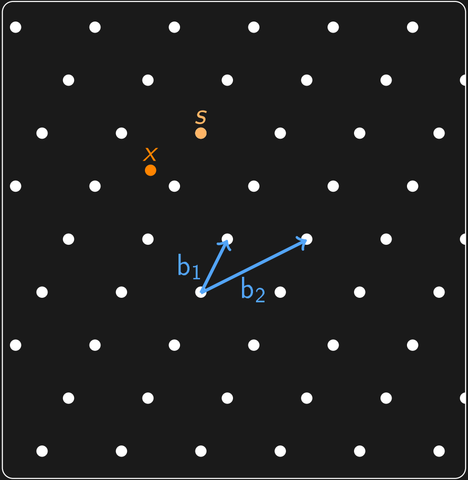

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The smaller the basis, the closer the solution

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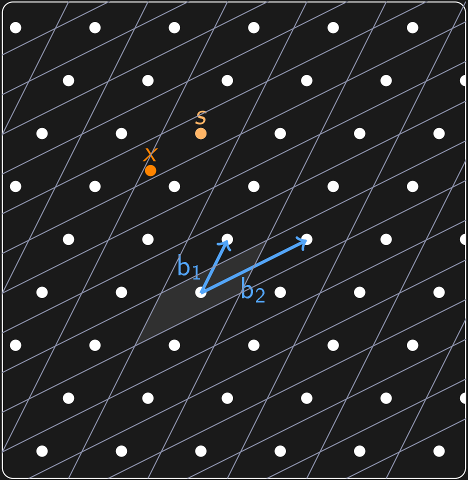

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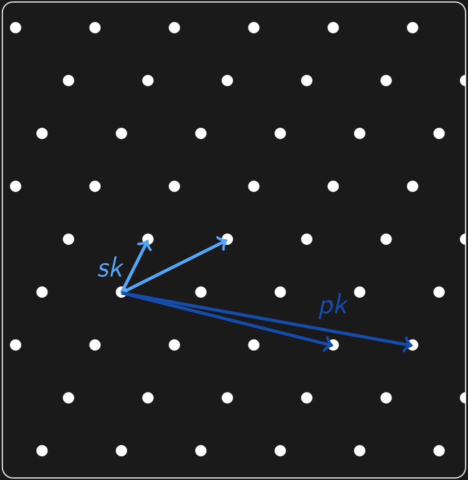
 $= \left\{ x_1 b_1 + x_2 b_2 \,\middle|\, |x_i| \leq \frac{1}{2} \right\}$

Hash-and-sign: first idea [GGH97]



KeyGen:

- pk = bad basis of $\mathcal{L}$
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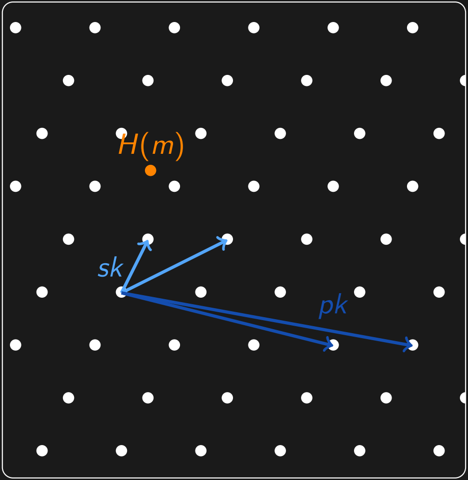

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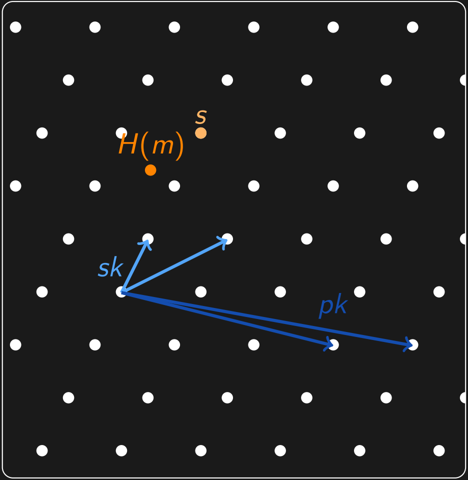

