Automorphisms and derivations: the grassmannian case

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TNN grassmannian and postroid varieties

A point P in the grassmannian $\mathcal{G}_{kn}(\mathbb{R})$ is **totally nonnegative** if its Plücker coordinates can be represented by the $k \times k$ minors of a $k \times n$ matrix A such that each of these $k \times k$ minors are nonnegative.

Cells are specified by stating precisely which Plücker coordinates are zero. If $\mathcal{F}$ is a subset of Plücker coordinates then $S^\circ_{\mathcal{F}}$ is the cell where minors in $\mathcal{F}$ are zero (and those not in $\mathcal{F}$ are nonzero, so positive).

If $S^\circ_{\mathcal{F}} \neq \emptyset$, then $\mathcal{F}$ defines a so-called *postroid variety*.

Quantum postroids

L-Lenagan-Nolan Let $\mathcal{F}$ be a family of Plücker coordinates and $\mathcal{F}_q$ the corresponding family of quantum Plücker coordinates. TFAE

- The totally nonnegative cell associated to $\mathcal{F}$ in $\mathcal{G}_{kn}^{\mathsf{tnn}}$ is non-empty.
- $\mathcal{F}_q$ is the set of all quantum minors that belong to torus-invariant prime in $\mathcal{G}_q(k, n)$.

When q is transcendental, $\mathcal{F}_q$ generates a (completely) prime ideal. The corresponding quotient can be thought of as a quantum postroid.

Why do we care about tnn cells / positroids?

1. Link with soliton solutions of KP equation.

2. Link with scattering amplitudes in the $N = 4$ SYM model.

3. They are fun!!

Today's aim: compute invariants of (q-)positroids. We will be modest an look at a specific case when $\mathcal{F} = \emptyset$. In this case, the quantum positroid is just the quantum grassmannian and we would like to compute its automorphism group, its Hochschild cohomology, its irreducible representations, etc.

Quantum 2×2 matrices

The coordinate ring of quantum 2×2 matrices

$$\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := K \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is generated by four indeterminates a, b, c, d subject to the following rules:

$$ab = qba, \qquad cd = qdc$$

$$ac = qca, \qquad bd = qdb$$

$$bc = cb, \qquad ad - da = (q - q^{-1})cb.$$

The **quantum determinant** $ad - qbc$ is a central element

The algebra of $m \times p$ quantum matrices.

$$R = O_q(\mathcal{M}_{m,p}) := K \begin{bmatrix} Y_{1,1} & \dots & Y_{1,p} \\ \vdots & & \vdots \\ Y_{m,1} & \dots & Y_{m,p} \end{bmatrix},$$

where each 2×2 sub-matrix is a copy of $O_q(M(2))$.

$O_q(\mathcal{M}_{m,p})$ is an iterated Ore extension with the indeterminates $Y_{i,\alpha}$ adjoined in the lexicographic order and so is a noetherian integral domain.

In the square case $(m = p = n)$

$$D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)}$$

is the **quantum determinant**. D_q is a central element.

Quantum minors of quantum matrices

They are the quantum determinants of square sub-matrices of $O_q(\mathcal{M}_{m,p})$.

More precisely, if $I \subseteq [\![1,m]\!]$ and $\Lambda \subseteq [\![1,p]\!]$ with $\mid I \mid = \mid \Lambda \mid$, the **quantum minor** associated with the rows I and columns Λ is

$$[I \mid \Lambda] := D_q(\mathcal{O}_q(M_{I,\Lambda})).$$

For example, $[12|23] = Y_{1,2}Y_{2,3} - qY_{1,3}Y_{2,2}$ is the quantum minor of R associated with the rows $1,2$, and the columns $2,3$.

