Reconstruction of inclusions and cracks in Calderón's inverse conductivity problem

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June 14th, Toulouse

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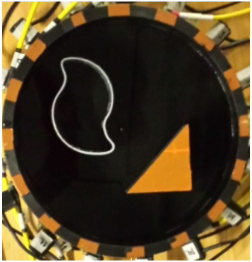
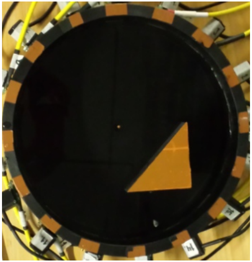
Electrical Impedance Tomography

Inverse problem of EIT: From several current–voltage measurements on surface electrodes, reconstruct the electrical conductivity distribution of an object.





The continuum model

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, bounded Lipschitz domain with connected complement
- $\Gamma \subseteq \partial\Omega$ arbitrarily small open boundary piece
- Conductivity $\gamma \in L^\infty_+(\Omega) := \{\varsigma \in L^\infty(\Omega;\mathbb{R}) \mid \inf \varsigma > 0\}$
- Boundary current density f
- Electric potential $u = u_f^\gamma$

Continuum Model

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega, \qquad \nu \cdot (\gamma \nabla u)|_{\partial\Omega} = \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

Local Neumann-to-Dirichlet map (current-to-voltage)

$$\Lambda(\gamma) : L^2_\diamond(\Gamma) \to L^2_\diamond(\Gamma), \quad f \mapsto u_f^\gamma|_\Gamma,$$

is compact and self-adjoint, with $\diamond$ denoting a zero-mean condition on Γ.

Forward and inverse problems

- **Forward problem**
  - $\Lambda : \gamma \mapsto \Lambda(\gamma)$

- **Calderón's inverse conductivity problem**

  For which classes of coefficients do we have:
  - $\Lambda(\gamma_1) = \Lambda(\gamma_2) \Rightarrow \gamma_1 = \gamma_2$?
  - An algorithm to evaluate $\Lambda^{-1} : \Lambda(\gamma) \mapsto \gamma$?

- **Inclusion/obstacle detection**

  Let $\gamma = \gamma_0 + \gamma_D$ and $D = \text{supp}(\gamma_D)$.
  - Can we reconstruct D from γ_0 and $\Lambda(\gamma)$?

Monotonicity inequalities

Lemma 1 (Kang–Seo–Sheen)

For $f \in L^2_\diamond(\Gamma)$ *and* $\gamma_1, \gamma_2 \in L^\infty_+(\Omega)$ *there are the following estimates:*

$$\int_\Omega \frac{\gamma_2}{\gamma_1}(\gamma_1 - \gamma_2)|\nabla u_f^{\gamma_2}|^2 \,\mathrm{d}x \leq \langle (\Lambda(\gamma_2) - \Lambda(\gamma_1))f, f\rangle_{L^2(\Gamma)} \leq \int_\Omega (\gamma_1 - \gamma_2)|\nabla u_f^{\gamma_2}|^2 \,\mathrm{d}x$$

This implies the following intuitive relation between conductivity and power:

$$\gamma_1 \geq \gamma_2 \text{ a.e. in } \Omega \qquad \Rightarrow \qquad \Lambda(\gamma_2) \geq \Lambda(\gamma_1).$$

Monotonicity-based reconstruction

- **Tamburrino–Rubinacci:** Bounds for inclusions using monotonicity inequalities.
- **Harrach–Ullrich:** For closed set C with connected complement, and for γ piecewise analytic with $-\beta_{\mathsf{L}} \leq \gamma - 1 \leq \beta_{\mathsf{U}}$:

$$\operatorname{supp}(\gamma - 1) \subseteq C \quad \text{if and only if} \quad \Lambda(1 - \beta_{\mathsf{L}}\chi_C) \geq \Lambda(\gamma) \geq \Lambda(1 + \beta_{\mathsf{U}}\chi_C).$$

Extreme inclusions

- For $\varsigma \in L^\infty_+(\Omega)$ and $C = C_0 \cup C_\infty$, let $\sigma = \sigma(\varsigma, C_0, C_\infty)$ denote

$$\sigma = \begin{cases} \varsigma & \text{in } \Omega \setminus C, \\ 0 & \text{in } C_0, \\ \infty & \text{in } C_\infty. \end{cases}$$

- The conductivity equation now becomes:

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \text{ in } \Omega \setminus C, \\ \nu \cdot (\sigma \nabla u) &= \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{on } \partial(\Omega \setminus C_0) \setminus \Gamma, \end{cases} \\ \nabla u &= 0 \text{ in } C_\infty^\circ, \\ \int_{\partial C_i} \nu \cdot (\sigma \nabla u)\, \mathrm{d}S &= 0 \text{ for each component } C_i \text{ of } C_\infty. \end{aligned}$$

Convergence of ND maps

Theorem 2

Define the ϵ-truncated version of σ, with $\epsilon > 0$, by

$$\sigma_\epsilon = \begin{cases} \varsigma & \text{in } \Omega \setminus C, \\ \epsilon\varsigma & \text{in } C_0, \\ \epsilon^{-1}\varsigma & \text{in } C_\infty. \end{cases}$$

Then the following estimate holds

$$\|u_f^{\sigma_\epsilon} - u_f^\sigma\|_{H^1(\Omega\setminus C_0)} \leq K\sqrt{\epsilon}\|f\|_{L^2(\Gamma)},$$

with $K > 0$ independent of f and ϵ. As a direct consequence

$$\|\Lambda(\sigma_\epsilon) - \Lambda(\sigma)\|_{\mathscr{L}(L^2_\diamond(\Gamma))} \leq K\sqrt{\epsilon}.$$