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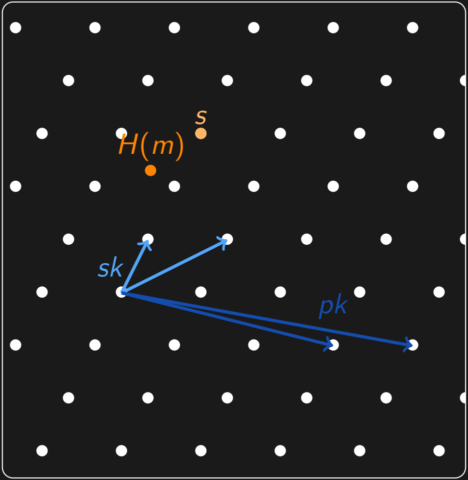

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Verify(s, pk):

- check that $s \in \mathcal{L}$
- check that $H(m) - s$ is small

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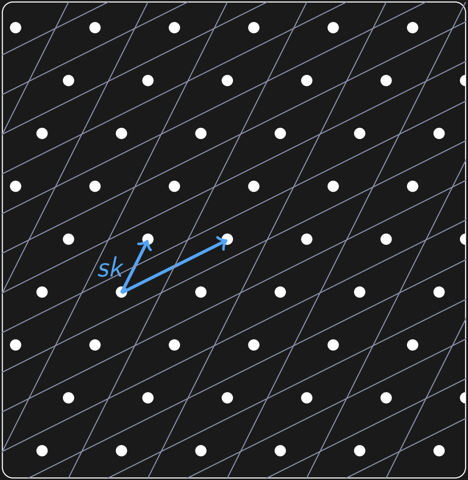
Attack on this first idea [NR06]



Parallelepiped attack:

[NR06] Nguyen and Regev. Learning a parallelepiped: Cryptanalysis of GGH and NTRU signatures. J. Cryptology

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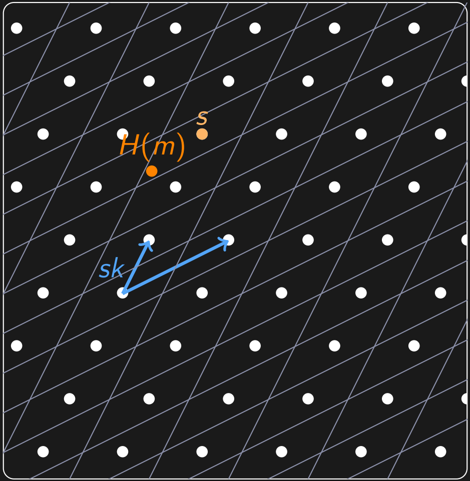

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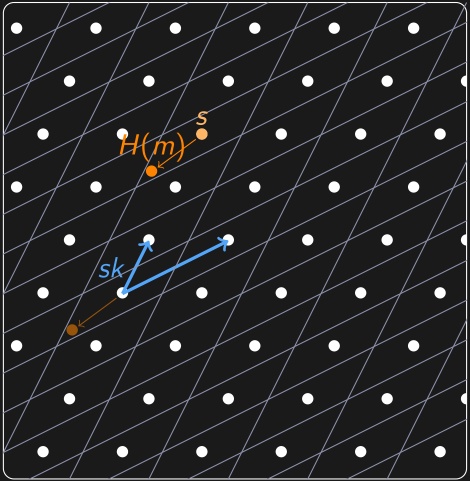

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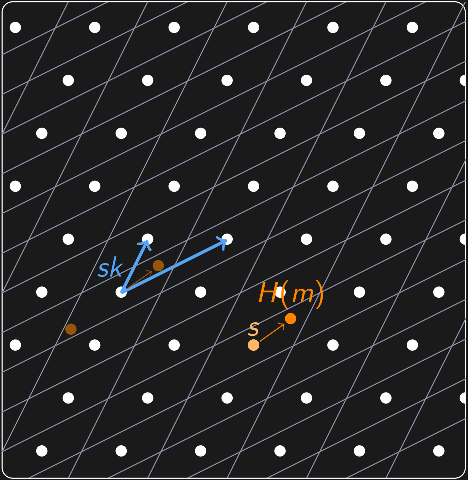

