A short and shallow introduction to Open Quantum Systems

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These are the lectures notes for a series of lectures given for the introductory school, held from Nov 26th to Dec 1st 2023, of the ***Quantum many-body systems out-of-equilibrium*** thematic trimester at Institut Henri Poincaré. The lectures consisted of 3 sessions of 1h30 and were given at the "Institut d'études scientifiques de Cargèse" in Corsica. I express my gratitude towards **Patrizia Vignolo** for the organization of this very pleasant school and inviting me to give these lectures.

There exists an abundant and excellent literature on the subject but to prepare these lectures, I mostly restricted myself to the following references:

- *The theory of open quantum systems*, **H.-P. Breuer and F. Petruccione** [1]. (One of) the bible on the subject.
- *Lecture notes on the Theory of Open Quantum Systems,* **D.A. Lidar** Lidar [2]. A self-contained course on the subject by one of the expert of the field.
- *Quantum noise,* **C. Gardiner, P. Zoller** Gardiner and Zoller [3]. Another classic text book with a focus on methods used in optics and stochastic quantum and/or classical differential equations.
- *Statistical Aspects of Quantum state Monitoring for (and by) Amateurs* **D. Bernard**. A non-published set of lectures that can be found here with a focus on continuous measurement.

Proper referencing of all the physical concepts and ideas introduced in these lectures is beyond the scope of these lectures. These notes have not been carefully reviewed and some sections are not even written yet. If you find some typos, I would appreciate you indicating them to me. If you happen to enjoy the notes, you can also drop me a message, that will give me motivation to improve them.

Prerequisites : The reader should be familiar with quantum mechanics of closed systems and especially the density matrix formalism. Knowledge of second quantification would help for some parts.

Introduction

The overarching philosophy of these lecture notes about open quantum systems revolves around the endeavor to generalize the concept of "quantum evolution". For closed systems, a very powerful general formalism that works in non-relativistic case is provided by the Schrödinger equation. However there are some situations that are not encompassed by the Schrödinger equation. One such instance is encountered when performing a projective measurement on the system. Despite being a perfectly legitimate quantum evolution, it eludes description through the unitary approach. Our objective is to broaden this description, terming the generalized quantum evolution as a *quantum map* or *channel*.

Having a generalized form for the evolution offers a solid foundation, enabling the derivation of theorems and properties with broad applicability. For closed systems, a central role is played by the generator of unitary evolutions in time, i.e. the *Hamiltonian*. Likewise, for open systems, once we obtain the generic form of a quantum map, it will be desirable to derive the generic form of the generator. This will be given by the *Gorini–Kossakowski–Sudarshan–Lindblad* (GKLS) equation, often just called the Lindblad or quantum master equation.

We will begin by the axiomatic approach, defining a set of reasonable axioms that a quantum map should adhere to and, subsequently, deriving the generic expression aligning with these axioms. Following this, we then proceed to derive the generator of a quantum map.

In a second part we will see how such Lindblad evolution arises from microsopic models such as system-bath coupling and measurements.

Finally, we will talk in the last part about stochastic processes in the context of both quantum noise induced by an harmonic oscillators bath and continuous measurements.

As said before, the subjects covered will not, by any mean, be exhaustive.

1 The axiomatic approach

1.1 Quantum channel

Definition. *Quantum channel.*

Let $\mathcal{H}_S$ be an Hilbert space and $\rho \in \text{End}(\mathcal{H}_S)$ denote density matrices (i.e. positive semi-definite, Hermitian operator of trace 1). We define a quantum map as an application Φ :

$$\Phi : \text{End}(\mathcal{H}_S) \to \text{End}(\mathcal{H}_S) \tag{1}$$
$$\rho \to \Phi(\rho), \tag{2}$$

which fulfills the following requirements :

- Φ is trace preserving, i.e $\text{tr}(\Phi(\rho)) = \text{tr}(\rho)$.
- Φ is linear, i.e $\Phi(a\rho_1 + b\rho_2) = a\Phi(\rho_1) + b\Phi(\rho_2)$.
- Φ is completely positive.

Perhaps the most unusual property is the complete positivity. The positivity of the map would be a natural requirement, i.e. that the positive eigenvalues of ρ remains positive under action of the map (as they should, as one can interpret them as probability weights). Complete-positivity is a stronger property that we define now :

Definition. *Complete-positivity (CP).*

Let $\mathcal{H}_R$ denotes an arbitrary auxiliary space of dimension k. A map Φ is said to be completely positive when for any $\mathcal{H}_R$, the trivial extension $\Phi \otimes \mathbb{I}_R$ is positive where $\mathbb{I}_R$ is the identity operator on $\mathcal{H}_R$.

The way to interpret a CP-map is that one cannot alter the positivity of a given density matrix by doing a trivial operation on a space that is disconnected from the system.

To make more sense of that, let's exhibit a map that is positive but not completely positive. The canonical example is the transpose map T.

Example. *The transpose map.*

As is well-known, T preserves the eigenvalues of a given matrix. Indeed, the eigenvalues of ρ are determined by solving the determinantal problem $\det(\rho - \lambda\mathbb{I}) = 0$ and such equation is invariant by action of the transpose. Thus the transpose is *positive*.

Now, we will show on an example that it is not completely-positive.

Let $\mathcal{H}_S$ be a Hilbert space of dimension 2. A generic density matrix can be written as a 2×2 matrix

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3}$$

The action of the transpose is

$$T(\rho) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{4}$$

Now consider additionally a space $\mathcal{H}_R$ of dimension 2. We denote the basis of $\mathcal{H}_S \otimes \mathcal{H}_R$ by $\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\}$. The action of the partial transpose on an element $|i_S, i_R\rangle \langle j_S, j_R|$ is $T \otimes \mathbb{I}_R (|i_S, i_R\rangle \langle j_S, j_R|) = |j_S, i_R\rangle \langle i_S, j_R|$, i.e only the indices belonging to $\mathcal{H}_S$ are swapped.