Corollary: There is an H^1-extension of u_f^σ onto the set C_0, satisfying

$$\|u_f^{\sigma_\epsilon} - u_f^\sigma\|_{H^1(\Omega)} \leq K\sqrt{\epsilon}\|f\|_{L^2(\Gamma)}.$$

Inverse problem for detection of extreme inclusions

- We denote a set of *admissible test inclusions* by

$$\mathcal{A} = \{C \Subset \Omega \mid C \text{ is the closure of an open set, has connected complement, and has Lipschitz boundary } \partial C\}.$$

- Let $\gamma_0 \in L^\infty_+(\Omega)$ satisfy the *unique continuation principle* (UCP), and suppose $0 \leq \gamma \leq \infty$ is measurable and $D = \operatorname{supp}(\gamma - \gamma_0) \in \mathcal{A}$.
- **Inverse problem:** Reconstruct D from knowledge of γ_0 and $\Lambda(\gamma)$.
- Some simplifying notation for some $C \in \mathcal{A}$:
  - $\Lambda_C^\emptyset = \Lambda(\sigma(\gamma_0, C, \emptyset))$
  - $\Lambda_\emptyset^C = \Lambda(\sigma(\gamma_0, \emptyset, C))$

General monotonicity method

- Suppose $D = D_0 \cup D_\infty \cup D_- \cup D_+$ is a disjoint union, with $D, D_0, D_\infty \in \mathcal{A}$ and $D_\pm$ are measurable sets.
- Define $0 \leq \gamma \leq \infty$ by

$$\gamma = \begin{cases} 0 & \text{in } D_0, \\ \infty & \text{in } D_\infty, \\ \gamma_- & \text{in } D_-, \\ \gamma_+ & \text{in } D_+, \\ \gamma_0 & \text{in } \Omega \setminus D. \end{cases}$$

- We assume γ satisfies a technical assumption near ∂D (next slide).

Theorem 3 (Candiani–Dardé–Garde–Hyvönen)

For any $C \in \mathcal{A}$, then

$$D \subseteq C \quad \textit{if and only if} \quad \Lambda_C^\emptyset \geq \Lambda(\gamma) \geq \Lambda_\emptyset^C.$$

Technical assumption near ∂D

- For each $x \in \partial D$, and every open neighbourhood W of x, there exists a relatively open set $V \subset D$ that intersects ∂D, and $V \subset \widetilde{D} \cap W$ for one set $\widetilde{D} \in \{D_0, D_\infty, D_-, D_+\}$.
  - If $\widetilde{D} = D_-$, there exists an open ball $B \subset V$ such that $\sup_B(\gamma_- - \gamma_0) < 0$.
  - If $\widetilde{D} = D_+$, there exists an open ball $B \subset V$ such that $\inf_B(\gamma_+ - \gamma_0) > 0$.

- **In non-technical terms:** The sign of $\gamma - \gamma_0$ cannot change arbitrarily often near any open part of ∂D. And, either a jump from γ_0 or a local strict increase or decrease from γ_0 near ∂D.

Illustration of numerical implementation

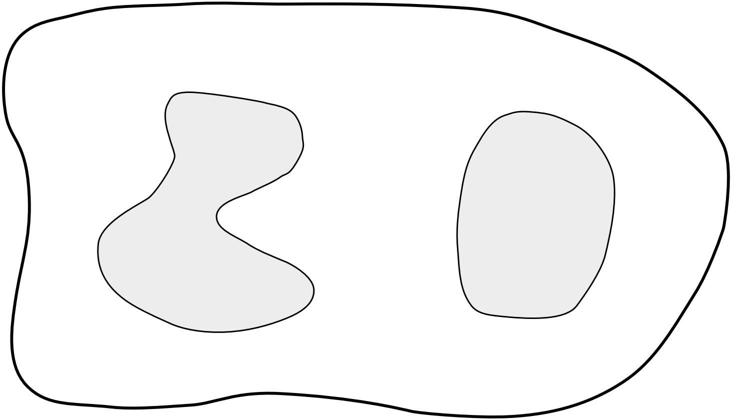
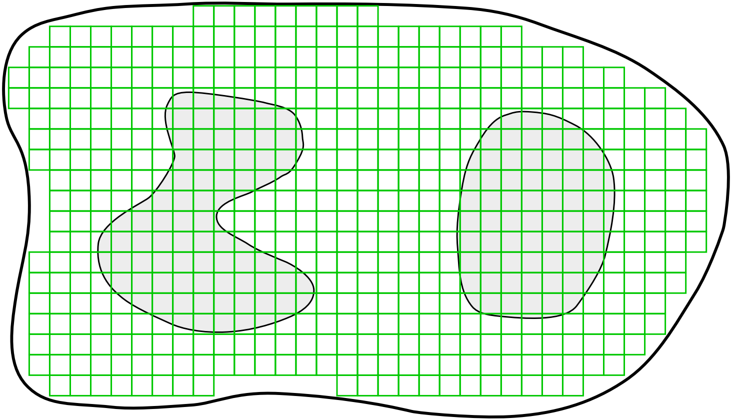