The quantum grassmannian $\mathcal{G}_q(k,n)$

The quantum grassmannian $\mathcal{G}_q(k,n)$ is the subalgebra of $O_q\left(\mathcal{M}_{k,n}\right)$ generated by the maximal $k \times k$ quantum minors

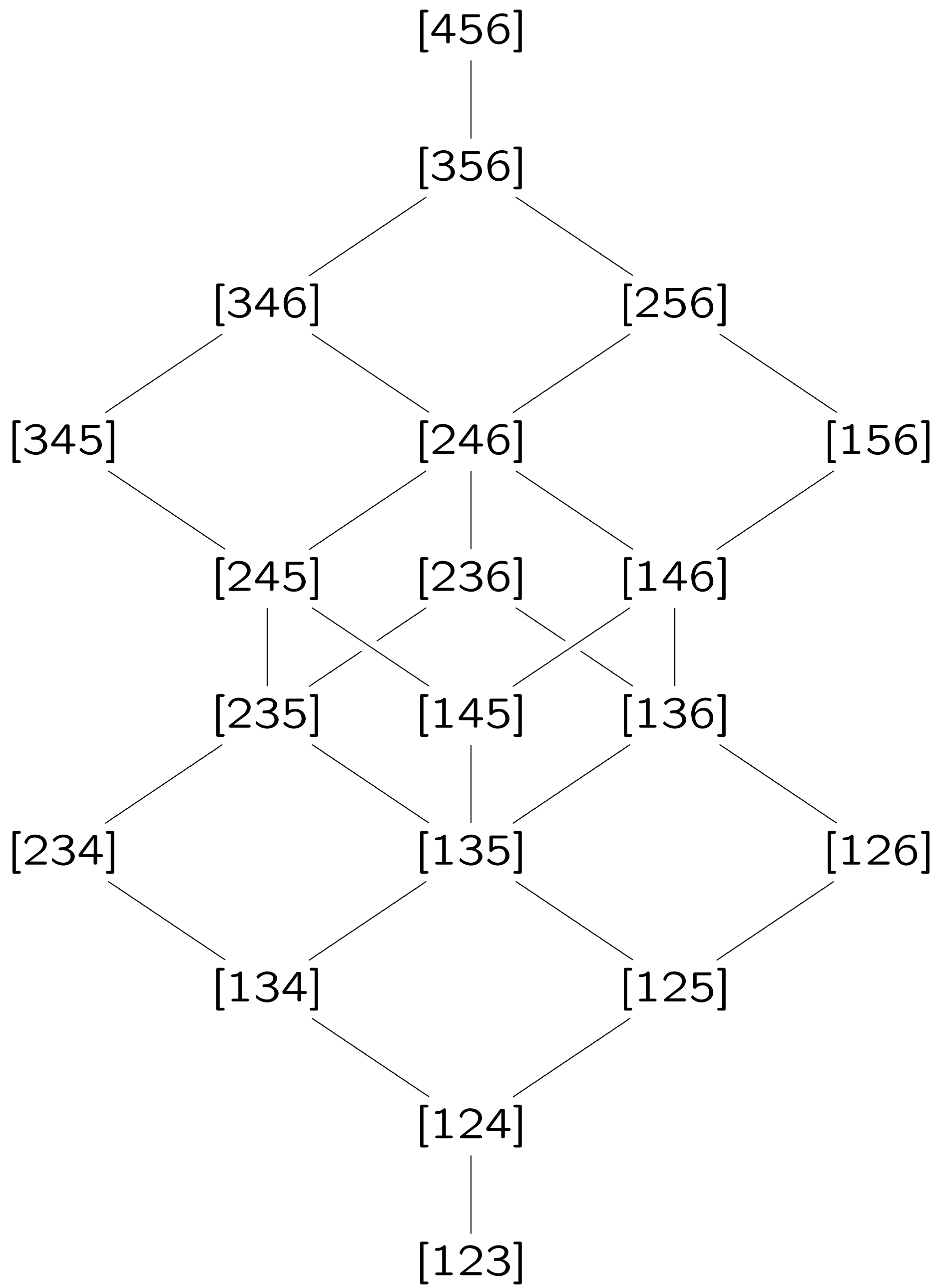
Denote by $[I]$ the quantum minor $[1 \ldots k|I]$. There is a torus action of $\mathcal{H} = (K^*)^n$ given by column multiplication. T

Example $\mathcal{G}_q(2,4)$ is generated by the six quantum minors $[12], [13], [14], [23], [24], [34]$.