Now consider the state $\rho = |\psi\rangle \langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$.

$$\rho = \frac{1}{2}\left(|0,0\rangle \langle 0,0| + |0,0\rangle \langle 1,1| + |1,1\rangle \langle 0,0| + |1,1\rangle \langle 1,1|\right), \tag{5}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$
$$T \otimes \mathbb{I}_R(\rho) = \frac{1}{2}\left(|0,0\rangle \langle 0,0| + |1,0\rangle \langle 0,1| + |0,1\rangle \langle 1,0| + |1,1\rangle \langle 1,1|\right), \tag{7}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

and it is easy to see that the eigenvalues of ρ are $\{1,0,0,0\}$ while the ones of $T\otimes\mathbb{I}_R(\rho)$ are $\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$, hence using the partial transpose on the system alone, we ended up with an *unphysical* state. Thus, the partial transpose is not a valid quantum operation.

Remark : It turns out that the partial transpose can be used as a *measure of entanglement* between two systems. This is called the PPT criterion for positive partial transpose.

Having defined the axioms that we wish our quantum map fulfills, it would be now desirable to be able to write a generic form for a quantum-map. This is what we'll see in the next subsection.

1.2 Kraus operator sum representation (OSR)

A set operators acting on $\mathcal{H}_S$, $\{K_\mu\}$ are said to be a Kraus operator sum representation (OSR) if $\sum_\mu K_\mu^\dagger K_\mu = \mathbb{I}$. The associated map is defined as

$$\Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger. \tag{9}$$

Theorem. *Being a quantum channel is equivalent to admitting a Kraus OSR.*

We will prove only one implication of the equivalence, that is that admitting a Kraus OSR $\Longrightarrow$ being a quantum channel. Let us check that it indeed fulfills the axioms:

Trace preservation: We indeed have that $\mathrm{tr}(\rho) = \mathrm{tr}\left(\sum_\mu K_\mu \rho K_\mu^\dagger\right)$ thanks to the cyclicity of the trace and the property $\sum_\mu K_\mu^\dagger K_\mu = \mathbb{I}$.

Linearity: The linearity is evident.

Complete-positivity

$$\langle\nu|\,\Phi\otimes\mathbb{I}_R(\rho)\,|\nu\rangle = \sum_\mu \langle\nu|\,(K_\mu\otimes\mathbb{I}_R)\,\rho\,\left(K_\mu^\dagger\otimes\mathbb{I}_R\right)|\nu\rangle = \sum_\mu \langle\nu_\mu|\,\rho\,|\nu_\mu\rangle \tag{10}$$

with $|\nu_\mu\rangle = \left(K_\mu^\dagger\otimes\mathbb{I}_R\right)|\nu\rangle$. Since ρ is positive, we have that $\sum_\mu \langle\nu_\mu|\,\rho\,|\nu_\mu\rangle > 0$, hence the CP property. See Choi [4] for the proof in the other direction.

Remark: If there is only a single Kraus operator in the sum, the quantum channel is an unitary evolution.

1.3 Gorini–Kossakowski–Sudarshan–Lindblad (GKLS) equation a.k.a the quantum master equation

Most of the times in quantum mechanics, we are interested in continuous time evolution of quantum states. The starting point for solving unitary evolution is usually the local generator form, i.e the Hamiltonian. The unitary is obtained through the exponential map $U(t) = e^{-iHt}$.

Likewise, it would be useful to have a local in time formulation for quantum channels, i.e., is it possible for a quantum channel Φ_t parametrized by time to generate it by application of the exponential map to a local operator $\mathcal{L}$: $\Phi_t = e^{\mathcal{L}t}$? We will see that this is indeed the case and the associated local evolution equation is the famous Gorini–Kossakowski–Sudarshan–Lindblad (GKLS) equation.

To do so, we first have to require an additional property from the map Φ_t which is the semi-group property :

Definition. *Semi-group.*

A quantum channel with a continuous parametrization Φ_t is said to have the semi-group property if

$$\Phi_t \circ \Phi_{t'} = \Phi_{t+t'}. \tag{11}$$

The semi-group property, when taken for an infinitesimal timestep dt provides the *Markov* property, i.e. that the evolution of the system at time t only depends on the state of the system at time t:

$$\rho_{t+dt} = \Phi_{t+dt}(\rho_0) = \Phi_{dt}(\rho_t). \tag{12}$$

This in general is no small assumption. It means that there is no *memory effect* induced by the external degrees of freedom on the evolution of the system. Hence, having the GKLS form is much stronger than having the Kraus ORS form.

We will now give an explicit generic form for Φ_{dt}. Define the time dependent Kraus operators as

$$\Phi_t = \sum_\mu K_\mu(t)\rho K_\mu^\dagger(t). \tag{13}$$

We will perform a small time expansion.

$$K_\mu = \varphi_\mu \mathbb{I} + N_\mu \sqrt{dt} + M_\mu dt. \tag{14}$$

Remark that since $\Phi_t \propto K_\mu K_\mu^\dagger$ we need to do the Taylor expansion in powers of $\sqrt{dt}$ to get the most general object of order dt. For $dt \to 0$, we must recover the identity map, thus

$$\sum_\mu |\varphi_\mu|^2 = 1, \tag{15}$$

and these numbers thus define a probability distribution. We denote the average with respect to this distribution by $\mathbb{E}^0[\bullet] := \sum_\mu |\varphi_\mu|^2 \bullet$. φ_μ can be interpreted as probability amplitudes.

Let us express the Kraus condition

$$\sum_\mu K_\mu^\dagger K_\mu = \sum_\mu \left(\varphi_\mu^* \mathbb{I} + N_\mu^\dagger \sqrt{dt} + M_\mu^\dagger dt\right) \left(\varphi_\mu \mathbb{I} + N_\mu \sqrt{dt} + M_\mu dt\right) \tag{16}$$

$$= \mathbb{I} + \sum_\mu \left(\varphi_\mu N_\mu^\dagger + \varphi_\mu^* N_\mu\right) \sqrt{dt} + \sum_\mu \left(N_\mu^\dagger N_\mu + M_\mu^\dagger + M_\mu\right) dt + O\left(dt^{3/2}\right) \tag{17}$$

leading to the conditions

$$\sum_\mu \left(\varphi_\mu N_\mu^\dagger + \varphi_\mu^* N_\mu\right) = 0, \tag{18}$$

$$\sum_\mu \left(N_\mu^\dagger N_\mu + M_\mu^\dagger + M_\mu\right) = 0. \tag{19}$$

These conditions mean that $\sum_\mu \varphi_\mu N_\mu^\dagger$ and $\sum_\mu M_\mu + \frac{1}{2} N_\mu^\dagger N_\mu$ are anti-Hermitian operators. For reason that will be clarified below, we give a name to the latter

$$\sum_\mu M_\mu + \frac{1}{2} N_\mu^\dagger N_\mu = -iH, \tag{20}$$

with H an Hermitian matrix $H^\dagger = H$.