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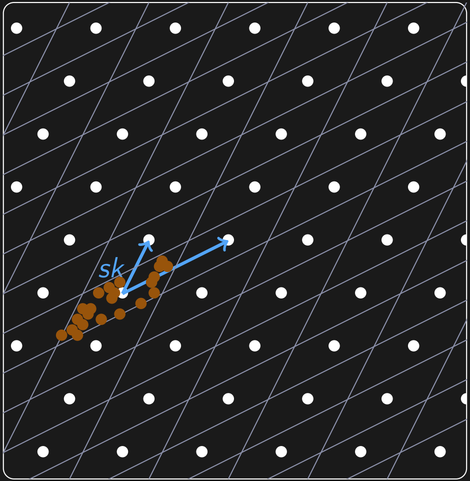

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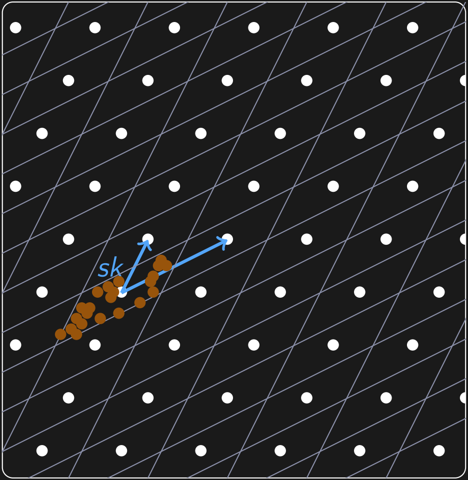

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From the shape of the parallelepiped, one can recover the short basis

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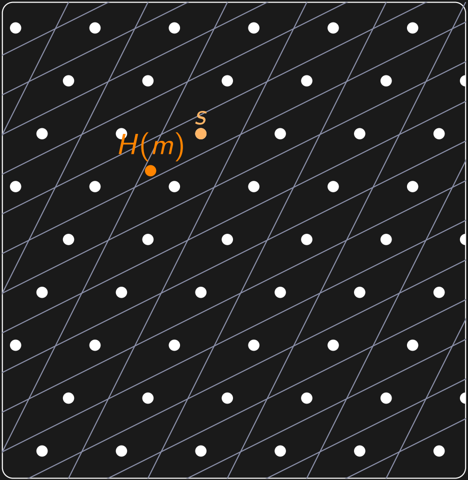
Preventing the attack [GPV08]



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[GPV08] Gentry, Peikert, and Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC.

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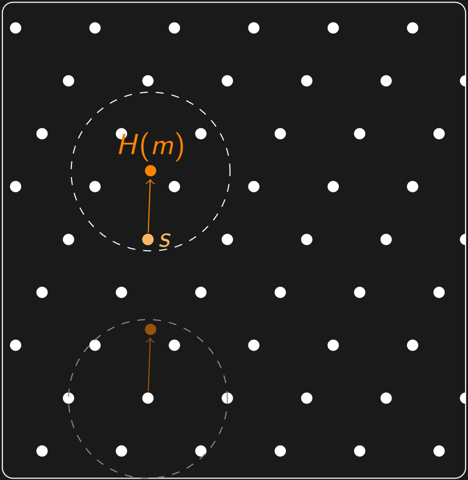

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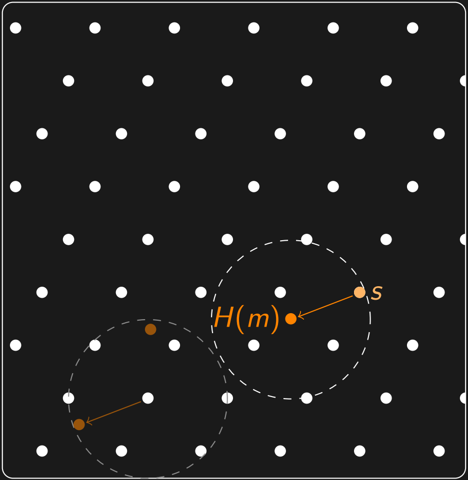