Illustration of numerical implementation



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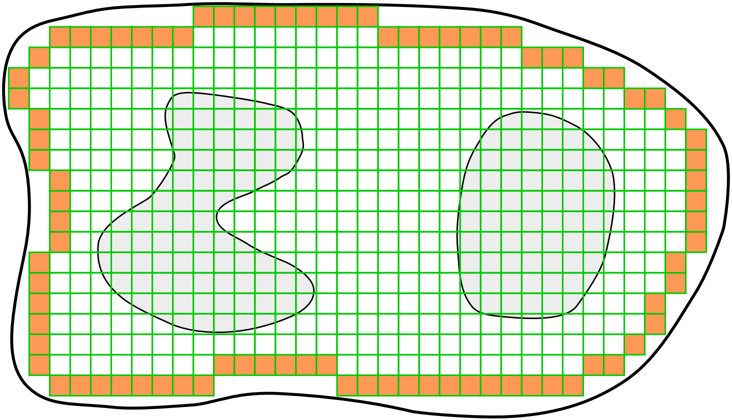


Illustration of numerical implementation

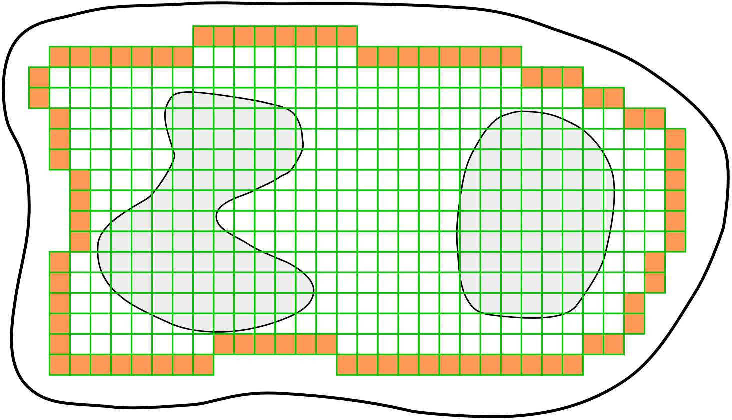


Illustration of numerical implementation

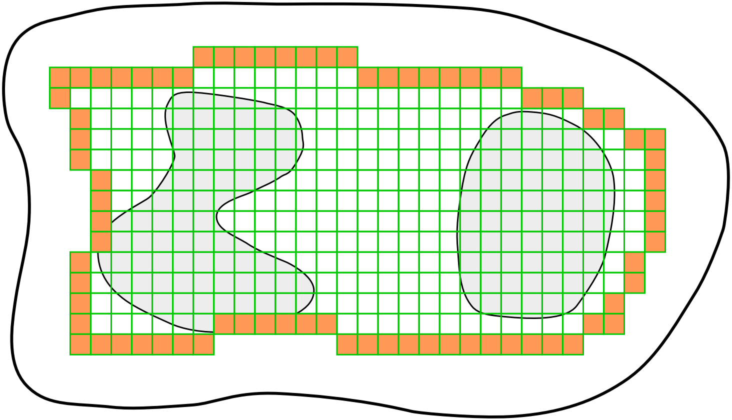
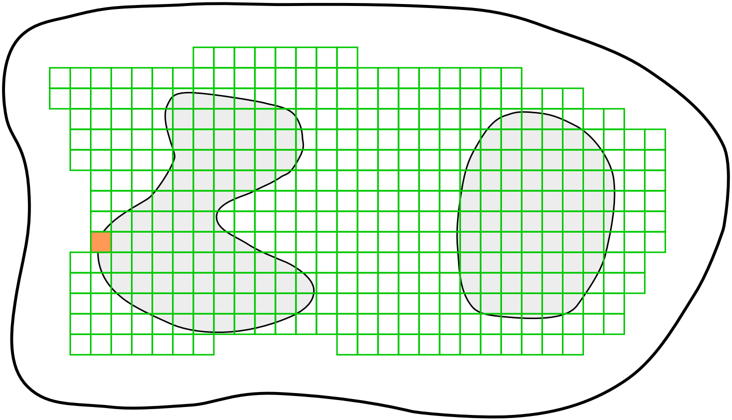


Illustration of numerical implementation



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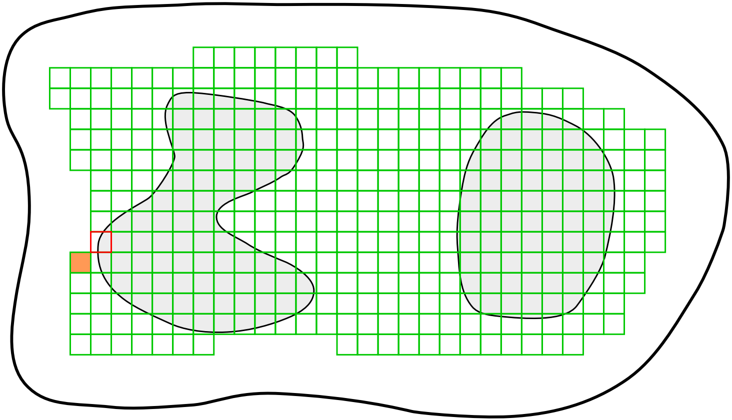