The infinitesimal time evolution on ρ_t Φ_{dt} is now given by

$$\rho_{t+dt} = \Phi_{dt}\left(\rho_t\right) = \sum_\mu \left(\varphi_\mu \mathbb{I} + N_\mu \sqrt{dt} + M_\mu dt\right) \rho_t \left(\varphi_\mu^* \mathbb{I} + N_\mu^\dagger \sqrt{dt} + M_\mu^\dagger dt\right) \tag{21}$$

$$= \rho_t + \sum_\mu \left(\varphi_\mu^* N_\mu \rho_t + \rho_t \varphi_\mu N_\mu^\dagger\right) \sqrt{dt} + \sum_\mu \left(N_\mu \rho_t N_\mu^\dagger + M_\mu \rho_t + \rho_t M_\mu^\dagger\right) dt. \tag{22}$$

We further impose cancellation of the term proportional to μ, i.e

$$\sum_\mu \varphi_\mu N_\mu^\dagger = ib, \; b \in \mathbb{R}. \tag{23}$$

Using additionally (20), we end up with the GKLS equation:

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \sum_\mu N_\mu \rho_t N_\mu^\dagger - \frac{1}{2}\left\{N_\mu^\dagger N_\mu, \rho_t\right\}, \tag{24}$$

$$=: \mathcal{L}(\rho_t), \tag{25}$$

where $\{,\}$ is the anticommutator. Most of the times, this equation is taken as the starting point of an open quantum system problem.

Remark: Had we not imposed the term proportional to $\sqrt{dt}$ to be 0, it would have become the dominant contribution and thus, we would have to rescale $\sqrt{dt} \to dt$ and discard the term proportional to dt^2. However, this would be equivalent to set $N_\mu \to 0$ in the previous expansion and this is not the most general case.

1.3.1 Dual evolution on operator / Heisenberg picture

In the spirit of the Heisenberg picture of closed quantum mechanics, Eq.(24) induces a dual evolution on operators. Indeed, let $\hat{O}$ be a quantum operator and $O(t)$ its expectation value $O(t) := \mathrm{tr}(\rho_t \hat{O})$. We define the time-evolved operator $\hat{O}(t)$ as the time-dependent operator such that

$$\mathrm{tr}\left(\rho_t \hat{O}\right) = \mathrm{tr}\left(\rho_0 \hat{O}(t)\right) \tag{26}$$

The time evolution of $\hat{O}(t)$ is given by

$$\mathrm{tr}\left(\rho_0 \frac{d}{dt}\hat{O}(t)\right) = \mathrm{tr}\left(\frac{d}{dt}\rho_t \hat{O}\right), \tag{27}$$

$$= \mathrm{tr}\left(\rho_t \mathcal{L}^*(\hat{O})\right), \tag{28}$$

$$\frac{d}{dt}\hat{O}(t) = \mathcal{L}^*(\hat{O})|_t \tag{29}$$

with

$$\mathcal{L}^*(\hat{O}) = i[H, \hat{O}] + \sum_\mu L_\mu^\dagger \hat{O} L_\mu - \frac{1}{2}\left\{L_\mu^\dagger L_\mu, \hat{O}\right\} \tag{30}$$

where the last expression was obtained using (**??**) and the cyclicity of the trace. Notice that the order of the daggers as changed in the part proportional to $L^\dagger, L$. Notice also that the identity is indeed invariant under the action of $\mathcal{L}^*$: $\mathcal{L}^*(\mathbb{I}) = 0$.

1.4 The GKLS equation contains classical continuous time Markov processes in its description

In this section, we show that all continuous time classical Markov processes can be described by a GKLS equation. First, we recall what a classical Markov process is. Let $\mathcal{C}$ be the space of classical configurations of a given system. A configuration is noted C. Let $p_C(t)$ be the classical probability of finding the system in C. A Markov process is a dynamical process which induces the following time evolution on $p_C(t)$:

$$\frac{d}{dt}p_C(t) = \sum_{C' \neq C} \left(M_{C' \to C}\, p_{C'}(t) - M_{C \to C'}\, p_C(t)\right), \tag{31}$$

where $M_{C' \to C} \geq 0$ is the transition rate from configuration C' to C. We will use the standard notation $M_{C'C} := M_{C' \to C}$ for $C' \neq C$ and $M_{CC} = -\sum_{C' \neq C} M_{CC'}$ so that

$$\frac{d}{dt}p_C(t) = \sum_{C'} M_{C'C}\, p_{C'}(t). \tag{32}$$

We now show that we can always associate to such continuous Markov process an equation of the form (**??**). To each classical configuration we associate an element of the Hilbert space $|C\rangle$. We now suppose that ρ is diagonal in that basis, i.e $\rho_t = \sum_C \rho_t^C |C\rangle\langle C|$.

Now consider the GKLS equation:

$$\frac{d}{dt}\rho_t = \sum_{C, C', C' \neq C} L_{C'C}\, \rho_t L_{C'C}^\dagger - \frac{1}{2}\left\{L_{C'C}^\dagger L_{C'C}, \rho_t\right\} \tag{33}$$

with $L_{C'C} = \sqrt{M_{C'C}}\, |C\rangle\langle C'|$ and $L_{C'C}^\dagger L_{C'C} = M_{C'C}\, |C'\rangle\langle C'|$ so that

$$\frac{d}{dt}\rho_t^C = \sum_{C' \neq C} M_{C'C}\, \rho_t^{C'} - M_{CC'}\, \rho_t^C, \tag{34}$$

And we see indeed that we can interpret ρ_t^C as a classical probability weight associated to the configuration C. However the GKLS contains more structure since it also describes the dynamics of off-diagonal elements as well as cases where L_μ can not be written as $|C\rangle\langle C'|$. However having the classical Markov process interpretation in mind is often practical for qualitative reasoning.

1.5 Examples

Before going on, let us see some examples.