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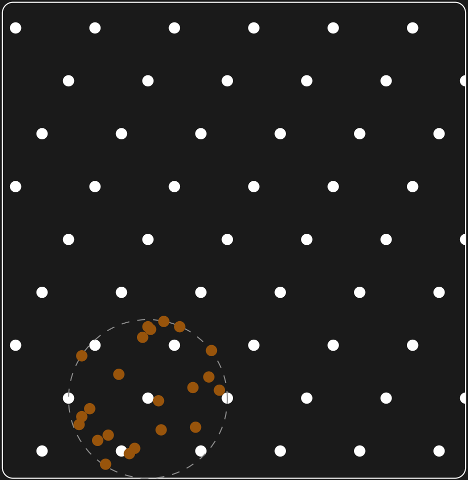

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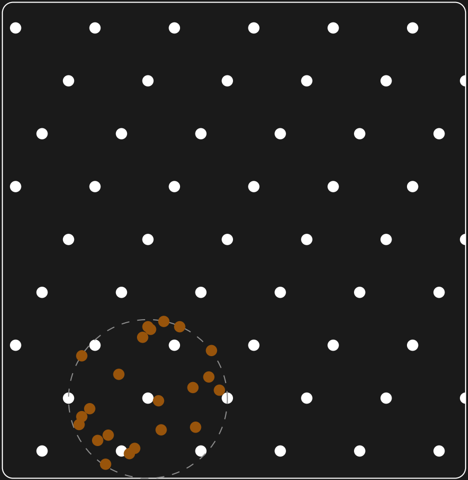
- sample $s \in \mathcal{L} \cap \mathcal{B}_r(x)$ (small radius r)

[GPV08] Gentry, Peikert, and Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC.

Preventing the attack [GPV08]



Idea: do not decode deterministically but randomly

Sign(m, sk):

- $x = H(m)$ (hash the message)
- sample $s \in \mathcal{L} \cap \mathcal{B}_r(x)$ (small radius r)

Lemma: if an adversary can forge signatures, then she can recover a short basis of $\mathcal{L}$ using only pk (in the ROM)

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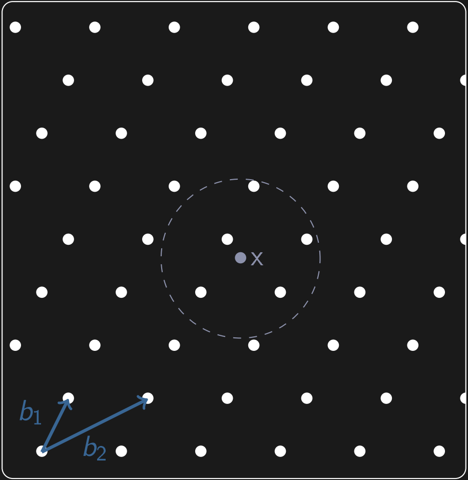
Sampling uniformly in a ball [PP21]

Input: center x, radius r

(and a short basis $(b_1, \ldots, b_n)$)

Output: $s \leftarrow \mathcal{U}(\mathcal{L} \cap \mathcal{B}_r(x))$



[PP21] Plançon and Prest. Exact Lattice Sampling from Non-Gaussian Distributions. PKC.

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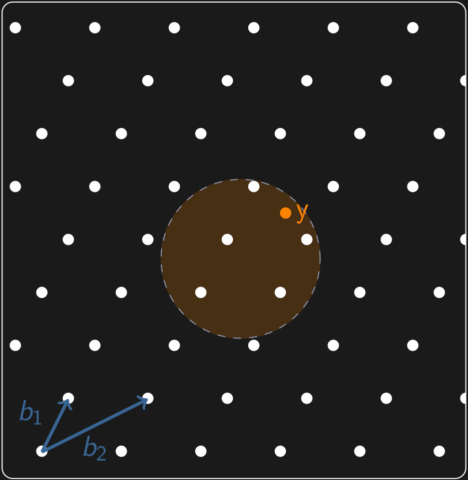

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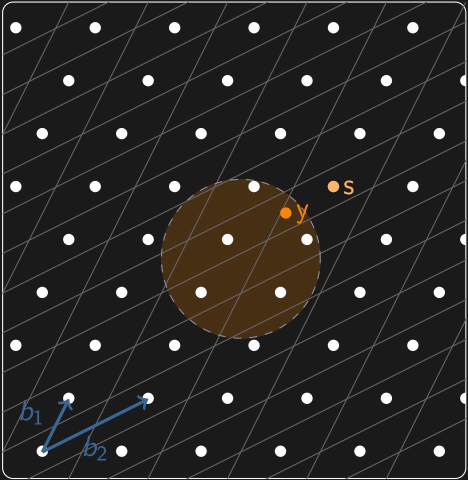