Most minors $q^{\bullet}$-commute, for example, $[12]\,[34] = q^2\,[34]\,[12]$, however, $[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$ and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Partial order:
$[i_1 < \cdots < i_k] \leq [j_1 < \cdots < j_k]$ whenever $i_s \leq j_s$ for all s.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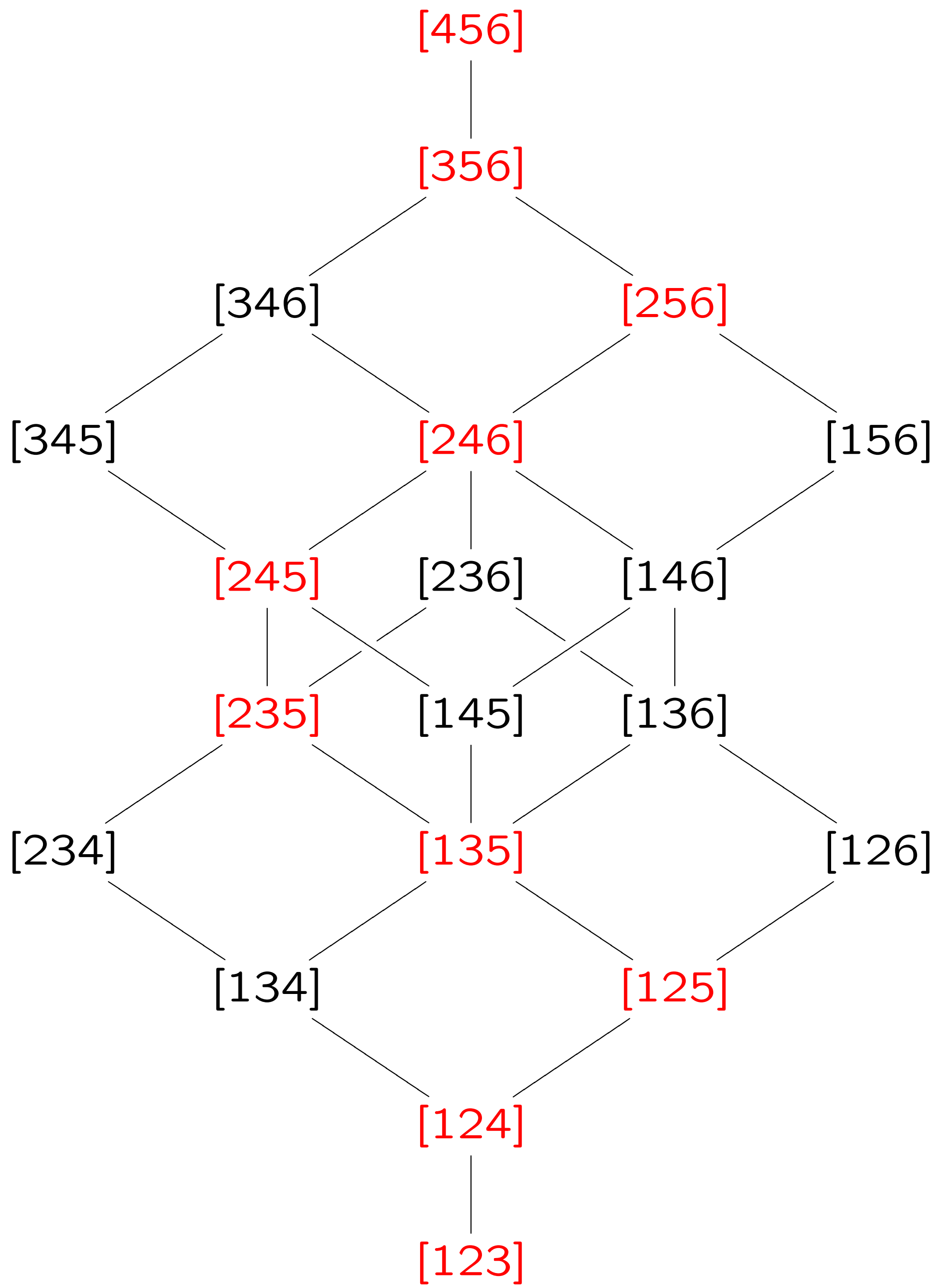
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