1.5.1 Spontaneous emission spin 1/2

Consider a spin 1/2 and denote the associated basis by $\{|\uparrow\rangle, |\downarrow\rangle\}$. Let H_0 be the Hamiltonian of the isolated system

$$H_0 = \omega_0 \sigma_z, \tag{35}$$

where σ_z is the z Pauli matrix. We want to describe the spin flip from up to down coming from, e.g., spontaneous emission of a photon. The spin flip can be described by an operator $L = \sqrt{\alpha}\,|\downarrow\rangle\langle\uparrow|$ where α is the rate at which the spontaneous emission occurs. We parametrize the density matrix as

$$\rho = \rho_{\uparrow\uparrow}|\uparrow\rangle\langle\uparrow| + \rho_{\uparrow\downarrow}|\uparrow\rangle\langle\downarrow| + \rho_{\downarrow\uparrow}|\downarrow\rangle\langle\uparrow| + \rho_{\downarrow\downarrow}|\downarrow\rangle\langle\downarrow|. \tag{36}$$

The evolution equation is given by

$$\frac{d}{dt}\rho = -i\omega_0[\sigma_z, \rho] + \alpha\left(|\downarrow\rangle\langle\uparrow|\rho|\uparrow\rangle\langle\downarrow| - \frac{1}{2}\{|\uparrow\rangle\langle\uparrow|, \rho\}\right). \tag{37}$$

In terms of components

$$\frac{d}{dt}\rho_{\uparrow\uparrow} = -\alpha\rho_{\uparrow\uparrow}, \tag{38}$$

$$\frac{d}{dt}\rho_{\downarrow\downarrow} = \alpha\rho_{\uparrow\uparrow}, \tag{39}$$

$$\frac{d}{dt}\rho_{\uparrow\downarrow} = -2i\omega_0\rho_{\uparrow\downarrow} - \frac{\alpha}{2}\rho_{\uparrow\downarrow}, \tag{40}$$

which admit as solutions

$$\rho_{\uparrow\uparrow}(t) = \rho_{\uparrow\uparrow}(0)e^{-\alpha t}, \tag{41}$$

$$\rho_{\downarrow\downarrow}(t) = 1 - \rho_{\uparrow\uparrow}(0)e^{-\alpha t}, \tag{42}$$

$$\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0)e^{-\left(i2\omega_0 + \frac{\alpha}{2}\right)t} \tag{43}$$

And we see that the stationary state is

$$\rho_\infty = |\downarrow\rangle\langle\downarrow|. \tag{44}$$

We see that this steady state is unique and completely independent of the initial state which is a consequence of the irreversible nature of the evolution.

1.5.2 Boundary driven free fermions

Suppose that we have a discrete 1d fermionic chain of size N. We consider a tight-binding Hamiltonian acting in the bulk :

$$H_0 = -\tau \sum_{j=1}^{N-1} c_j^\dagger c_{j+1} + \text{H.c.} \tag{45}$$

We will study a simple out-of-equilibrium situation by driving the system out-of-equilibrium by adding Lindblad operators that act only on the boundaries, i.e. the first and last sites. The total evolution of the density matrix is then given by

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \sum_{a\in\{1,N\}} \alpha_a \mathcal{L}_a^+(\rho_t) + \beta_a \mathcal{L}_a^-(\rho_t) \tag{46}$$

with

$$\mathcal{L}_a^+(\rho_t) = c_a^\dagger \rho_t c_a - \frac{1}{2}\left\{c_a c_a^\dagger, \rho_t\right\}, \tag{47}$$

$$\mathcal{L}_a^-(\rho_t) = c_a \rho_t c_a^\dagger - \frac{1}{2}\left\{c_a^\dagger c_a, \rho_t\right\}. \tag{48}$$

At the level of classical Markov processes, the interpretation of $\mathcal{L}_a^+$ is that it injects a fermion on site a with rate α_a when it is empty while $\mathcal{L}_a^-$ destroys a fermion on site a with rate β_a when it is occupied. Thus, when the rates

$(\alpha_1, \beta_1) \neq (\alpha_N, \beta_N)$ we have an imbalance and expect an out-of-equilibrium situation. In general in these boundary driven problems, one is interested in the computation of the out-of-equilibrium steady state.

In this example, the problem is considerably simplified by the fact the the equations of motion close at the level of two-point functions. Indeed, let $G_{ij} := \mathrm{tr}\left(\rho_t c_j^\dagger c_i\right)$. One can check that, indeed,

$$\frac{d}{dt} G_{i,j} = -i\tau \left(G_{i,j+1} + G_{i,j-1} - G_{i+1,j} - G_{i-1,j}\right) \tag{49}$$

$$- \sum_{a \in \{1,N\}} \frac{1}{2} \left(\delta_{i,a} + \delta_{j,a}\right) \left(\alpha_a + \beta_a\right) G_{i,j} - \delta_{i,a} \delta_{j,a} \alpha_a. \tag{50}$$

The stationary state is the solution to $\frac{d}{dt} G_{i,j} = 0$. Because the problem is quadratic, we have a system of N^2 (actually less because G_{ij} is Hermitian) coupled equations to solve. However, for this particular case, we are able to exhibit the solution analytically.

For simplification, let us fix $\alpha_1 = \beta_N = \frac{1+\mu}{2}$, $\alpha_N = \beta_1 = \frac{1-\mu}{2}$ and $\tau = 1$. One can check that the following solution solves the stationary equations:

$$G_{i,j} = \delta_{i,j} \left(\frac{1}{2} + \frac{\mu}{6} \delta_{i,1} - \frac{\mu}{6} \delta_{i,N}\right) - i \delta_{i,j-1} \frac{\mu}{3} + i \delta_{i,j+1} \frac{\mu}{3}. \tag{51}$$

We see that apart, from the diagonal and the first off-diagonal, all the elements are 0. The density profile is completely flat apart from the two boundary sites.

The current operator is defined through the conservation equation taken in the bulk:

$$\frac{d}{dt} G_{i,i} = -i\tau \left(G_{i,i+1} + G_{i,i-1} - G_{i+1,i} - G_{i-1,i}\right), \tag{52}$$

$$=: -\left(J_{i+1} - J_i\right), \tag{53}$$

$$J_i := i\tau \left(G_{i-1,i} - G_{i,i-1}\right), \tag{54}$$

$$= \frac{2\mu}{3}. \tag{55}$$

As expected, the current is proportional to the bias. Notice however that it is *independent of system size* which is a signature of *ballistic* transport.

2 Constructive approaches: a few (important) examples

2.1 System bath coupling

We start with a powerful theorem.