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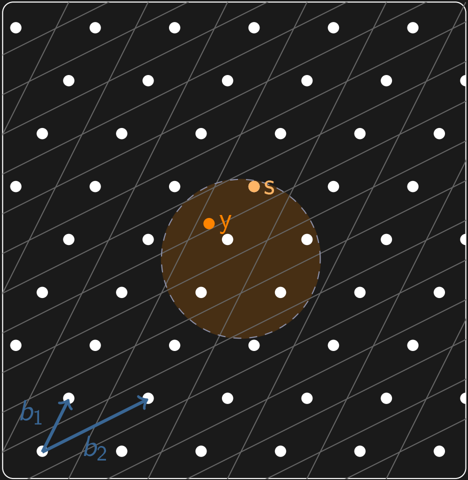

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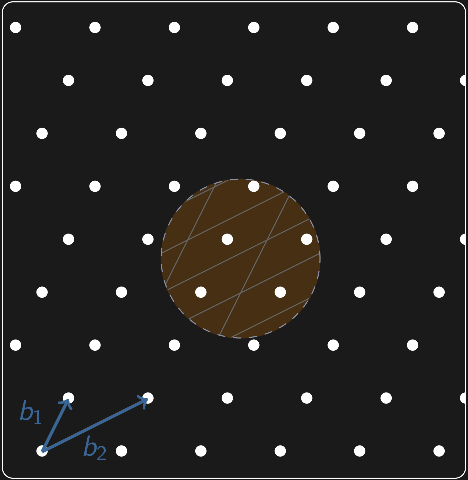

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Algo:

- Sample $y \leftarrow \mathcal{U}(\mathcal{B}_{r'}(x))$
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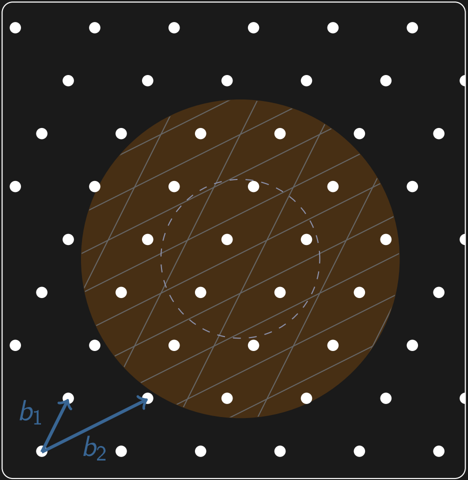

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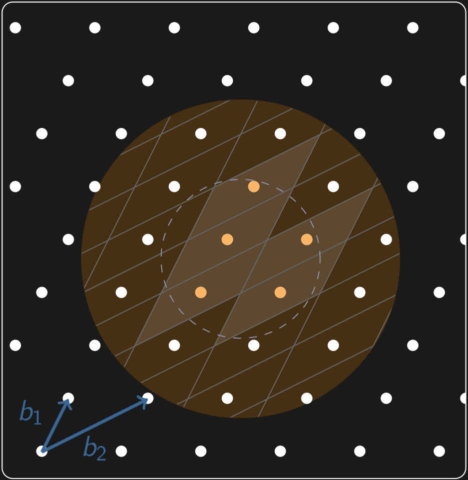

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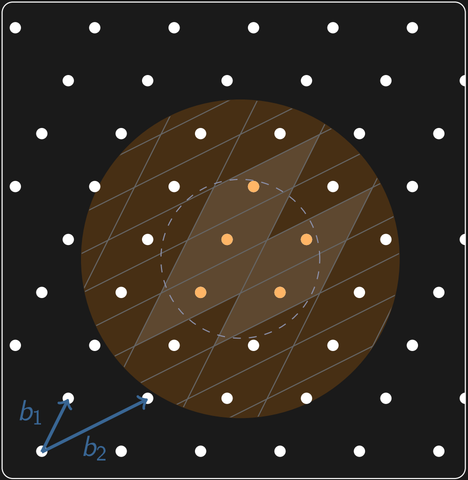
polynomial time if $r \geq 2n^2 \cdot \max_i \|\mathbf{b}_i\|$



