Arnol'd cat map lattices

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Séminaire de Physique Théorique
IDP, Tours
2 February 2023

Based on arXiv:2208.03267 and forthcoming.
In collaboration with Minos Axenides and Emmanuel Floratos; supported by CNRS IEA program "Chaotic behavior of closed quantum systems" under contract 318687.

Outline

What are Arnol'd cat map lattices?

They are lattices of Arnol'd cat maps, that are coupled between them, in a way defined by the structure of the lattice.

- What are Arnol'd cat maps?
- What does it mean that they're coupled in lattices?
- Why should we care?

What are Arnol'd cat maps?

There's just one Arnol'd cat map. It's defined by the mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} q \\ p \end{pmatrix}_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}_n \mod 1$$

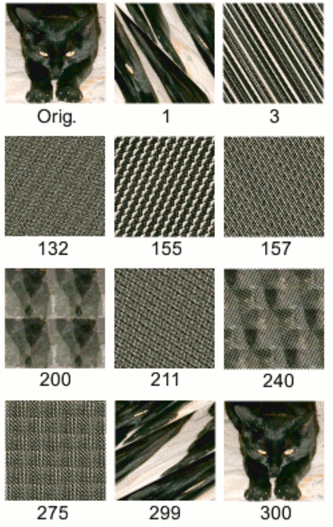
It can, also, equivalently, be defined as

$$\begin{pmatrix} q & , & p \end{pmatrix}_{n+1} = \begin{pmatrix} q & , & p \end{pmatrix}_n \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \mod 1$$

It was invented by V. I. Arnol'd in *Problèmes ergodiques de la mécanique classique* (1967).
The mod 1 operation implies that, in fact, it's a mapping from the 2-torus to itself: $\mathbb{T}^2 \rightarrow \mathbb{T}^2$.
The element "2" can be, either in the lower right or the upper left corner, it doesn't matter.

Why is it called "cat map"?

The "iterated cat" (The map acts on rational coordinates)



Cat maps, beyond Arnol'd

Applications, from $\mathbb{T}^2 \to \mathbb{T}^2$,

$$\left(\begin{array}{c} q \\ p \end{array}\right)_{n+1} = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\left(\begin{array}{c} q \\ p \end{array}\right)_n \mod 1$$

with $ad - bc = 1$ and $a + d > 2$ are, now, generically, called "cat maps".

What does it mean that they're coupled in lattices?

It means that, instead of having one map, from the 2-torus to the 2-torus, we have many maps, coupled between them–a many-body system; a field theory.
There are two ways to couple such maps:

- In phase space.
- In configuration space.

How to do so, in a particular way, that takes into account certain symmetries, is the subject of this talk.

Why should we care?

There are many reasons to care about the Arnol'd cat map in particular and coupling many of them, more generally.

- The Arnol'd cat map was the first system–proposed by Arnol'd in 1967–that displayed chaotic behavior in the simplest possible setting. In particular, it displayed *mixing*. It's the mod 1 that's crucial.
- It describes an accelerated observer in the near horizon geometry of an extremal black hole.
- Its dynamical properties can be used to describe the properties of the near horizon geometry itself.
- Many more applications in information theory most recently (steganography) as well as in fluid mechanics.
- It can be understood as describing an harmonic oscillator in an inverted potential, whose runaway behavior is cured by the periodic boundary conditions; which implies that chaotic systems can be imagined as Hamiltonian systems this way.

The symmetry in phase space: Covariance under symplectic transformations

The single Arnol'd cat map is an element of $\mathrm{SL}_2[\mathbb{Z}]$. This means that it is an element of $\mathrm{Sp}_2[\mathbb{Z}]$, since

$$\mathsf{J} = \mathsf{M}^{\mathrm{T}}\mathsf{J}\mathsf{M}$$

where

$$\mathsf{J} = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$$

and M is the Arnol'd cat map. (This property holds for any element of $\mathrm{SL}_2[\mathbb{Z}]$; it is, also, an element of $\mathrm{Sp}_2[\mathbb{Z}]$.) This identification no longer holds for more than one map: It's not true that an element of $\mathrm{SL}_{2n}[\mathbb{Z}]$ is, also, an element of $\mathrm{Sp}_{2n}[\mathbb{Z}]$.