Theorem. *(Stinespring) Let Φ be a quantum channel acting on $\rho \in \mathrm{End}(\mathcal{H}_S)$. There always exists a Hilbert space $\mathcal{H}_R$, a state $\rho_R \in \mathrm{End}(\mathcal{H}_R)$ and a unitary operator U acting on $\mathcal{H}_S \otimes \mathcal{H}_R$ such that*

$$\Phi(\rho) = \mathrm{tr}_R \left(U \rho \otimes \rho_R U^\dagger\right) \tag{56}$$

where $\mathrm{tr}_R(\bullet)$ denotes the partial trace on $\mathcal{H}_R$.

This means that, in principle, it is always possible to see the evolution of a given quantum channel as resulting from the unitary evolution of a bigger system of which we have traced out some degrees of freedom that we call the *bath* degrees of freedom. Now we will show how to explicitly make such a construction.

Consider an Hamiltonian of the form

$$H = H_S + H_B + H_I. \tag{57}$$

S refers to the system, B to the bath and I is some interaction term between the two. It will turn out to be convenient to work in the interacting picture defined as follows. Let $\rho(t)$ designates the density matrix in the Schrödinger picture, define the density matrix and the Hamiltonian in the interaction picture as

$$\rho_I(t) := e^{i(H_S+H_B)t} \rho(t) e^{-i(H_S+H_B)t}, \tag{58}$$

$$H_I(t) := e^{i(H_S+H_B)t} H_I e^{-i(H_S+H_B)t}, \tag{59}$$

and we have

$$\frac{d}{dt}\rho_I(t) = -i\left[H_I(t), \rho_I(t)\right]. \tag{60}$$

Observables in the interaction picture become time-dependent,

$$O_I(t) := e^{i(H_S+H_B)t} O e^{-i(H_S+H_B)t}. \tag{61}$$

For notational convention, we will drop the I index from now on. We will assume that the initial state is factorized:

$$\rho(0) = \rho_S(0) \otimes \rho_B. \tag{62}$$

The integral form of (60) is

$$\rho(t) = \rho(0) - i\int_0^t ds\left[H_I(s), \rho(s)\right]. \tag{63}$$

Inserting this integral form into (60), we get

$$\frac{d}{dt}\rho(t) = -i[H_I(t), \rho(0)] - \int_0^t ds\left[H_I(t), [H_I(s), \rho(s)]\right]. \tag{64}$$

To get the evolution on the system alone we take the partial trace over the bath degrees of freedom that we denote by tr_B. We note ρ_S the reduced density matrix of the system alone, $\rho_S := \mathrm{tr}_B(\rho)$:

$$\frac{d}{dt}\rho_S(t) = -i\mathrm{tr}_B\left([H_I(t), \rho(0)]\right) - \mathrm{tr}_B\left(\int_0^t ds\left[H_I(t), [H_I(s), \rho(s)]\right]\right). \tag{65}$$

To simplify what will follow, we will suppose that $\forall t\ \mathrm{tr}_B\left([H_I(t), \rho(0)]\right) = 0$. This means

$$\frac{d}{dt}\rho_S(t) = -\mathrm{tr}_B\left(\int_0^t ds\left[H_I(t), [H_I(s), \rho(s)]\right]\right). \tag{66}$$

We will now do a series of approximations that will allow to treat (66). The first one is the *Born approximation* or *weak coupling* approximation. It consists in saying that, at any time, the effect of the system of the bath is negligible and thus, we may always write the density matrix in a factorized form

$$\rho(t) = \rho_S(t) \otimes \rho_B. \tag{67}$$

We additionally make the replacement $s \to t - s$ for reasons of readibility that will be made clear below:

$$\frac{d}{dt}\rho_S(t) = -\mathrm{tr}_B\left(\int_0^t ds\left[H_I(t), [H_I(t-s), \rho_S(t-s) \otimes \rho_B]\right]\right). \tag{68}$$

In this equation, there are implicitly two-times correlators evaluated on the initial distribution of the bath ρ_B, i.e objects of the form $\mathrm{tr}_B\left(O_B(t)O_B(t-s)\rho_B\right)$. The next approximation, the *Markov approximation* will suppose that there is a separation of time scales between the relaxation time τ_B in the bath and the typical time at which the evolution takes place in the system τ_S. This tells us that the integral is non-zero for $s \propto \tau_B$. We are thus allowed to do two things: i) Replace $\rho_S(s)$ by $\rho_S(t)$. ii) Take the upper bound for the integral to $+\infty$. This gives

$$\frac{d}{dt}\rho_S(t) = -\mathrm{tr}_B\left(\int_0^\infty ds\left[H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]\right]\right). \tag{69}$$

The advantage of having the lower bound to $-\infty$ is that the evolution equation doesn't depend on the initial conditions anymore.

We now suppose that the interaction between the bath and the system is of the form (in the Schrödinger picture)

$$H_I = \sum_m A_m^\dagger \Gamma_m + A_m \Gamma_m^\dagger \tag{70}$$

where Γ_m are operators acting on the bath only and A_m are eigenoperators of H_S satisfying

$$\left[H_S, A_m^\dagger\right] = \omega_m A_m^\dagger, \quad \left[H_S, A_m\right] = -\omega_m A_m. \tag{71}$$

$$H_I(t) = \sum_m A_m^\dagger e^{i\omega_m t}\Gamma_m(t) + A_m e^{-i\omega_m t}\Gamma_m^\dagger(t). \tag{72}$$

The double commutator in (69) gives rise to 4 terms. Let's zoom in on one of them :

$$\mathrm{I} := \mathrm{tr}_B\left(\int_0^\infty ds H_I(t)H_I(t-s)\rho_S(t)\otimes\rho_B\right), \tag{73}$$

$$= \mathrm{tr}_B\Bigg(\int_0^\infty ds \sum_{m,m'} \left(A_m^\dagger e^{i\omega_m t}\Gamma_m(t) + A_m e^{-i\omega_m t}\Gamma_m^\dagger(t)\right) \tag{74}$$

$$\left(A_{m'}^\dagger e^{i\omega_{m'}(t-s)}\Gamma_{m'}(t-s) + A_{m'}e^{-i\omega_{m'}(t-s)}\Gamma_{m'}^\dagger(t-s)\right)\rho_S(t)\otimes\rho_B\Bigg). \tag{75}$$