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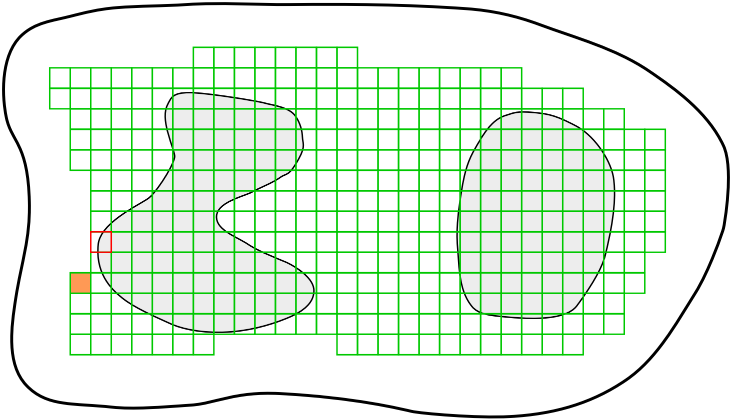
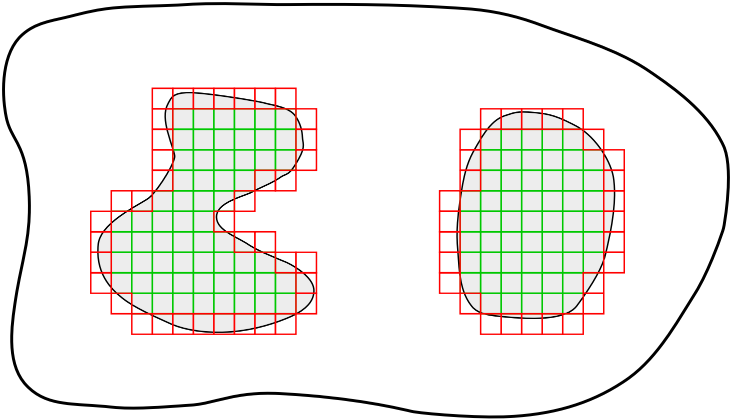


Illustration of numerical implementation



Proof of $D \subseteq C \Rightarrow \Lambda_C^\emptyset \geq \Lambda(\gamma) \geq \Lambda_\emptyset^C$

- Define the ϵ-truncation of γ, with $\epsilon > 0$, as

$$\gamma_\epsilon = \begin{cases} \epsilon\gamma_0 & \text{in } D_0, \\ \epsilon^{-1}\gamma_0 & \text{in } D_\infty, \\ \gamma_- & \text{in } D_-, \\ \gamma_+ & \text{in } D_+, \\ \gamma_0 & \text{in } \Omega \setminus D. \end{cases} \qquad \gamma_{\alpha,C} = \begin{cases} \alpha\gamma_0 & \text{in } C, \\ \gamma_0 & \text{in } \Omega \setminus C. \end{cases}$$

- Let $0 < \epsilon_0 < 1$ be small enough that $\epsilon_0\gamma_0 \leq \gamma$ in D_- and $\epsilon_0^{-1}\gamma_0 \geq \gamma$ in D_+.
- Assume $D \subseteq C$ and $0 < \epsilon \leq \epsilon_0$ then $\gamma_{\epsilon,C} \leq \gamma_\epsilon \leq \gamma_{\epsilon^{-1},C}$. By monotonicity:

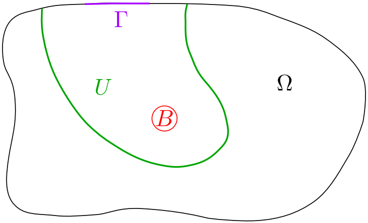
$$\Lambda(\gamma_{\epsilon,C}) \geq \Lambda(\gamma_\epsilon) \geq \Lambda(\gamma_{\epsilon^{-1},C}).$$

Letting $\epsilon \to 0$ gives $\Lambda_C^\emptyset \geq \Lambda(\gamma) \geq \Lambda_\emptyset^C$.

On the unique continuation principle

For $U \subseteq \overline{\Omega}$ relatively open and connected, we say that $\varsigma \in L^\infty_+(\Omega)$ satisfies the weak **unique continuation principle** (UCP) in U for the conductivity equation if:

- If $\nabla \cdot (\varsigma \nabla v) = 0$ in U° and $v \equiv 0$ in an open set $B \subseteq U$, then $v \equiv 0$ in U.
- If $\nabla \cdot (\varsigma \nabla v) = 0$ in U° with vanishing Cauchy data on $\partial U \cap \Gamma$, then $v \equiv 0$ in U.



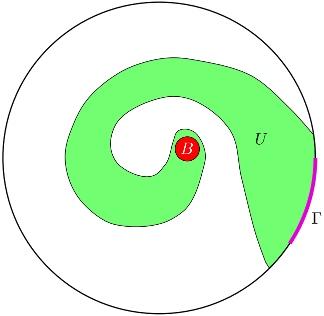
- This is e.g. satisfied for:
  - $d = 2$: L^∞_+
  - $d \geq 3$: $L^\infty_+ \cap$ Lipschitz
  - $d \geq 2$: Piecewise analytic (allows discontinuities)

Localised potentials

Lemma 4 (Gebauer)

Let $U \subset \overline{\Omega}$ be a relatively open connected set that intersects Γ. Let $B \subset U$ be an open set and assume $\varsigma \in L^\infty_+(\Omega)$ satisfies the UCP in U. Then there are sequences $(f_i) \subset L^2_\diamond(\Gamma)$ and $(u_i) \subset H^1_\diamond(\Omega)$ with $u_i = u^\varsigma_{f_i}$ such that

$$\lim_{i\to\infty} \int_B |\nabla u_i|^2 \, \mathrm{d}x = \infty \qquad \text{and} \qquad \lim_{i\to\infty} \int_{\Omega\setminus U} |\nabla u_i|^2 \, \mathrm{d}x = 0.$$



Simultaneous localization of power densities

Lemma 5

For $\varsigma \in L^\infty_+(\Omega)$ *and* $(f_i) \subset L^2_\diamond(\Gamma)$*, suppose that* $u_i = u^{\varsigma}_{f_i}$ *satisfies*

$$\lim_{i\to\infty} \int_B |\nabla u_i|^2 \,\mathrm{d}x = \infty \qquad \textit{and} \qquad \lim_{i\to\infty} \int_{\Omega\setminus U} |\nabla u_i|^2 \,\mathrm{d}x = 0.$$

If $C = C_0 \cup C_\infty$ *with* $C \subset \Omega \setminus \overline{U}$ *and* $\hat{u}_i = u^{\sigma}_{f_i}$ *with* $\sigma = \sigma(\varsigma, C_0, C_\infty)$*, then it also holds*

$$\lim_{i\to\infty} \int_B |\nabla \hat{u}_i|^2 \,\mathrm{d}x = \infty \qquad \textit{and} \qquad \lim_{i\to\infty} \int_{\Omega\setminus U} |\nabla \hat{u}_i|^2 \,\mathrm{d}x = 0.$$

If $\mathrm{supp}(\varsigma_1 - \varsigma_2) \subset \Omega \setminus \overline{U}$ the localisation for ς_1 is transferred to ς_2 (Harrach–Ullrich).

