EEG source localisation and cortical mapping Mathematics of electrical imaging: modeling, theory and implementation, Toulouse

Nemaire Masimba^{1,2}

¹Institut de Mathématiques de Bordeaux (IMB), ²IHU-Liryc

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This research was undertaken when I was a member of the FACTAS team at the Inria Centre at Université Côte d'Azur and the Analysis team of IMB. This work was done jointly with Paul Asensio, Jean-Michel Badier, Juliette Leblond and Jean-Paul Marmorat.

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- Typically it is assumed that the head is a nested layered conductor with distinct layers (brain, skull, scalp) having different but constant electric conductivities.
- Using EEG data, a Cauchy data completion problem (recovery of the electric potential and normal currents) can be solved on the outer surfaces of the scalp, skull and cortex (inverse cortical mapping problem).

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- We will take the view that the inverse problems of source localisation and cortical mapping are different aspects of the same problem and we aim to solve them simultaneously.
- The hope is that this will improve numerical accuracy and provide context in the analysis of the recovery results.

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 - the Newton potential maps the (distributional) divergence of the members of space continuously.
 - the space is uniformly smooth.
- We will present a method for solving the inverse problems of source localisation and cortical mapping simultaneously with the source modelled as vector measures supported on the white matter fibres.

• Let $\gamma:[0,l] o \mathbb{R}^3$ be a Lipschitz mapping and let $S:=\gamma([0,l]).$ If γ is such that

$$\mathcal{H}_1(\gamma([a,b]))=b-a, \quad orall [a,b] \subset [0,l],$$

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then γ is an orientable rectifiable curve, where \mathcal{H}_1 is the 1-dimensional Hausdorff measure. • On S define the vector measure \mathbf{R}_{γ} through the relation

$$\langle {\sf R}_{\boldsymbol{\gamma}}, {\sf f}
angle = \int_0^{\prime} {\sf f}({\boldsymbol{\gamma}}(t)) \cdot {\boldsymbol{\gamma}}'(t) \, dt, \quad {
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• Hence for each white matter fibre we can define a collection of vector measure $\mathbf{R}_{h\gamma} = h(\gamma(t))\gamma'\mathcal{H}_1$, with $\gamma(t)$ being an arclength parametrisation of the white matter fibre and $h \in L^1(\gamma'\mathcal{H}_1)$.

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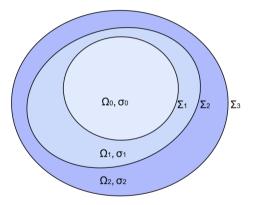
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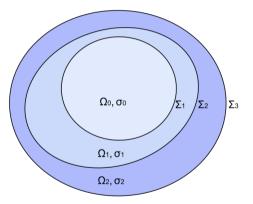
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- In the source localisation we aim to recover $h(\gamma(t))$.

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For each Σ_i, σ_i⁻ and σ_i⁺ are σ inside and outside Σ_i, respectively, φ⁻ and φ⁺ are the non-tangential limits of φ approaching from inside and outside, respectively.

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 Regularity of solutions to elliptic PDEs imply that φ satisfies the following transmission conditions on each Σ_i, i = 1,2,3

$$\phi^- = \phi^+$$

$$\sigma^- \partial_\nu \phi^- = \sigma^+ \partial_\nu \phi^+$$

• Using a result of Gezelowitz, Sarvas showed that ϕ associated with μ_0 , at $x \in \mathbb{R}^3 \setminus \mathrm{supp}(\mu_0)$ are:

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where \mathcal{H}_2^i is the 2-dimensional Hausdorff measure on Σ_i , ν is the unit outer normal on Σ_i .

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- It can be shown that all divergence-vector measure $\mathbf{R}_{h\gamma}$ are silent, that is all vector measures such that,

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Definitions and preliminaries

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- A concrete example are all vector measures $\mathbf{R}_{h\gamma}$ with h = 1 and $\gamma(0) = \gamma(I)$.
- The existence of silent sources implies non-uniqueness of solutions to the inverse source localisation problem hence it is ill-posed.

Problem 1

Given EEG data f and parameters $\alpha, \beta, \lambda, \lambda_i > 0$, i = 1, 2, 3, find

$$(\boldsymbol{\mu}_0, \phi_1, \phi_2, \phi_3) = \operatorname*{arg \, inf}_{(\boldsymbol{\mu}_0, \phi_1, \phi_2, \phi_3)} \mathcal{T}_{f, \lambda}(\boldsymbol{\mu}_0, \phi_1, \phi_2, \phi_3),$$

where

$$\begin{split} \mathcal{T}_{f,\lambda} &:= \alpha \, \| \boldsymbol{\mathcal{F}}_1(\boldsymbol{\mu}_0, \phi_1, \phi_2, \phi_3) - f \|^2 + \beta \, \| \boldsymbol{\mathcal{F}}_2(\boldsymbol{\mu}_0, \phi_1, \phi_2, \phi_3) \|^2 \\ &+ \lambda \Re \Big(\, \| \boldsymbol{\mu}_0 \|_{[\mathcal{M}(\boldsymbol{\Sigma}_0)]^3} \, \Big) + \sum_{i=1}^3 \lambda_i \, \| \phi_i \|_{L^2(\boldsymbol{\Sigma}_i)}^2 \, . \end{split}$$

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with some initial guess for k = 0 by solving the problems

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• This is a minimising sequence for which $\mathcal{T}_{f,\lambda}(\mu_0^{\{k\}}, \phi_1^{\{k\}}, \phi_2^{\{k\}}, \phi_3^{\{k\}})$ converges at least linearly because of the uniform smoothness.

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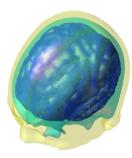
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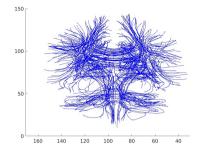
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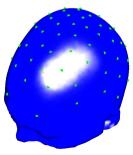
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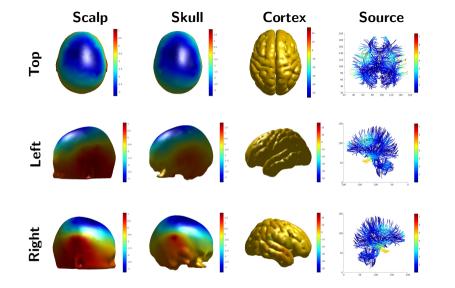


Figure: Source recovery with real auditory EEG data for the N100 response.

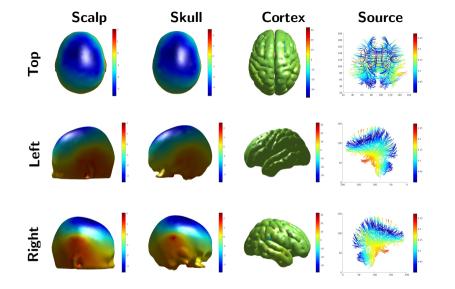


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- There is a need to improve the computational efficiency.
- Of particular interest is to study if it is possible to change the regularisation parameters at each step of the alternating minimisation algorithm in order to get a norm-minimising equivalent solution in the end.

Thank you :)