And we see the explicit contribution of the bath two-times correlators $\mathrm{tr}_B\left(\Gamma_m^\dagger(t)\Gamma_{m'}(t-s)\rho_B\right)$. We will assume that the bath is not *squeezed,* meaning that the pair creation and annihilation correlators are 0. Furthermore, we will suppose that the bath correlators are translationally invariant in time so that they are just functions of s. Finally, we make our third and last approximation, the *rotating-wave approximation* which stipulates that terms in the sum with $\omega_m \neq \omega_{m'}$ oscillate fast compared to the time scale of typical variations of the system τ_S. We then end up with

$$\mathrm{I} = \sum_m \left(A_m^\dagger A_m \int_0^\infty ds e^{i\omega_m s}\mathrm{tr}_B\left(\Gamma_m(s)\Gamma_m^\dagger(0)\rho_B\right) + A_m A_m^\dagger \int_0^\infty ds e^{-i\omega_m s}\mathrm{tr}_B\left(\Gamma_m^\dagger(s)\Gamma_m(0)\rho_B\right)\right)\rho_S(t). \tag{76}$$

Now introducing

$$\int_0^\infty ds e^{i\omega_m s}\mathrm{tr}_B\left(\Gamma_m(s)\Gamma_m^\dagger(0)\rho_B\right) := \frac{1}{2}K_m + i\delta_m, \tag{77}$$

$$\int_0^\infty ds e^{-i\omega_m s}\mathrm{tr}_B\left(\Gamma_m(0)\Gamma_m^\dagger(s)\rho_B\right) := \frac{1}{2}K_m - i\delta_m, \tag{78}$$

$$\int_0^\infty ds e^{-i\omega_m s}\mathrm{tr}_B\left(\Gamma_m^\dagger(s)\Gamma_m(0)\rho_B\right) := \frac{1}{2}G_m + i\epsilon_m, \tag{79}$$

$$\int_0^\infty ds e^{i\omega_m s}\mathrm{tr}_B\left(\Gamma_m^\dagger(0)\Gamma_m(s)\rho_B\right) := \frac{1}{2}G_m - i\epsilon_m. \tag{80}$$

$$\frac{d}{dt}\rho_S(t) = \sum_m \Big[- i\delta_m\left[A_m^\dagger A_m, \rho_S\right] - i\epsilon_m\left[A_m A_m^\dagger, \rho_S\right] \tag{81}$$

$$+ G_m\left(A_m^\dagger \rho_S A_m - \frac{1}{2}\left\{A_m A_m^\dagger, \rho_S\right\}\right) \tag{82}$$

$$+ K_m\left(A_m \rho_S A_m^\dagger - \frac{1}{2}\left\{A_m^\dagger A_m, \rho_S\right\}\right)\Big]. \tag{83}$$

And we see that the we indeed obtain terms of the Lindblad form.

Remark that there is an asymmetry between the process with the quantum jump A_m and $A_m^\dagger$. This asymmetry naturally comes from the asymmetry of the correlations in the bath (Think for instance of the creation and annihilation of a photon in the vacuum which is strongly different from the process where we first annihilate and create). Interestingly, this gives a way of realizing *classical* asymmetric stochastic processes such as the ASEP (asymmetric simple exclusion process).

2.1.1 Summary of the approximations used

We recall the main approximations used in the previous derivation

Rotating wave approximation Terms oscillating at $\omega_S + \omega_B$ can be neglected.

Born approximation Frequency scales associated with the system environment coupling is small in scale compared to the system and environment's own frequency scales. This is equivalent to saying that the coupling between the system and the bath is weak.

Markov approximation System-environment coupling is time independent over short timescales. The Markov approximation also assumes that relaxation times of the bath are fast compared to typical timescale of the dynamics within the system. For a thermal bath, the lifetime of the correlation in the bath are set by $\tau_T := \frac{\hbar}{2\pi k_B T}$. For $T = 1K$, $\tau_T \approx 10^{-12}s$.

2.2 Measurements

Definition. *Positive Operator Valued Measure (POVM)*

A set of operators $\{F_s\}$ on a Hilbert space $\mathcal{H}$ constitute a POVM if they satisfy the relation

$$\sum F_s^\dagger F_s = \mathbb{I}. \tag{84}$$

In the context of measurements, we can associate to a given POVM a measurement process as follows : s indexes the possible outputs of the measurement and the state after the measurement is updated to

$$\rho \to \frac{F_s \rho F_s^\dagger}{\pi(s)}, \tag{85}$$

where $\pi(s)$ is the probability associated with the result s given by $\pi(s) := \text{tr}\left(F_s \rho F_s^\dagger\right)$. The POVM definition ensures that the density matrix stay positive semi-definite, Hermitian of trace 1.

For the usual measurement of an observable $O = \sum_s \lambda_s \Pi_s$ with λ_s the (possibly degenerate) eigenvalues associated to O and Π_s the corresponding projectors, the F_s are simply the Π_s. However, the POVM definition encompasses more general cases such as weak measurements as we will see.

If we average over the results of the measurement, we get

$$\mathbb{E}[\rho] = \sum_s \pi(s) \frac{F_s \rho F_s^\dagger}{\pi(s)}, \tag{86}$$

$$= \sum_s F_s \rho F_s^\dagger, \tag{87}$$

where we denoted the average with $\mathbb{E}[]$. We see that we naturally get a Kraus operator structure with F_s being the Kraus operators.

Suppose now that the we repeat the measurements and they occur with some probability rate γ, i.e at each time step dt there is a probability γdt for a measurement to occur and $(1-\gamma)\, dt$ for a measurement to not take place. Additionally, we always discard the results of the measurements, i.e we average over the different measurement results (we discard the $\mathbb{E}[]$ in what follows to lighten notations). The density matrix gets updated according to

$$\rho_{t+dt} = \gamma dt \sum_s F_s \rho_t F_s^\dagger + (1-\gamma) dt \rho_t, \tag{88}$$

$$\frac{d}{dt}\rho_t = \gamma \sum_s F_s \rho F_s^\dagger + \frac{1}{2}\{F_s^\dagger F_s, \rho\} dt, \tag{89}$$

where in the second line we used that $\sum_s F_s^\dagger F_s = \mathbb{I}$. We see that this is indeed of the Lindblad form.