Improved monotonicity principles

Lemma 6

Let $C = C_0 \cup C_\infty$, $\varsigma, \varsigma_1, \varsigma_2 \in L^\infty_+(\Omega)$, *and* $f \in L^2_\diamond(\Gamma)$.

- *Different background conductivity (*$\sigma_1 = \sigma(\varsigma_1, C_0, C_\infty)$ *and* $\sigma_2 = \sigma(\varsigma_2, C_0, C_\infty)$*):*

$$\int_{\Omega\setminus C} \frac{\varsigma_2}{\varsigma_1}(\varsigma_1 - \varsigma_2)|\nabla u_2|^2 \, \mathrm{d}x \leq \langle (\Lambda_2 - \Lambda_1) f, f\rangle_{L^2(\Gamma)} \leq \int_{\Omega\setminus C} (\varsigma_1 - \varsigma_2)|\nabla u_2|^2 \, \mathrm{d}x.$$

- *With and without perfectly conducting inclusions (*$\sigma_1 = \sigma(\varsigma, C_0, C_\infty)$ *and* $\sigma_2 = \sigma(\varsigma, C_0, \emptyset)$*):*

$$\int_{C_\infty} \varsigma|\nabla u_2|^2 \, \mathrm{d}x \leq \langle (\Lambda_2 - \Lambda_1) f, f\rangle_{L^2(\Gamma)} \leq K \int_{C_\infty} |\nabla u_2|^2 \, \mathrm{d}x,$$

  where $K > 0$ *is independent of* f.

- *With and without perfectly insulating inclusions (*$\sigma_1 = \sigma(\varsigma, \emptyset, C_\infty)$ *and* $\sigma_2 = \sigma(\varsigma, C_0, C_\infty)$*):*

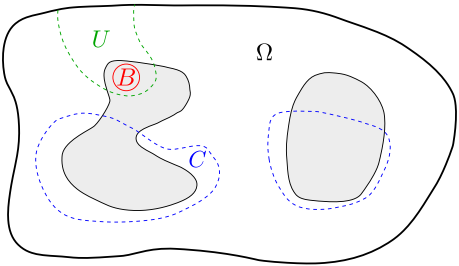
$$\int_{C_0} \varsigma|\nabla u_1|^2 \, \mathrm{d}x \leq \langle (\Lambda_2 - \Lambda_1) f, f\rangle_{L^2(\Gamma)} \leq \int_{C_0} \varsigma|\nabla u_2|^2 \, \mathrm{d}x.$$

Proof of $D \not\subseteq C \Rightarrow \neg(\Lambda_C^{\emptyset} \geq \Lambda(\gamma) \geq \Lambda_{\emptyset}^{C})$

- Assume $D \not\subseteq C$, i.e. $D \setminus C$ contains an open ball B that can be connected to Γ via a relatively open connected set $U \subset \overline{\Omega}$.
- We may assume that $\overline{U} \cap C = \emptyset$ and either of the following four options holds:

$$\text{(a): } \overline{U} \cap (D \setminus D_+) = \emptyset, \qquad \text{(b): } \overline{U} \cap (D \setminus D_-) = \emptyset,$$
$$\text{(c): } \overline{U} \cap (D \setminus D_\infty) = \emptyset, \qquad \text{(d): } \overline{U} \cap (D \setminus D_0) = \emptyset.$$

- In the following we consider case (d).



Proof of $D \not\subseteq C \Rightarrow \neg(\Lambda_C^{\emptyset} \geq \Lambda(\gamma) \geq \Lambda_{\emptyset}^{C})$

- Recall the definition of γ, and now introduce also some new auxiliary conductivities:

$$\gamma := \begin{cases} 0 & \text{in } D_0 \\ \infty & \text{in } D_\infty \\ \gamma_- & \text{in } D_- \\ \gamma_+ & \text{in } D_+ \\ \gamma_0 & \text{in } \Omega \setminus D \end{cases} \qquad \gamma_1 := \begin{cases} \gamma_- & \text{in } D_- \\ \gamma_+ & \text{in } D_+ \\ \gamma_0 & \text{in } \Omega \setminus (D_- \cup D_+) \end{cases}$$

  and $\gamma_2 := \sigma(\gamma_1, \emptyset, D_\infty)$ and $\gamma_C := \sigma(\gamma_0, C, \emptyset)$.
- We will now estimate each of the following terms:

$$\Lambda_C^{\emptyset} - \Lambda(\gamma) = [\Lambda_C^{\emptyset} - \Lambda(\gamma_0)] + [\Lambda(\gamma_0) - \Lambda(\gamma_1)] + [\Lambda(\gamma_1) - \Lambda(\gamma_2)] + [\Lambda(\gamma_2) - \Lambda(\gamma)]$$

- Let (f_i) simultaneously localize potentials $u_{0,i}$, $u_{1,i}$, $u_{2,i}$, and $u_{C,i}$ in B along the set U.