What could be the guiding principle for constructing elements of $\mathrm{Sp}_{2n}[\mathbb{Z}]$, that we can identify as coupled Arnol'd cat maps?

Intermezzo on $\mathrm{Sp}_{2n}[\mathbb{Z}]$

A $2n \times 2n$ integer-valued matrix M is an element of $\mathrm{Sp}_{2n}[\mathbb{Z}]$ iff

$$\mathrm{J} = \mathrm{M}^{\mathrm{T}}\mathrm{J}\mathrm{M}$$

where now

$$\mathrm{J} = \begin{pmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix}$$

We can decompose M into blocks

$$\mathrm{M} = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix}$$

whence we deduce that

$$\begin{array}{l} \mathrm{A}^{\mathrm{T}}\mathrm{D} - \mathrm{C}^{\mathrm{T}}\mathrm{B} = I_{n\times n} \\ \mathrm{A}^{\mathrm{T}}\mathrm{C} = \mathrm{C}^{\mathrm{T}}\mathrm{A} \\ \mathrm{B}^{\mathrm{T}}\mathrm{D} = \mathrm{D}^{\mathrm{T}}\mathrm{B} \end{array}$$

Intermezzo on $\mathrm{Sp}_{2n}[\mathbb{Z}]$

We can, in fact, show that any matrix, M, element of $\mathrm{Sp}_{2n}[\mathbb{Z}]$, can be written as

$$\mathsf{M} = \begin{pmatrix} I_{n\times n} & 0 \\ \mathsf{S}_{\mathrm{L}} & I_{n\times n} \end{pmatrix} \begin{pmatrix} \mathsf{U}^{\mathrm{T}} & 0 \\ 0 & \mathsf{U}^{-1} \end{pmatrix} \begin{pmatrix} I_{n\times n} & \mathsf{S}_{\mathrm{R}} \\ 0 & I_{n\times n} \end{pmatrix}$$

where $\mathsf{S}_{\mathrm{R,L}}$ are integer symmetric matrices and U an invertible, integer matrix, which can be determined, given the matrices $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$, as follows:

$$\begin{aligned} \mathsf{U} &= \mathsf{A}^{\mathrm{T}} \\ \mathsf{S}_{\mathrm{L}} &= \mathsf{C}\mathsf{A}^{-1} \\ \mathsf{S}_{\mathrm{R}} &= \mathsf{A}^{-1}\mathsf{B} \end{aligned}$$

These relations hold, iff A is invertible, with integer entries. If this isn't the case, it is possible to redefine M so that A does have the desired properties.

Intermezzo on $\mathrm{Sp}_{2n}[\mathbb{Z}]$

For the single Arnol'd cat map we have $n = 1$ and

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} 1_{2\times 2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

How can we generalize this to more than one cat maps?

The relation between the Arnol'd cat map and the Fibonacci sequence

The key observation is that the Arnol'd cat map is related in a very special way to the Fibonacci sequence, defined by the recursion relation

$$f_{n+1} = f_n + f_{n-1}$$

with $f_0 = 1 = f_1$. (For other initial conditions, the integers, generated by this sequence, are known as the "Lucas numbers".)
The reason is that the Fibonacci sequence can be written in matrix form as

$$\left(\begin{array}{c} f_n \\ f_{n+1} \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right) \left(\begin{array}{c} f_{n-1} \\ f_n \end{array}\right) \equiv \mathcal{A} \left(\begin{array}{c} f_{n-1} \\ f_n \end{array}\right)$$

The relation between the Arnol'd cat map and the Fibonacci sequence

We note that $\det \mathcal{A} = -1$; $\mathcal{A}$ satisfies the relation

$$\mathcal{A}^{\mathrm{T}} \mathsf{J} \mathcal{A} = -\mathsf{J}$$

while

$$\mathcal{A}^2 = \mathsf{M}$$

the Arnol'd cat map, introduced previously– which implies that $\mathsf{M} \in \mathrm{Sp}_2[\mathbb{Z}]$: $\mathsf{M}^{\mathrm{T}} \mathsf{J} \mathsf{M} = \mathsf{J}$
Therefore we remark a very particular relation between the position and the momentum of the "particle", that evolves under the Arnol'd cat map, M :

$$\mathcal{A} \left(\begin{array}{c} f_n \\ f_{n+1} \end{array} \right) = \mathsf{M} \left(\begin{array}{c} f_{n-1} \\ f_n \end{array} \right) = \left(\begin{array}{c} f_{n+1} \\ f_{n+2} \end{array} \right)$$

They are both elements of the Fibonacci sequence! We don't need two sequences, $\{q_n\}$ and $\{p_n\}$, to label the points in phase space, on which the Arnol'd cat map acts, only one: The Fibonacci sequence!