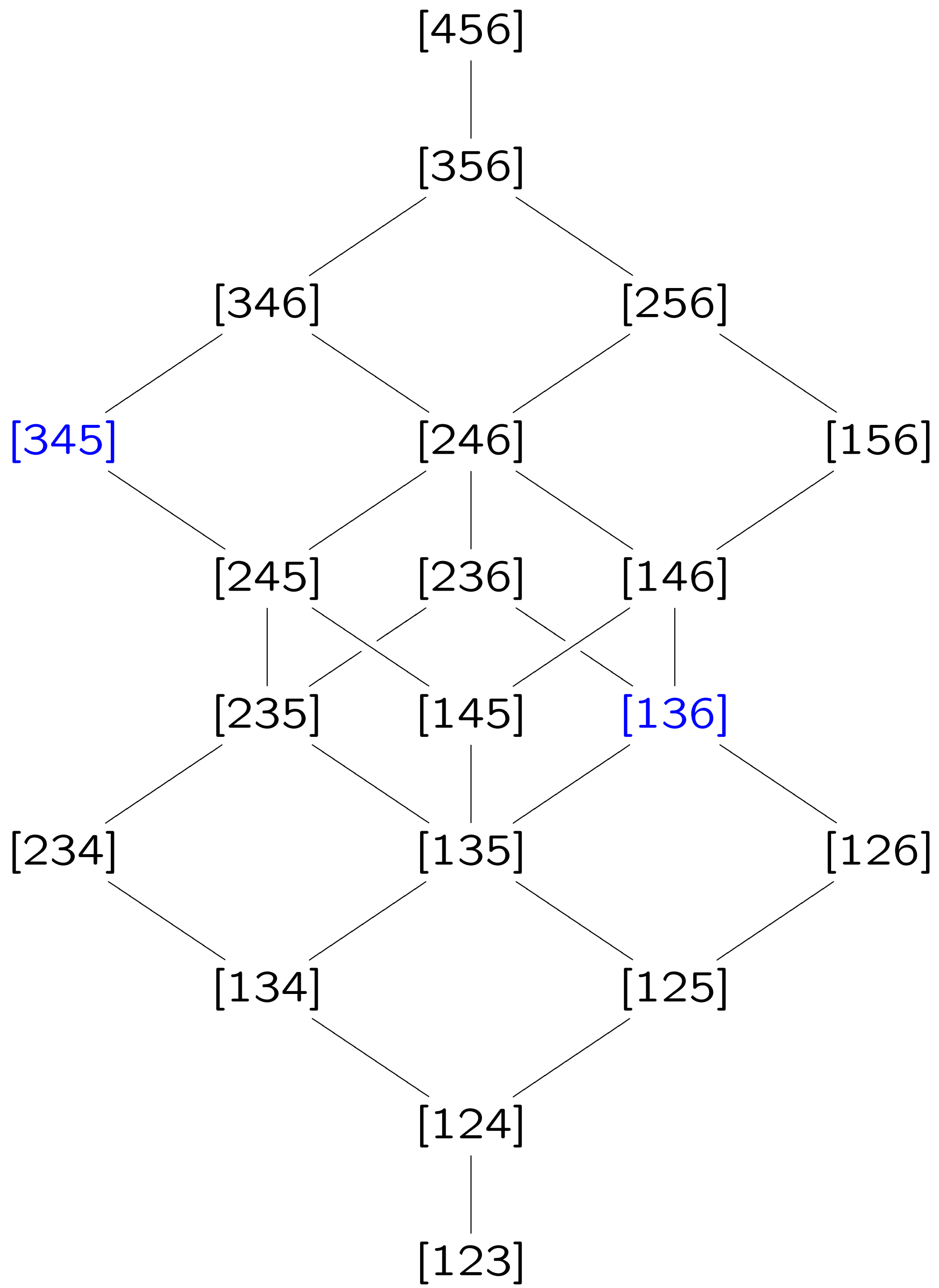
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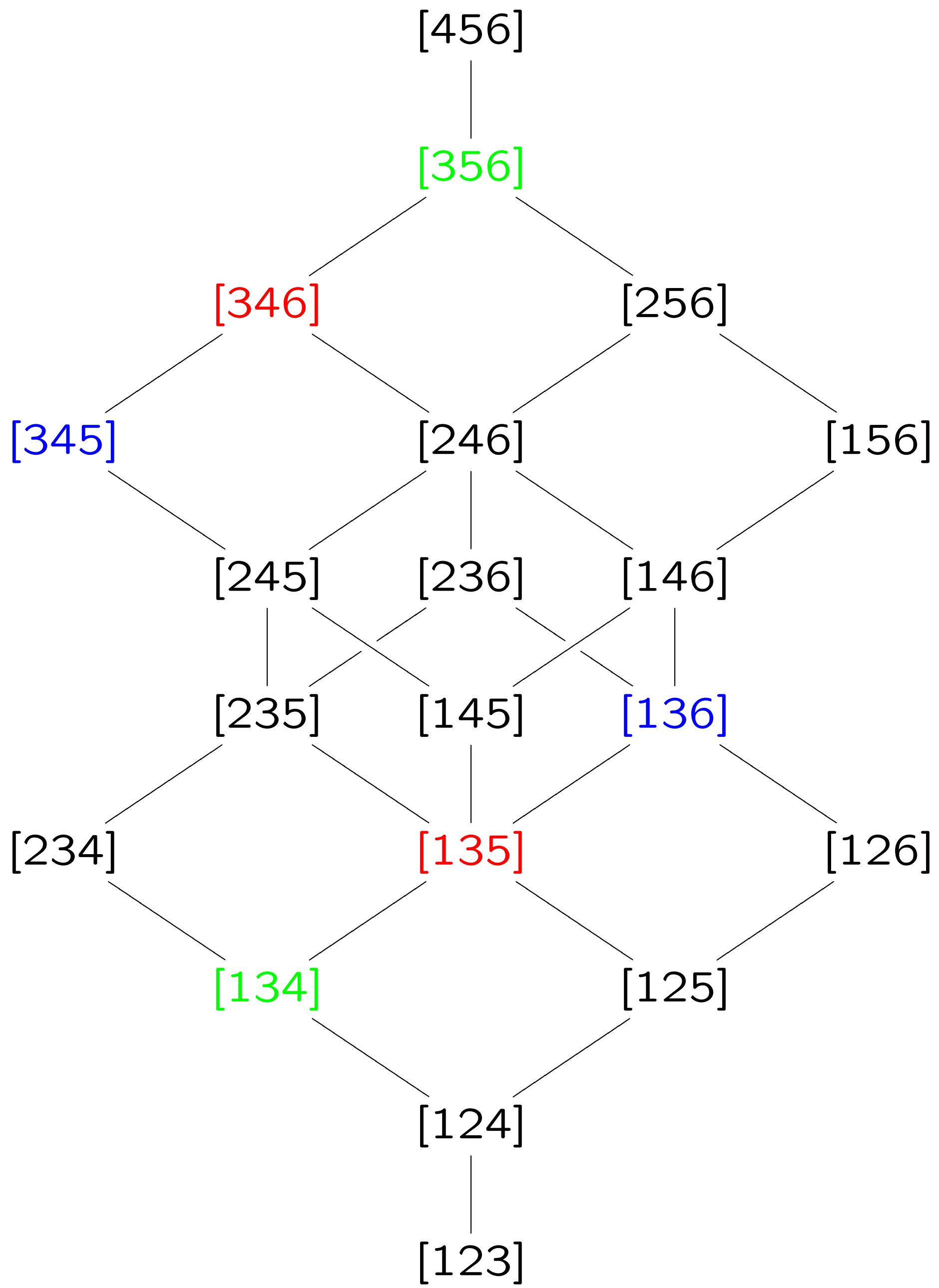
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Noncommutative dehomogenisation

- Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be an $\mathbb{N}$-graded algebra and $x \in R_1$ be a nonzerodivisor that is normal (ie. $xR = Rx$)

- Then $S := R[x^{-1}]$ is $\mathbb{Z}$-graded

- Set $\mathrm{Dhom}(R, x) := S_0 \quad (= R_0 + R_1x^{-1} + R_2x^{-2} + \dots)$, the **noncommutative dehomogenisation of** R **at** x.

- For $r \in R$, write $xr = \sigma(r)x$, with σ an automorphism of R.

- $R[x^{-1}] \cong \mathrm{Dhom}(R, x)[y, y^{-1}; \sigma]$ (where y is x in disguise)

Noncommutative dehomogenisation of $\mathcal{G}_q(k,n)$ at $[12...k]$

- In $\mathcal{G}_q(k,n)$ the quantum Plücker coordinate $u := [12...k]$ $q^\bullet$-commutes with each $[I]$ and so is normal. Consequently, the Ore localisation at the powers of u exists and

- **Theorem (Lenagan-Rigal)** Dhom$(\mathcal{G}_q(k,n), u) \cong O_q\left(\mathcal{M}_{k,n-k}\right)$

The dehomogenisation equality for $\mathcal{G}_q(k,n)$

- The dehomogenisation equality

$$\mathcal{G}_q(k,n)[u^{-1}] = O_q\left(\mathcal{M}_{k,n-k}\right)[y, y^{-1}; \sigma]$$

  can be used either to get properties of quantum matrices from the quantum grassmannian or, vice versa, to get properties of the quantum grassmannian from quantum matrices.

- Today, we use the known automorphism group of quantum matrices to calculate the automorphism group of the quantum grassmannian.

Obvious automorphisms of 2x2 quantum matrices

- Recall $O_q(\mathcal{M}_2) = K\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with

$$ab = qba, \quad cd = qdc, \quad ac = qca, \quad bd = qdb,$$

$$bc = cb, \quad ad - da = (q - q^{-1})bc.$$

- The torus $\mathcal{H} := (K^*)^4$ acts on $\mathcal{O}_q(\mathcal{M}_2)$ so that $h := (\alpha_1, \alpha_2; \beta_1, \beta_2)$ multiplies row i by α_i and column j by β_j

- Transposition (flip over the diagonal) gives an automorphism of $\mathcal{O}_q(\mathcal{M}_2)$ because b and c satisfy the same commutation rules.

Obvious automorphisms of quantum matrices

- Recall $O_q(\mathcal{M}_2) = K \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with

$$ab = qba, \quad cd = qdc, \quad ac = qca, \quad bd = qdb,$$

$$bc = cb, \quad ad - da = (q - q^{-1})bc.$$

- Similarly, the torus $\mathcal{H} := (K^*)^{k+n}$ acts by automorphisms on $\mathcal{O}_q(M_{kn})$ so that $h := (\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_n)$ multiplies row i by α_i and column j by β_j.

- When $k = n$, we also have the transpose automorphism.

The automorphism group of quantum matrices

From now on, q is not a root of unity.

- **Theorem** The automorphism group of $\mathcal{O}_q(\mathcal{M}_{m,n})$ is $\mathcal{H} := (K^*)^{(m+n)}$ when $m \neq n$, and $(K^*)^{2n} \rtimes \langle \tau \rangle$ when $m = n$

- **History:** Alev and Chamarie did the 2×2 case (1992).
  Conjecture of Andruskiewitsch-Dumas (2003)
  L-Lenagan: nonsquare case and the 3×3 case (2007, 2013).
  Yakimov:the $n \times n$ case in general (2013).

Obvious automorphisms of the quantum grassmannian $\mathcal{G}_q(k,n)$

- Recall that $\mathcal{O}_q(G_{kn})$ is the subalgebra of $\mathcal{O}_q(M_{kn})$ generated by the $k \times k$ quantum minors $[i_1 < \cdots < i_k]$.

- The torus $\mathcal{H} := (K^*)^n$ of column automorphisms of $\mathcal{O}_q(M_{kn})$ acts on $\mathcal{O}_q(G_{kn})$ by restriction so that

$$(h_1, \ldots, h_n) \circ [i_1 < \cdots < i_k] = h_{i_1} \cdots h_{i_k} [i_1 < \cdots < i_k]$$

Strategy for the quantum grassmannian

- Given any automorphism ρ of $\mathcal{O}_q(G_{kn})$ show that by adjusting ρ by elements of $\mathcal{H}$ we can assume that the quantum Plücker coordinates $[1 \dots k]$ and $[n-k+1 \dots n]$ are fixed by ρ.

- With this assumption, we may extend ρ to act on the left hand side of the dehomogenisation equality

$$\mathcal{O}_q(G_{kn})(u^{-1}) = \mathcal{O}_q(M_{k,n-k})[y, y^{-1}; \sigma]$$

  and this transfers to an action on the right hand side.