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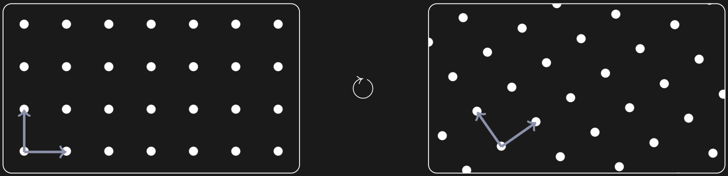
Summary

Hash-and-sign signature scheme:

- requires a lattice $\mathcal{L}$ + a short basis B_s + a bad basis B_p;
- provably secure if recovering a short basis from B_p is hard.

How to generate a hard lattice?



Objective

What we want: An algorithm KeyGen such that

- KeyGen computes
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- computing a short basis of $\mathcal{L}$ from B_p is hard with overwhelming probability

Worst-possible basis

There is a basis B_0 of $\mathcal{L}$ that can be computed
in poly time from any other basis B

$\Rightarrow B_0$ is a worst possible basis

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Input: any basis B of $\mathcal{L}$

Algorithm:

- Compute LLL-reduced basis $C = (c_1, \ldots, c_n)$
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  - $\max_i \|c_i\| \leq 2^n \cdot \min_{C'} \max_i \|c_i'\|$ (C' ranging over all bases of $\mathcal{L}$)

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- sample many vectors $v_j \leftarrow \mathcal{U}(\mathcal{L} \cap \mathcal{B}_r)$ (with $r = 2n^2 \cdot 2^n \cdot \min_{C'} \max_i \|c'_i\|$)
  - until they generate $\mathcal{L}$
  - poly time because $r \geq 2n^2 \cdot \max_i \|c_i\|$

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- extract a basis B_0 from the v_j's
  - linear algebra $\Rightarrow$ poly time

Worst-possible basis

There is a random basis B_0 of $\mathcal{L}$ that can be computed in poly time from any other basis B

$\Rightarrow$ B_0 is a worst possible distribution over bases

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- computing a short basis from B_p is hard
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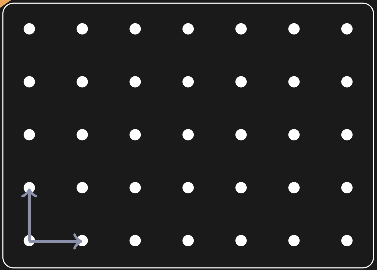
Using lattice isomorphism [DW22,BGPS23]



$\mathcal{L}_0 = \mathbb{Z}^n$

[DW22] Ducas and van Woerden. On the lattice isomorphism problem, quadratic forms [...] Eurocrypt

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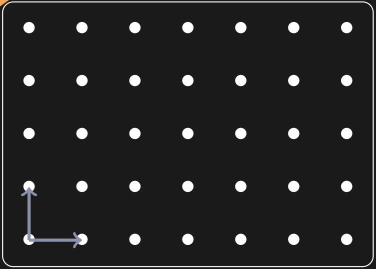

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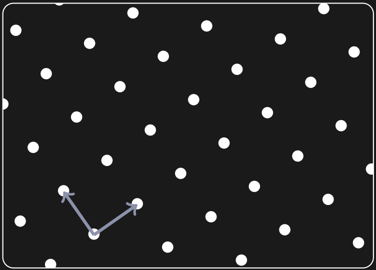
rotate $\longrightarrow$ (choose O orthogonal matrix)



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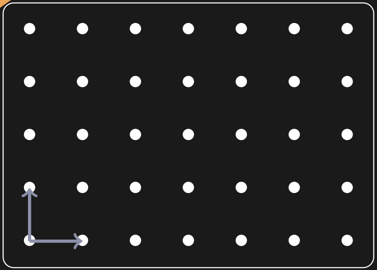

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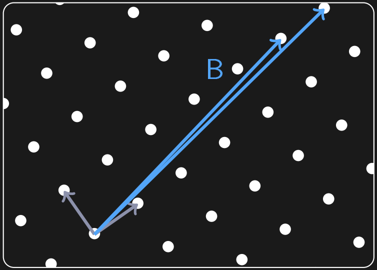