3 Stochastic formalism

In the examples of Sec.2, quantum maps were obtained in both cases by averaging over some external degrees of freedom (dof). For the system bath coupling, it was the external bath degrees of freedom while for the measurements, it was the measurements results. What would happen if we kept track of these dofs ? We will see that in both case, it is possible, under certain assumptions to obtain an efficient effective description of these additional dofs in terms of stochastic processes.

3.1 Quantum white noise

We will illustrate concepts of quantum noise on a idealized problem. Consider the following Hamiltonian:

$$H = H_S + H_B + H_I, \tag{90}$$

where B is a set of bosonic modes indexed by ω with linear energy ω:

$$H_B = \int_{-\infty}^{\infty} d\omega \omega b_\omega^\dagger b_\omega, \tag{91}$$

with $[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$. Although the integral reaches to $-\infty$, one has in practice to imagine all the modes far away from some resonance value ω_c are not relevant and do not contribute to the physics of the problem. The integration range is then just a mere calculational convenience.

The interaction is taken to be

$$H_I = i \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \gamma(\omega) \left(b_\omega^\dagger c - c^\dagger b_\omega \right) \tag{92}$$

where c and $c^\dagger$ are unspecified operators acting on the system. In the Markov approximation, the coupling $\kappa(\omega)$ is taken to be frequency independent, $\gamma(\omega) = \gamma$.

The Heisenberg equations of motion on the bath mode and on a given generic system operator a are given by

$$\frac{d}{dt} b_\omega(t) = -i\omega b_\omega(t) + \frac{\gamma}{\sqrt{2\pi}} c(t), \tag{93}$$

$$\frac{d}{dt} a(t) = i[H_S, a] + \gamma \int \frac{d\omega}{\sqrt{2\pi}} \left(b_\omega^\dagger [a, c] - [a, c^\dagger] b_\omega \right). \tag{94}$$

The solution of the former is

$$b_\omega(t) = e^{-i\omega(t-t_0)} b_\omega(t_0) + \frac{\gamma}{\sqrt{2\pi}} \int_{t_0}^{t} dt' e^{-i\omega\left(t-t'\right)} c(t'). \tag{95}$$

In what follows we will note $b_\omega(t_0) = b_\omega$.

Inserting the previous solution into (94), we obtain:

$$\frac{d}{dt} a(t) = i[H_S, a] \tag{96}$$

$$+ \gamma \int \frac{d\omega}{\sqrt{2\pi}} d\omega \left(e^{i\omega(t-t_0)} b_\omega^\dagger [a, c] - [a, c^\dagger] e^{-i\omega(t-t_0)} b_\omega \right) \tag{97}$$

$$+ \gamma^2 \int \frac{d\omega}{2\pi} \int_{t_0}^{t} dt' \left(e^{i\omega\left(t-t'\right)} c^\dagger(t')[a, c] - e^{-i\omega\left(t-t'\right)} [a, c^\dagger] c(t') \right). \tag{98}$$

Our particular choice of bath spectrum and coupling allows to simplify greatly this expression:

$$\frac{d}{dt} a(t) = i[H_S, a] + \left(\gamma b_{\mathrm{str}}^\dagger(t) + \frac{\gamma^2}{2} c^\dagger \right) [a, c] - [a, c^\dagger] \left(\gamma b_{\mathrm{str}}(t) + \frac{\gamma^2}{2} c(t) \right), \tag{99}$$

where we introduced the field

$$b_{\mathrm{str}}(t) := \int \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega(t-t_0)} b_\omega. \tag{100}$$

(Notice the factor of 1/2 which comes from the fact that $\int_{t_0}^{t} c(t')\delta(t-t')dt' = \frac{1}{2} c(t)$ since the Dirac distribution is located at the upper integration limit.) The b_{str} operator fulfill the commutation relation

$$\left[b_{\mathrm{str}}(t), b_{\mathrm{str}}^\dagger(t') \right] = \delta(t - t'). \tag{101}$$

An identity that we will use later and notice now is that for $t' > t$,

$$[a(t), b_{\mathrm{str}}(t')] = 0. \tag{102}$$

This is simply a causality statement, since the value of $a(t)$ can only depend, by construction, on its previous values and previous values of b_{str}.

The statistics of these operators are imposed by fixing the bath distribution ρ_B:

$$\mathrm{tr} \left(b_{\mathrm{str}}^\dagger(t) b_{\mathrm{str}}(t') \rho_B \right) = \int \frac{d\omega d\omega'}{2\pi} e^{i\omega'(t'-t_0)} e^{-i\omega(t-t_0)} \mathrm{tr} \left(b_\omega^\dagger b_{\omega'} \rho_B \right). \tag{103}$$

The *quantum white noise* limit is obtained by imposing that

$$\mathrm{tr} \left(b_\omega^\dagger b_{\omega'} \rho_B \right) = \alpha \delta(\omega - \omega'). \tag{104}$$

The crucial assumption here is that α is *independent* of ω which would not be the case for e.g. a thermal state. This leads to

$$\operatorname{tr}\left(b_{\mathrm{str}}^{\dagger}(t)b_{\mathrm{str}}(t')\rho_B\right) = \alpha\delta\left(t-t'\right), \tag{105}$$

$$\operatorname{tr}\left(b_{\mathrm{str}}(t)b_{\mathrm{str}}^{\dagger}(t')\rho_B\right) = \left(\alpha+1\right)\delta\left(t-t'\right). \tag{106}$$

These relations encourage us to interprete $b_{\mathrm{str}}(t)$ as some kind of *white noise*. However, one should keep in mind that they are non-commuting objects, hence the name *quantum white noise*. In the following we will set $t_0 \to 0$ for simplification.

Let W be the "quantum Wiener process" associated to this quantum white noise:

$$W_t := \int_0^t b_{\mathrm{str}}(t')dt'. \tag{107}$$

The statistics of this object is defined through the distribution of the bath ρ_B. We have that

$$\left[W_{t_1}, W_{t_2}^{\dagger}\right] = \min\left(t_1, t_2\right) \tag{108}$$

$$\operatorname{tr}\left(W_t\rho_B\right) = \operatorname{tr}\left(W_t^2\rho_B\right) = 0, \tag{109}$$

$$\operatorname{tr}\left(W_{t_1}^{\dagger}W_{t_2}\rho_B\right) = \alpha\min\left(t_1, t_2\right), \tag{110}$$

$$\operatorname{tr}\left(W_{t_1}W_{t_2}^{\dagger}\rho_B\right) = \left(\alpha+1\right)\min\left(t_1, t_2\right). \tag{111}$$

We will assume that these are all the correlations required to fix the statistics of W_t, i.e we assume that the distribution of the bath is *Gaussian*.