The relation between the Arnol'd cat map and the Fibonacci sequence

Furthermore, we can prove, by induction, that

$$\mathrm{M}^n = \mathcal{A}^{2n} = \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix}$$

These properties mean that anything we can describe using the Arnol'd cat map, we can describe using the Fibonacci sequence and vice versa.
This implies, in particular, that we can describe the coupling between Arnol'd cat maps as the coupling between Fibonacci sequences!

Coupling Fibonacci sequences consistent with symplectic covariance

This can be achieved as follows: Write

$$f_{m+1} = a_1 f_m + b_1 f_{m-1} + c_1 g_m + d_1 g_{m-1}$$
$$g_{m+1} = a_2 g_m + b_2 g_{m-1} + c_2 f_m + d_2 f_{m-1}$$

where the a_i, b_i, c_i, d_i are integers, $f_0 = 0 = g_0$ and $f_1 = 1 = g_1$ are the initial conditions and $m = 1, 2, 3, \ldots$. As we shall deduce below, the coupling between $k-$Fibonacci sequences can be deduced as a special case. We wish to understand what constraints symplectic covariance imposes.

Coupling Fibonacci sequences consistent with symplectic covariance

We write these equations in the form:

$$X_{m+1} \equiv \begin{pmatrix} f_m \\ g_m \\ f_{m+1} \\ g_{m+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_1 & d_1 & a_1 & c_1 \\ d_2 & b_2 & c_2 & a_2 \end{pmatrix} \underbrace{\begin{pmatrix} f_{m-1} \\ g_{m-1} \\ f_m \\ g_m \end{pmatrix}}_{X_m}$$

Focus on

$$\mathrm{D} \equiv \begin{pmatrix} b_1 & d_1 \\ d_2 & b_2 \end{pmatrix} \qquad \mathrm{C} \equiv \begin{pmatrix} a_1 & c_1 \\ c_2 & a_2 \end{pmatrix}$$

and write the equations for the coupled Fibonacci sequences as

Coupling Fibonacci sequences consistent with symplectic covariance

$$X_{m+1} = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ \mathsf{D} & \mathsf{C} \end{pmatrix} X_m$$

Now impose the constraint, inspired by the corresponding property of the Fibonacci sequence

$$\begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ \mathsf{D} & \mathsf{C} \end{pmatrix}^{\mathrm{T}} \mathsf{J} \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ \mathsf{D} & \mathsf{C} \end{pmatrix} = -\mathsf{J}$$

which implies

$$\mathsf{D} = I_{n\times n} \qquad \mathsf{C} = \mathsf{C}^{\mathrm{T}}$$

Therefore $a_1 = k_1$, $a_2 = k_2$, $c_1 = c_2 = c$. In terms of these parameters, the recursion relations take the form

$$\begin{aligned} f_{m+1} &= k_1 f_m + f_{m-1} + c g_m \\ g_{m+1} &= k_2 g_m + g_{m-1} + c f_m \end{aligned}$$

Coupling Fibonacci sequences consistent with symplectic covariance: The $k-$Fibonacci sequence

This, in turn, can be identified as describing a particular coupling between a k_1- and a k_2-Fibonacci sequence. The $k-$Fibonacci sequence is the generalization of the Fibonacci sequence, defined by the relation

$$f_{n+1} = kf_n + f_{n-1}$$

where k is an integer.