Proof of $D \not\subseteq C \Rightarrow \neg(\Lambda_C^{\emptyset} \geq \Lambda(\gamma) \geq \Lambda_{\emptyset}^{C})$

- Using the improved monotonicity inequalities:

$$\langle(\Lambda_C^{\emptyset} - \Lambda(\gamma_0))f_i, f_i\rangle_{L^2(\Gamma)} \leq \sup_C(\gamma_0) \int_C |\nabla u_{C,i}|^2 \, \mathrm{d}x \to 0$$

$$\langle(\Lambda(\gamma_0) - \Lambda(\gamma_1))f_i, f_i\rangle_{L^2(\Gamma)} \leq \sup_{D_- \cup D_+}(\gamma_1 - \gamma_0) \int_{D_- \cup D_+} |\nabla u_{0,i}|^2 \, \mathrm{d}x \to 0$$

$$\langle(\Lambda(\gamma_1) - \Lambda(\gamma_2))f_i, f_i\rangle_{L^2(\Gamma)} \leq K \int_{D_\infty} |\nabla u_{1,i}|^2 \, \mathrm{d}x \to 0$$

$$\langle(\Lambda(\gamma_2) - \Lambda(\gamma))f_i, f_i\rangle_{L^2(\Gamma)} \leq -\inf_{D_0}(\gamma_0) \int_{D_0} |\nabla u_{2,i}|^2 \, \mathrm{d}x \to -\infty$$

- In total this gives $\lim_{i\to\infty} \langle(\Lambda_C^{\emptyset} - \Lambda(\gamma))f_i, f_i\rangle_{L^2(\Gamma)} = -\infty$, i.e. $\Lambda_C^{\emptyset} \not\geq \Lambda(\gamma)$.

Degenerate and singular inclusions

Extension of the method with Hyvönen:

- A nonnegative function w on $\mathbb{R}^d$ is called an A_2 Muckenhoupt weight, if w and $1/w$ are locally integrable and satisfy

$$\exists C > 0,\ \forall B \text{ open ball in } \mathbb{R}^d : \left(⨍_B w \, \mathrm{d}x \right) \left(⨍_B \frac{1}{w} \, \mathrm{d}x \right) \leq C.$$

- If Σ is a Lipschitz hypersurface, then w can locally behave as

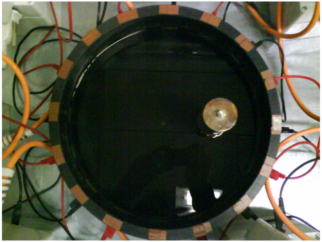
$$\text{dist}(\,\cdot\,, \Sigma)^s, \quad s \in (-1, 1).$$

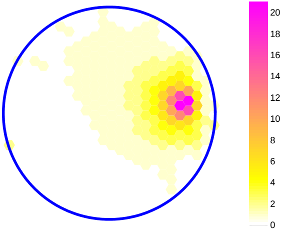
  Or near a point x_0 as

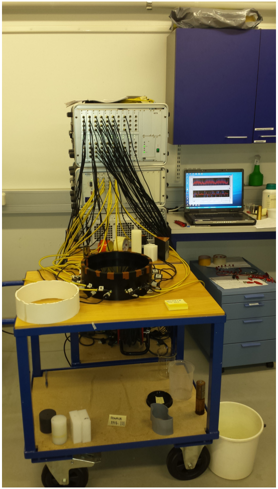
$$\text{dist}(\,\cdot\,, x_0)^s, \quad s \in (-d, d).$$

- We can allow γ to be the restriction of an A_2 weight in the interior of D, and still recover D with the Monotonicity Method.

Examples - real data







Kuopio impedance tomograph

Lipschitz cracks

Definition 7

A collection of cracks χ lies in the class $\mathcal{X}$ if for some $N \in \mathbb{N}_0$

$$\chi = \bigcup_{i=1}^{N} \sigma_i,$$

where the $\sigma_i \subset \Omega$ are $(d-1)$ dimensional orientable Lipschitz surfaces with non-empty Lipschitz boundary $\partial\sigma_i$, and with

$$\text{dist}(\sigma_i, \sigma_j) > 0 \text{ for } i \neq j \text{ and } \text{dist}(\sigma_i, \partial\Omega) > 0 \text{ for all } i.$$

We refer to $D \in \mathcal{X}$ as "a (D_0, D_∞) collection of cracks" if:

- $D = D_0 \cup D_\infty$ for $D_0, D_\infty \in \mathcal{X}$,
- $\text{dist}(D_0, D_\infty) > 0$,
- each crack in D_0 is perfectly insulating,
- and each crack in D_∞ is perfectly conducting.

Conductivity problem with cracks

Conductivity problem where D is a (D_0, D_∞) collection of cracks.

$$
\begin{aligned}
-\nabla \cdot (\gamma_0 \nabla u) &= 0 \quad \text{in } \Omega \setminus \overline{D}, \\
\gamma_0 \frac{\partial u}{\partial \nu} &= \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{on } \partial\Omega \setminus \overline{\Gamma}, \end{cases} \\
\gamma_0 \frac{\partial u}{\partial n} &= 0 \quad \text{on } D_0, \\
u &\text{ is locally constant on } D_\infty, \\
\int_{D_i} \left[\gamma_0 \frac{\partial u}{\partial n} \right] \mathrm{d}S &= 0 \quad \text{for each component } D_i \text{ of } D_\infty.
\end{aligned}
$$

Conductivity problem with cracks

- Weak problem for electric potential $u = u^{D_\infty}_{D_0,f}$:

$$\int_\Omega \gamma_0 \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Gamma f v|_\Gamma \, \mathrm{d}S, \qquad \forall v \in \mathcal{H}^{D_\infty}_{D_0}.$$

where

$$\mathcal{H}^{D_\infty}_{D_0} = \{v \in H^1_\diamond(\Omega \setminus \overline{D_0}) \mid v \text{ is locally constant on } D_\infty\}.$$

Note that ∇u extends to an L^2-function in all of Ω.