- In this equality, y and u are essentially the same element, and so ρ fixes y and a $k \times k$ quantum minor $(= [n-k+1 \dots n]u^{-1})$.

Strategy for the quantum grassmannian 2

- Now ρ acts on

$$\mathcal{O}_q(M_{k,n-k})[y, y^{-1}; \sigma]$$

  and fixes y.

- Show that ρ takes $\mathcal{O}_q(M_{k,n-k})$ to itself. Now we know how ρ acts on the right hand side of the dehomogenisation equality as we know the automorphism group of quantum matrices.

- Use the dehomogenisation equality

$$\mathcal{O}_q(G_{kn})(u^{-1}) = \mathcal{O}_q(M_{k,n-k})[y, y^{-1}; \sigma]$$

  to transfer this information back to $\mathcal{O}_q(G_{kn})$.

The automorphism group of the quantum grassmannian

- **Theorem** The automorphism group of $\mathcal{O}_q(G_{24})$ is $(K^*)^4 \rtimes \langle\tau\rangle$ where $h = (h_1, h_2, h_3, h_4)$ acts on $[ij]$ by multiplying by $h_i h_j$ and τ is the diagram automorphism which fixes $[12], [13], [24]$ and $[34]$ and interchanges $[14]$ and $[23]$.

- **Theorem** The automorphism group of $\mathcal{O}_q(G_{kn})$ is $(K^*)^n$ when $2k \neq n$ and $(K^*)^n \rtimes \langle\tau\rangle$ when $2k = n$ (here, τ is the diagram automorphism).

Automorphisms and grading

Let $A = \oplus_{i \in \mathbb{N}} A_i$ be a $\mathbb{N}$ -graded K-algebra with $A_0 = K$. Assume that A is a domain generated as an algebra by $x_1, \ldots, x_n$, and that $A_1 = Kx_1 + \cdots + Kx_n$. We set $A_{\geq d} := \oplus_{i \geq d} A_i$.

Assume that, for all i, there exist j and $q_{ij} \in K^*$ such that $x_i x_j = q_{ij} x_j x_i$.

Let σ be an automorphism of A and x be a nonzero homogeneous element of degree d of A.

Then $\sigma(x) = y_d + y_{>d}$, where $y_d \in A_d \setminus \{0\}$ and $y_{>d} \in A_{>d}$.

Automorphisms and normal elements

Lemma Let $A = \oplus_{i=0}^{\infty} A_i$ be a graded algebra that is a domain with A_0 equal to the base field and A generated in degree one. Suppose that $a = a_1 + \cdots + a_m$ is a normal element with $a_i \in A_i$ for each i. Then a_1 is a normal element.

Automorphisms and normal elements: UFD case

Chatters: An element p of a noetherian domain R is *prime* if (i) $pR = Rp$, (ii) pR is a height one prime ideal of R, and (iii) R/pR is an integral domain. A noetherian domain R is a *unique factorisation domain* if R has at least one height one prime ideal, and every height one prime ideal is generated by a prime element.

Lemma: Let $A = \oplus_{i=0}^{\infty} A_i$ be a graded algebra that is a domain with A_0 equal to the base field. Suppose also that A is a unique factorisation domain. Let a be a homogeneous element of degree one that is normal.
Then a generates a prime ideal of height one.

Proof Let P be a prime that is minimal over the ideal aR. By the noncommutative principal ideal theorem, the height of P is one. Hence, $P = pR$ for some normal element p, as R is a UFD.

Thus, a is a (right) multiple of p. By degree considerations, p must have degree one and a must be a scalar multiple of p. Thus, a and p generate the same ideal, which is the prime ideal P.

Back to $\mathcal{G}_q(k, n)$

Set $[u] = [1, \ldots, k]$. This is a prime normal element and $u[I] = q^{d(I)}[I]u$ with $d(I) :=\mid I \setminus (I \cap u) \mid$.

Lemma: Suppose that $a = \sum a_I[I] \neq 0$, with $a_I \in K$, is a linear combination of quantum Plücker coordinates that is a normal element. Then $d(I)$ is the same for each I that has $a_I \neq 0$.