Coupling Fibonacci sequences consistent with symplectic covariance

This particular coupling is determined by the condition that the square of the evolution matrix is an element of $\mathrm{Sp}_4[\mathbb{Z}]$:

$$\mathsf{A} = \left(\begin{array}{cc} 0 & 1 \\ 1 & \mathsf{C} \end{array} \right) \Rightarrow \mathsf{M} = \mathsf{A}^2 = \left(\begin{array}{cc} 1 & \mathsf{C} \\ \mathsf{C} & 1 + \mathsf{C}^2 \end{array} \right) \qquad (1)$$

The role of the coupling is played by the integer c.
We remark that, if $c = 0$ and $k_1 = k_2 = 1$, we recover two, decoupled, Arnol'd cat maps; if $c \neq 0$, and $k_1 = k_2 = 1$ we can, thereby, identify two "coupled" Arnol'd cat maps, while, If $k_1 = k_2 = k$, the system decouples into two, independent, $(k + c)$, resp. $(k - c)$-cat maps, for $f_m \pm g_m$.

Coupling Fibonacci sequences consistent with symplectic covariance

Now we can define the coupling matrix for n sequences as

$$\mathsf{C} = \mathsf{K} + \mathsf{PG} + \mathsf{GP}^{\mathrm{T}}$$

where K and G are diagonal matrices and P is the "shift" operator.
The corresponding $2n \times 2n$ evolution matrix, A is given by

$$\mathsf{A} = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ I_{n\times n} & \mathsf{C} \end{pmatrix}$$

and satisfies the relation $\mathsf{A}^{\mathrm{T}}\mathsf{JA} = -\mathsf{J}$.

Coupling Fibonacci sequences consistent with symplectic covariance

Its square,

$$\mathsf{M} = \mathsf{A}^2 = \left(\begin{array}{cc} I_{n\times n} & \mathsf{C} \\ \mathsf{C} & I_{n\times n} + \mathsf{C}^2 \end{array} \right)$$

therefore satisfies the relation $\mathsf{M}^{\mathrm{T}}\mathsf{J}\mathsf{M} = \mathsf{J}$, showing that $\mathsf{M} \in \mathrm{Sp}_{2n}[\mathbb{Z}]$. Since A is symmetric, (from the property that $\mathsf{C} = \mathsf{C}^{\mathrm{T}}$), M is positive definite and its eigenvalues come in pairs, $(\lambda, 1/\lambda)$, with $\lambda > 1$. This property implies that, for all matrices K and G this system of coupled maps is hyperbolic.

Coupling Fibonacci sequences consistent with symplectic covariance

It is possible to decompose the classical evolution matrix M in terms of the generators of the symplectic group

$$\mathrm{M} = \begin{pmatrix} I_{n\times n} & 0_{n\times n} \\ \mathrm{C} & I_{n\times n} \end{pmatrix} \begin{pmatrix} I_{n\times n} & \mathrm{C} \\ 0_{n\times n} & I_{n\times n} \end{pmatrix}$$

Moreover each factor generates, for any symmetric, integer, matrix C, an abelian subgroup of $\mathrm{Sp}_{2n}[\mathbb{Z}]$. These factors are called "left" (resp. "right") translations.
An important special case arises if we impose translation invariance along the chain, i.e. $K_I = K$ and $G_I = G$ for all $I = 1, 2, \ldots, n$.

Periodic behavior

Since the entries of M are integers, it acts in a particular way on rational points. A particularly interesting class of such points is defined by

$$\boldsymbol{x}_0 = (k_1/N, k_2/N, \ldots, k_n/N, l_1/N, l_2/N, \ldots, l_n/N)$$

where $0 \leq k_I \leq N, 0 \leq l_I \leq N$ and k_I, l_I, N integers.
On such points the map

$$(\boldsymbol{k}/N, \boldsymbol{l}/N)_{m+1} = (\boldsymbol{k}/N, \boldsymbol{k}/N)_m \mathrm{M} \bmod, 1$$

take the form

$$(\boldsymbol{k}, \boldsymbol{l})_{m+1} = (\boldsymbol{k}, \boldsymbol{l})_m \mathrm{M} \bmod N$$