- The local ND map is denoted $\Lambda^{D_\infty}_{D_0}$.

Conductivity problem with cracks

A different way to understand the crack problems:

$$\mathcal{H}_{\emptyset}^{D_\infty} \subseteq \mathcal{H}_{D_0}^{D_\infty} \subseteq \mathcal{H}_{D_0}^{\emptyset}.$$

And in the inner product $\langle u, v\rangle_* = \int_\Omega \gamma_0 \nabla u \cdot \nabla v \, \mathrm{d}x$:

- $u_{\emptyset,f}^{D_\infty}$ is the orthogonal projection of $u_{D_0,f}^{D_\infty}$ onto $\mathcal{H}_{\emptyset}^{D_\infty}$.
- $u_{D_0,f}^{D_\infty}$ is the orthogonal projection of $u_{D_0,f}^{\emptyset}$ onto $\mathcal{H}_{D_0}^{D_\infty}$.

Monotonicity reconstruction of cracks

Theorem 8 (Garde–Vogelius)

Let D be a (D_0, D_∞) collection of cracks. Given any $C \in \mathcal{A}$, then

$$D \subset C \qquad \text{if and only if} \qquad \Lambda_C^{\emptyset} \geq \Lambda_{D_0}^{D_\infty} \geq \Lambda_{\emptyset}^{C}.$$

Theorem 9 (Garde–Vogelius)

Let $D \in \mathcal{X}$.

- *Given any $\chi \in \mathcal{X}$, then*

$$\chi \subseteq D \qquad \text{if and only if} \qquad \Lambda_D^{\emptyset} \geq \Lambda_{\chi}^{\emptyset}.$$

- *Given any $\chi \in \mathcal{X}$, then*

$$\chi \subseteq D \qquad \text{if and only if} \qquad \Lambda_{\emptyset}^{\chi} \geq \Lambda_{\emptyset}^{D}.$$

Monotonicity reconstruction of cracks

More complicated setting:

- No open set inside inclusion to localise in.
- Usual monotonicity inequalities become trivial when collapsing inclusions to zero volume.
- Less general uniqueness results; we now assume $\gamma_0 \in C^2(\overline{\Omega})$ and positive.

Monotonicity reconstruction of cracks

The key idea for the “difficult direction” of the proof, is to construct:

- Sequence of Neumann boundary values (f_n),
- Sequence of potentials $\widetilde{u}_n = u^{\Sigma_\infty}_{\Sigma_0, f_n}$,
- Sequence of potentials $\widehat{u}_n = u^{\Sigma_\infty}_{\emptyset, f_n}$,

such that $\widetilde{u}_n$ and $\widehat{u}_n$ localise (blow up) in a set intersecting Σ_∞, and *also* the difference

$$\widetilde{u}_n - \widehat{u}_n$$

localise in the same way.

Localised potentials

We will need the constructive version of localised potentials.

Lemma 10 (Gebauer '08)

Let H, K_1, and K_2 be Hilbert spaces, let $A_j \in \mathscr{L}(K_j, H)$ for $j = 1, 2$, and assume that A_2^* is injective. Assume that there exists $y_0 \in R(A_1)$ such that $y_0 \notin R(A_2)$. For $n \in \mathbb{N}$ we define

$$\xi_n = \left(A_2 A_2^* + \tfrac{1}{n} I\right)^{-1} y_0$$

and

$$x_n = \frac{\xi_n}{\|A_2^* \xi_n\|_{K_2}^{3/2}}.$$

Then

$$\lim_{n\to\infty} \|A_1^* x_n\|_{K_1} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \|A_2^* x_n\|_{K_2} = 0.$$

An auxiliary operator

- Let $V \in \mathcal{A}$.
- Let Σ be a $(\Sigma_0, \Sigma_\infty)$ collection of cracks.

For $F \in L^2(V)^d$ we define $w = w^{\Sigma_\infty}_{\Sigma_0,F} \in \mathcal{H}^{\Sigma_\infty}_{\Sigma_0}$ as the unique solution of:

$$\int_\Omega \gamma_0 \nabla w \cdot \nabla v \,\mathrm{d}x = \int_V F \cdot \nabla v \,\mathrm{d}x, \qquad \forall v \in \mathcal{H}^{\Sigma_\infty}_{\Sigma_0}.$$

We define $L^{\Sigma_\infty}_{\Sigma_0}(V) : L^2(V)^d \to L^2_\diamond(\Gamma)$ as

$$L^{\Sigma_\infty}_{\Sigma_0}(V)F = w^{\Sigma_\infty}_{\Sigma_0,F}|_\Gamma.$$

Then

$$(L^{\Sigma_\infty}_{\Sigma_0}(V))^* f = \nabla u^{\Sigma_\infty}_{\Sigma_0,f}|_V, \qquad f \in L^2_\diamond(\Gamma).$$

Ranges

Lemma 11

Let $V \in \mathcal{A}$ *and let* Σ *be a* $(\Sigma_0, \Sigma_\infty)$ *collection of cracks.*

- *If* $\Sigma_0 \Subset V$ *then* $R(L_{\emptyset}^{\Sigma_\infty}(V)) = R(L_{\Sigma_0}^{\Sigma_\infty}(V))$.
- *If* $\Sigma_\infty \Subset V$ *then* $R(L_{\Sigma_0}^{\emptyset}(V)) = R(L_{\Sigma_0}^{\Sigma_\infty}(V))$.