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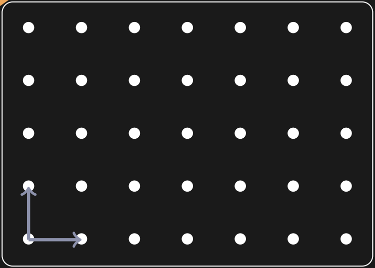

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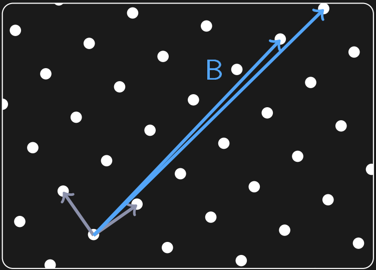

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Lattice Isomorphism Problem (LIP) assumption

recovering O from B is hard

$\Leftrightarrow$ computing a shortest basis of $\mathcal{L}$ is hard

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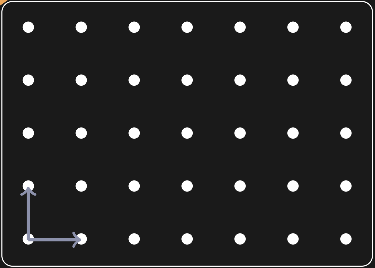

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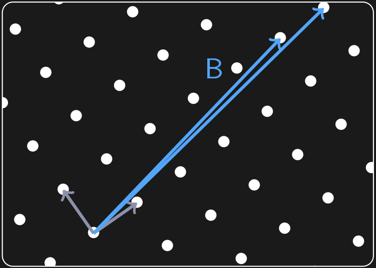

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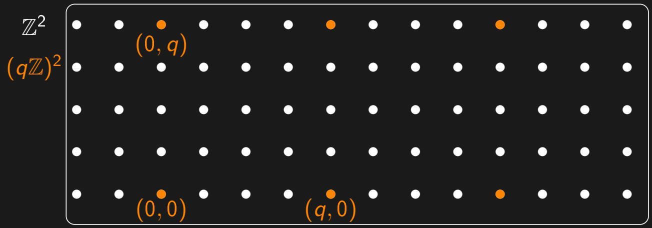
- Hawk: hash-and-sign + (module) LIP [DPPW23]

[DPPW23] Ducas, Postlethwaite, Pulles, van Woerden. Hawk: Module LIP makes lattice signatures [...] Asiacrypt

NTRU [HPS98]



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[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

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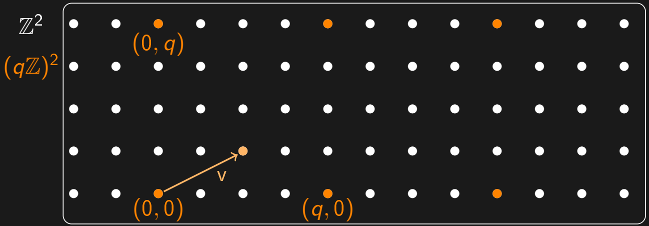

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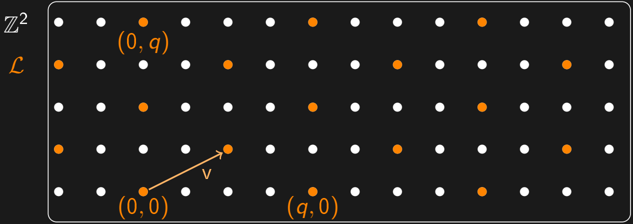

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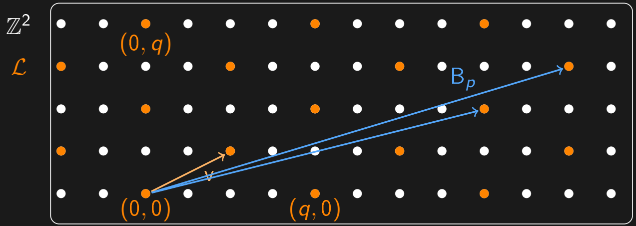