The points $((\boldsymbol{k}, \boldsymbol{l}))$ belong to the $2n-$torus, $\mathbb{T}^{2n}[N]$.
The matrix $\mathrm{M} \bmod N \in \mathrm{Sp}_{2n}[\mathbb{Z}_N]$ and has period $T[N]$, where $T[N]$ is defined as the smallest integer such that

$$\mathrm{M}^{T[N]} \equiv I_{n\times n} \bmod N$$

The issue of the periods $T[N]$

The period, $T[N]$ controls the "mixing" properties of the map. So it is useful to know how it scales with N. What is fascinating is that it seems to be a "random" function of N,

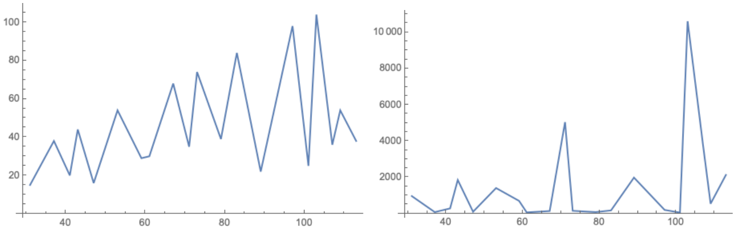


Figure: Period $T(N)$ for $N = p_{11}$ (the eleventh prime) to p_{31}, (the thirty-first prime) for $l = 1$, $n = 1$ and 2. We remark the dramatic change from $n = 1$, one map, to $n = 2$, two coupled maps. This reflects the dramatic increase in size of the order of the group.

The issue of the periods, $T[N]$

What is interesting is that, even for large N, the period $T[N]$ can be much smaller than N. Indeed, for the single Arnol'd cat map, Falk and Dyson showed (1982) that, when $N = f_q$, a Fibonacci number, that's a prime, then $T[N = f_q] = 2q$. This means very fast mixing.
How about coupled Arnol'd cat maps?

The period for coupled Arnol'd cat maps

The key observations:

- $$\mathcal{A}^{(n)} = \begin{pmatrix} 0 & I \\ I & \mathsf{C} \end{pmatrix} \Rightarrow \mathsf{M} = [\mathcal{A}^{(n)}]^2 = \begin{pmatrix} I & \mathsf{C} \\ \mathsf{C} & I + \mathsf{C}^2 \end{pmatrix}$$
- $$\mathsf{M}^m = \begin{pmatrix} \mathsf{C}_{2m-1} & \mathsf{C}_{2m} \\ \mathsf{C}_{2m} & \mathsf{C}_{2m+1} \end{pmatrix}$$
- The sequence of matrices C_m, is defined by the matrix C, the initial conditions $\mathsf{C}_0 = 0_{n\times n}, \mathsf{C}_1 = I_{n\times n}$ and the recursion relation

$$\mathsf{C}_{m+1} = \mathsf{C}\mathsf{C}_m + \mathsf{C}_{m-1}$$

The period for coupled Arnol'd cat maps

Therefore the period, $T[N]$, is determined by the relations $\mathsf{C}_{2T(N)} \equiv 0 \bmod N$, $\mathsf{C}_{2T(N)-1} \equiv I_{n\times n} \bmod N$ and $\mathsf{C}_{2T(N)+1} = \mathsf{C}\cdot\mathsf{C}_{2T(N)} + \mathsf{C}_{2T(N)-1} \equiv I_{n\times n} \bmod N$. The C_m are matrices, that are *matrix polynomials* in the matrix C; in fact they are the Fibonacci polynomials, defined by

$$\mathsf{F}_{m+1}(x) = x\mathsf{F}_m(x) + \mathsf{F}_{m-1}(x)$$

with initial conditions, $\mathsf{F}_0(x) = 0$ and $\mathsf{F}_1(x) = 1$, when the argument is a matrix!
Using these, instead of the evolution operator M, is more efficient, since C and the Fibonacci polynomials constructed from it, has half the size of M.

Locality in configuration space

Locality isn't an obvious property in phase space–it is much clearer what it means in configuration space, where the dynamical equations are those of Newton, rather than those of Hamilton. Straightforward calculation implies that these take the form

$$\boldsymbol{q}_{m+1} - 2\boldsymbol{q}_m + \boldsymbol{q}_{m-1} = \boldsymbol{q}_m \mathsf{C}^2 \bmod N$$

These describe *inverted* harmonic oscillators–but whose "runaway" behavior is "cured" by the periodic boundary conditions!