Back to $\mathcal{G}_q(k,n)$

Lemma: Let ρ be an automorphism of $\mathcal{G}_q(k,n)$. Then $\rho([u])_1 = \lambda[u]$, for some $\lambda \in K^*$.

Lemma: Let ρ be an automorphism of $\mathcal{G}_q(k,n)$. Then $\rho([u]) = \lambda[u]$, for some $\lambda \in K^*$.

Set $[w] := [n-k+1,\ldots,n]$, the extreme rightmost quantum Plücker coordinate.

Corollary Let ρ be an automorphism of $\mathcal{G}_q(k,n)$. Then there exists $h \in \mathcal{H}$ such that $(h \circ \rho)([u]) = u$ and $(h \circ \rho)([w]) = [w]$.

Using dehomogenisation

Let ρ be an automorphism of $\mathcal{G}_q(k,n)$. Set $[u] = [1 \dots k]$ and $[w] = [n-k+1, \dots, n]$. At the expense of adjusting ρ by an element of $\mathcal{H}$, we can, and will, assume that $\rho([u]) = [u]$ and $\rho([w]) = [w]$.

The automorphism ρ now extends to $\mathcal{G}_q(k,n)[[u]^{-1}]$, and so to $\mathcal{O}_q(M(k,n-k))[y^{\pm 1};\sigma]$, by the dehomogenisation equality and we know that $\rho(y) = y$.

Proposition Assume $\rho([u]) = [u]$ and $\rho([w]) = [w]$. ρ extends to an automorphism of $\mathcal{G}_q(k,n)[[u]^{-1}] = \mathcal{O}_q(M(k,n-k))[y^{\pm 1};\sigma]$ such that $\rho(y) = y$ and $\rho(\mathcal{O}_q(M(k,n-k))) = \mathcal{O}_q(M(k,n-k))$.

Key point for previous proposition: two gradings

Recall $y = u$ and set $v = [w][u]^{-1}$

First, set $T_i := \{a \in T \mid yay^{-1} = q^i a\}$. One can easily check $x_{ij} \in T_1$.

Lemma (i) $T = \bigoplus_{i=1}^{\infty} T_i$ (ii) $\rho(T_i) = T_i$.

Set $T^{(i)} := \{a \in T \mid vav^{-1} = q^{-i}a\}$. It is easy to show that $x_{ij} \in T^{(0)} \cup T^{(1)}$, $y = u \in T^{(k)}$ and $y^{-1} \in T^{(-k)}$.

Lemma (i) $T = \bigoplus_{i \in \mathbb{Z}} T^{(i)}$ (ii) $\rho(T^{(i)}) = T^{(i)}$.

Lemma $\left(T^{(0)} \cup T^{(1)}\right) \cap T_1 \subseteq \mathcal{O}_q(M(k, n-k))$.

Derivations

The strategy can be adapted to compute derivations as well.

Let D be a derivation of $\mathcal{G}_q(k,n)$ and suppose that $D([I]) = b_0 + \cdots + b_t$ is the homogeneous decomposition of $D([I])$. Then $b_0 = 0$.

Let D be a derivation of $\mathcal{G}_q(k,n)$ where $2k \le n$, and let $[w] = [n-k+1,\ldots,n]$ be the rightmost quantum Plücker coordinate. Suppose that $D([u]) = a_1 + \cdots + a_s$ is the homogeneous decomposition of $D([u])$ and that $D([w]) = b_1 + \cdots + b_t$ is the homogeneous decomposition of $D([I])$. Then a_1 is a scalar multiple of $[u]$ and b_1 is a scalar multiple of $[w]$.

Derivations

For each $i = 1, \ldots, n$, there is a derivation D_i whose action on quantum Plücker coordinates is given by $D_i([I]) = \delta(i \in I)[I]$.

Let $2 \leq k \leq n-2$. Then any derivation of $\mathcal{G}_q(k,n)$ is equal, modulo inner derivations, to a linear combination of $D_1, \ldots, D_n$. Furthermore, these n derivations are linearly independent modulo the inner derivations.

Conjecture: $HH^1(U_q^+(\mathfrak{g}))$ is a free Z-module of rank the rank of $\mathfrak{g}$.