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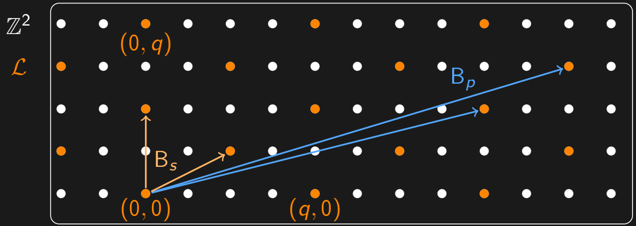

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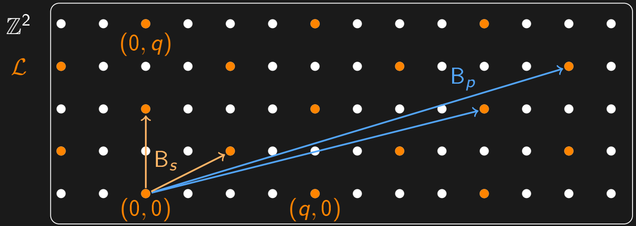

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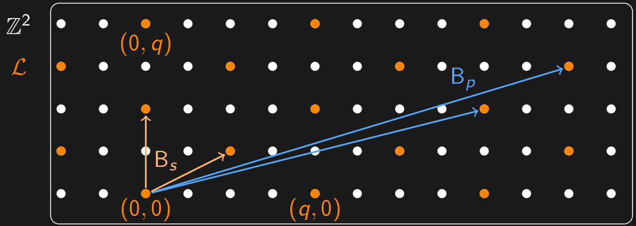

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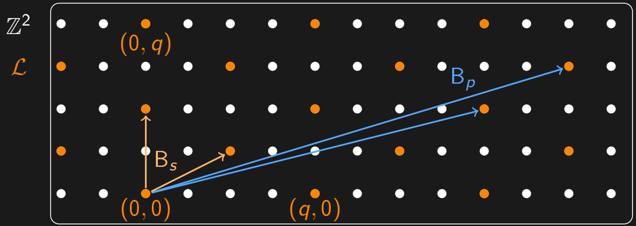

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- **Falcon:** hash-and-sign + NTRU

[Falcon] Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Prest, Ricosset, Seiler, Whyte, Zhang. NIST standard

Short Integer Solution [Ajt96]

Short Integer Solution (SIS) assumption

Let $\boxed{A} \leftarrow \mathcal{U}(\mathbb{Z}_q^{m\times n})$ $(m > n\log q)$ and

$$\mathcal{L}(A) := \{x \in \mathbb{Z}^m \,|\, xA = 0 \bmod q\}.$$

Finding a short basis of $\mathcal{L}(A)$ is hard with overwhelming probability.

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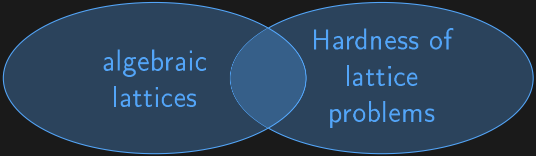
- GPV: hash-and-sign + SIS [GPV08]

[GPV08] Gentry, Peikert, Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. STOC

Conclusion

Some questions that interest me



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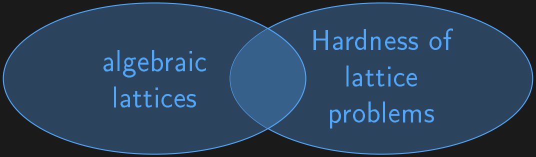

Some concrete questions: (come ask me if you want to know more)

- can we generate a random prime ideal $\mathfrak{p}$ in a number field K together with a short element in it?
- can we re-randomize an NTRU instance?

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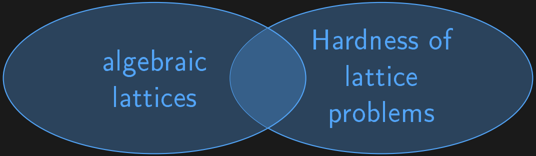

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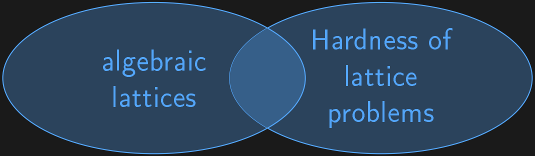

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Thank you