Measures of chaos: Lyapunov exponents

The Arnol'd cat map (as well as the many-body systems obtained by coupling many of these maps in the way presented above) is a chaotic system. This property can be highlighted by computing the spectrum of the Lyapunov exponents, available in closed form

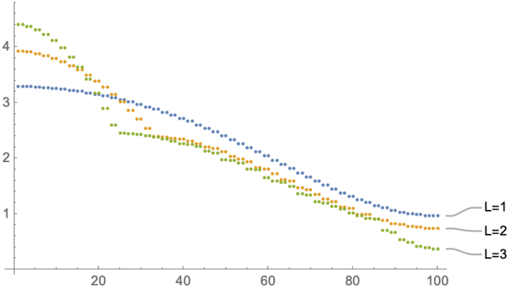


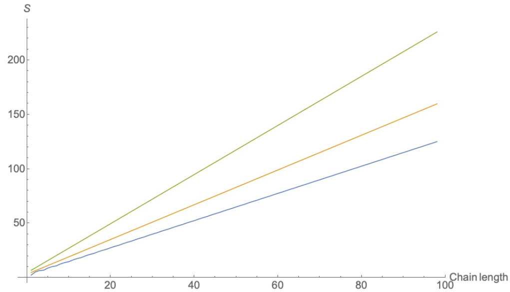
Figure: Histogram of the sorted Lyapunov spectra, $\lambda_{\pm}^{(L)}$, for uniform couplings, namely, $K = 3, G_I = G = 1, n = 101$ and $L = 1, 2, 3$.

Measures of chaos: The Kolmogorov-Sinai entropy

The Kolmogorov-Sinai entropy is, in fact, a *rate of change* of entropy. It is defined as the sum of the *positive* Lyapunov exponents

$$S_{\mathrm{KS}} = \sum_{I=0}^{n-1} \lambda_{+,I}$$

and is expected to scale as the volume of the system:



Measures of chaos

The reason it is useful, is that, by dimensional analysis, $1/S_{\mathrm{K-S}}$ has the dimensions of time–so it is expected that the mixing time should be proportional to $1/S_{\mathrm{K-S}}$; th only qestion being what is the proportionality constant. This isn't a trivial issue and, in general, we would expect that the mixing time is faster, the greater the K-S entropy, but there are known counterexamples depending on the choice of the initial probability distribution.
For black holes and quantum many-body systems, it has been conjectured that the mixing time is the scrambling time and these, in turn, are described by $1/S_{\mathrm{K-S}}$. What the proportionality coefficient is, remains to be found.

Conclusions and outlook

- How to couple Arnol'd cat maps isn't a well-defined statement, unless we specify what the symmetries are supposed to be. In this work we have shown how to couple n Arnol'd cat maps (as well as their generalizations), under the assumption that the evolution operator for the n maps is an element of $\mathrm{Sp}_{2n}[\mathbb{Z}]$ and showed how locality can be tuned for such systems.
  The rule is the following: The evolution operator, M, of n cat maps, that is an element of $\mathrm{Sp}_{2n}[Z]$, is given by the expression

$$\mathrm{M} = \left(\begin{array}{cc} I_{n\times n} & \mathrm{C} \\ \mathrm{C} & I_{n\times n} + \mathrm{C}^2 \end{array} \right)$$

  where C is an $n \times n$ symmetric matrix, that takes integer values. If the operations are done mod N, then the system has fascinating features.

Conclusions and outlook

- This construction is particularly appropriate for addressing the dynamics of $n-$body chaotic systems, that do not have an integrable limit. Nevertheless everything is under analytical control.
- They do show non-trivial conservation laws, that deserve closer scrutiny.
- This is, also, relevant for constructing toy models of the near horizon geometry itself of extremal black holes. They, already, pass, certain, non-trivial checks.

Conclusions and outlook

- These systems are classical; their quantum dynamics requires generalizing the construction for the single Arnol'd cat map, that describes single-particle probes of a fixed geometry.
- Their application to information processing has a broader range of applicability to (quantum) computing problems.
- These are lattice field theories, so their scaling limits are of particular interest, also!
- *The best is yet to come!*