The proof becomes much more complicated due to different function spaces involved.

Ranges

Lemma 12

Let Σ *be a* $(\Sigma_0, \Sigma_\infty)$ *collection of cracks. Assume that* $\Sigma_0 \Subset V$ *and* $\Sigma_\infty \Subset W$ *for* $V, W \in \mathcal{A}$ *with* $\mathrm{dist}(V, W) > 0$.

- *If* $\Sigma_0 \neq \emptyset$ *then there exists a sequence* (f_n) *in* $L^2_\diamond(\Gamma)$ *such that*

$$\lim_{n\to\infty} \langle (\Lambda_W^\emptyset - \Lambda_\emptyset^\emptyset) f_n, f_n \rangle = 0,$$
$$\lim_{n\to\infty} \langle (\Lambda_\emptyset^\emptyset - \Lambda_\emptyset^W) f_n, f_n \rangle = 0,$$
$$\lim_{n\to\infty} \langle (\Lambda_{\Sigma_0}^{\Sigma_\infty} - \Lambda_\emptyset^{\Sigma_\infty}) f_n, f_n \rangle = \infty.$$

- *Analogous result for* Σ_∞.

Key part of the proof

- Define $A = L_{\Sigma_0}^{\Sigma_\infty}(V) - L_{\emptyset}^{\Sigma_\infty}(V)$.
- Then $A^* f = \nabla(u_{\Sigma_0,f}^{\Sigma_\infty} - u_{\emptyset,f}^{\Sigma_\infty})|_V$.
- By unique continuation and zero mean conditions on Γ:

$$A^* f = 0 \quad \text{if and only if} \quad u_{\Sigma_0,f}^{\Sigma_\infty} = u_{\emptyset,f}^{\Sigma_\infty}.$$

- From non-invisibility of cracks for full ND map: $A \neq 0$.
- There is a $g \in R(A) \setminus \{0\}$ such that also $g \in R(L_{\emptyset}^{\Sigma_\infty}(V)) = R(L_{\Sigma_0}^{\Sigma_\infty}(V))$ but $g \notin R(L_{\emptyset}^{\Sigma_\infty}(W))$ (last part from usual proof of localised potentials).
- Using constructive version of localised potentials, there is a sequence (f_n) such that

$$\lim_{n\to\infty} \|(L_{\emptyset}^{\Sigma_\infty}(W))^* f_n\|_{L^2(W)^d} = 0,$$

$$\lim_{n\to\infty} \|(L_{\emptyset}^{\Sigma_\infty}(V))^* f_n\|_{L^2(V)^d} = \infty,$$

$$\text{and} \quad \lim_{n\to\infty} \|A^* f_n\|_{L^2(V)^d} = \infty.$$

Difficult direction of main result

- To prove

$$\Lambda_C^{\emptyset} \geq \Lambda_{D_0}^{D_\infty} \geq \Lambda_{\emptyset}^{C} \qquad \text{implies} \qquad D \subset C,$$

  we assume the contrapositive, i.e. $D \not\subset C$.

- We have either of two cases:
  - (a): There are $V, W \in \mathcal{A}$ with $\text{dist}(V, W) > 0$ and non-empty $\chi \in \mathcal{X}$, such that

$$\chi \subseteq D_0, \qquad \chi \Subset V, \qquad C \subseteq W, \qquad \text{and} \qquad D_\infty \Subset W.$$

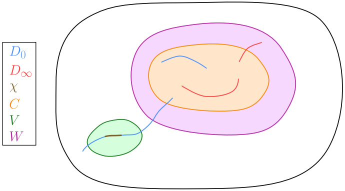
  - (b): There are $V, W \in \mathcal{A}$ with $\text{dist}(V, W) > 0$ and non-empty $\chi \in \mathcal{X}$, such that

$$\chi \subseteq D_\infty, \qquad \chi \Subset W, \qquad C \subseteq V, \qquad \text{and} \qquad D_0 \Subset V.$$

Difficult direction of main result

- Focusing on case (a):

$$\begin{aligned}\Lambda_C^{\emptyset} - \Lambda_{D_0}^{D_\infty} &= (\Lambda_C^{\emptyset} - \Lambda_{\emptyset}^{\emptyset}) + (\Lambda_{\emptyset}^{\emptyset} - \Lambda_{\emptyset}^{D_\infty}) + (\Lambda_{\emptyset}^{D_\infty} - \Lambda_{D_0}^{D_\infty}) \\ &\leq (\Lambda_W^{\emptyset} - \Lambda_{\emptyset}^{\emptyset}) + (\Lambda_{\emptyset}^{\emptyset} - \Lambda_{\emptyset}^{W}) + (\Lambda_{\emptyset}^{D_\infty} - \Lambda_{\chi}^{D_\infty}).\end{aligned}$$



- From our lemma, there is a sequence (f_n) so that

$$\lim_{n\to\infty} \langle (\Lambda_W^{\emptyset} - \Lambda_{\emptyset}^{\emptyset}) f_n, f_n \rangle = \lim_{n\to\infty} \langle (\Lambda_{\emptyset}^{\emptyset} - \Lambda_{\emptyset}^{W}) f_n, f_n \rangle = 0$$

$$\lim_{n\to\infty} \langle (\Lambda_{\emptyset}^{D_\infty} - \Lambda_{\chi}^{D_\infty}) f_n, f_n \rangle = -\infty.$$

- Hence $D \not\subseteq C$ implies $\Lambda_C^{\emptyset} \not\geq \Lambda_{D_0}^{D_\infty}$.

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