3.1.1 Itō versus Stratonovich

For some (quick) introduction on classical stochastic calculus and Itō vs Stratonovich, see App A. We will see in what follows that the concept of stochastic calculus and the Itō and Stratonovich prescriptions translate to the quantum case.

Itō prescription Let $g(t)$ be a generic system operator. The quantum Itō integral is defined through the following prescription:

$$\int_0^t g(t')dW_{t'} := \lim_{n\to\infty}\sum_{i=1}^{n} g\left(t_i\right)\left(W_{t_{i+1}} - W_{t_i}\right) \tag{112}$$

with $t = n\Delta t$ and $t_i = i\Delta t$.

Because of causality the increment $\left(W_{t_{i+1}} - W_{t_i}\right)$ commutes with $g(t_i)$, so that

$$\int_0^t g(t')dW_{t'} = \int_0^t dW_{t'}g(t'). \tag{113}$$

Stratonovich prescription Let '$\tilde{t}_i := \frac{1}{2}\left(t_i + t_{i+1}\right)$ be an intermediate time between t_i and t_{i+1}. We denote this convention with a $\circ$:

$$\int_0^t g(t')\circ dW_{t'} := \lim_{n\to\infty}\sum_{i=1}^{n} g\left(\tilde{t}_i\right)\left(W_{t_{i+1}} - W_{t_i}\right), \tag{114}$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} g\left(\tilde{t}_i\right)\left(W_{t_{i+1}} - W_{\tilde{t}_i}\right) + \sum_{i=1}^{n} g\left(\tilde{t}_i\right)\left(W_{\tilde{t}_i} - W_{t_i}\right)\right]. \tag{115}$$

Let Δg be the increment from t_i to $\tilde{t}_i$ in the Itō prescription, $\Delta g := g(\tilde{t}_i) - g(t_i)$

From our previous definition, we have

$$\Delta g = -\gamma\left[g, c^{\dagger}\right]\left(W_{\tilde{t}_i} - W_{t_i}\right) + \gamma\left(W_{\tilde{t}_i}^{\dagger} - W_{t_i}^{\dagger}\right)\left[g, c\right] + O\left(\tilde{t}_i - t_i\right). \tag{116}$$

(For our purpose, the deterministic terms $O\left(\tilde{t}_i - t_i\right)$ won't matter so we don't write them explicitly). This leads to

$$\begin{aligned}\int_0^t g(t') \circ dW_{t'} = \lim_{n\to\infty} \Bigg[&\sum_{i=1}^n g\left(\tilde{t}_i\right)\left(W_{t_{i+1}} - W_{\tilde{t}_i}\right) + \sum_{i=1}^n g\left(t_i\right)\left(W_{\tilde{t}_i} - W_{t_i}\right) \\ &- \gamma \sum_{i=1}^n \left[g, c^\dagger\right]\left(W_{\tilde{t}_i} - W_{t_i}\right)\left(W_{\tilde{t}_i} - W_{t_i}\right) \\ &+ \gamma \sum_{i=1}^n \left(W^\dagger_{\tilde{t}_i} - W^\dagger_{t_i}\right)\left(W_{\tilde{t}_i} - W_{t_i}\right)[g, c] \Bigg]\end{aligned} \tag{117}$$

We regroup the first two terms and use the Itō rules to get

$$\int_0^t g(t') \circ dW_{t'} = \int_0^t g(t') dW_{t'} + \gamma \frac{\alpha}{2} \int_0^t dt' [g, c]. \tag{118}$$

Similarly, one can show

$$\int_0^t dW_{t'} \circ g(t') = \int_0^t g(t') dW_{t'} + \gamma \frac{(\alpha+1)}{2} \int_0^t dt' [g, c], \tag{119}$$

$$\int_0^t g(t') \circ dW^\dagger_{t'} = \int_0^t g(t') dW^\dagger_{t'} - \gamma \frac{(\alpha+1)}{2} \int_0^t dt' [g, c^\dagger], \tag{120}$$

$$\int_0^t dW_{t'} \circ g(t') = \int_0^t g(t') dW^\dagger_{t'} - \gamma \frac{\alpha}{2} \int_0^t dt' [g, c^\dagger]. \tag{121}$$

These relations carry on for the integrands:

$$\begin{aligned} g(t) \circ dW_t &= g(t) dW_t + \gamma \frac{\alpha}{2} [g, c] dt, \\ dW_t \circ g(t) &= g(t) dW_t + \gamma \frac{(\alpha+1)}{2} [g, c] dt, \\ g(t) \circ dW^\dagger_t &= g(t) dW^\dagger_t - \gamma \frac{(\alpha+1)}{2} [g, c^\dagger] dt, \\ dW^\dagger_t \circ g(t) &= g(t) dW^\dagger_t - \gamma \frac{\alpha}{2} [g, c^\dagger] dt. \end{aligned} \tag{122}$$

Using these notations, we can rewrite (99) as an SDE taken in the *Stratonovich* convention as

$$\begin{aligned} da(t) = -i[a, H_S] dt + \frac{\gamma^2}{2}\left(c^\dagger [a, c] - [a, c^\dagger] c\right) dt, \\ + \gamma \left(dW^\dagger_t \circ [a, c] - [a, c^\dagger] \circ dW_t\right). \end{aligned} \tag{123}$$

3.1.2 Itō SDE

We now show that the following quantum SDE written in Itō convention is consistent with (99):

$$\begin{aligned} da = &- i\left[a, H_S\right] dt + \frac{\gamma^2}{2}(\alpha+1)\left(2c^\dagger a c - a c^\dagger c - c^\dagger c a\right) dt \\ &+ \frac{\gamma^2}{2}\alpha\left(2 c a c^\dagger - a c c^\dagger - c c^\dagger a\right) dt \\ &- \gamma\left[a, c^\dagger\right] dW_t + \gamma dW^\dagger_t \left[a, c\right]. \end{aligned} \tag{124}$$

equipped with the Itō rules $dW_t^2 = \left(dW^\dagger_t\right)^2 = 0$, $dW_t dW^\dagger_t = (\alpha+1)dt$, $dW^\dagger_t dW_t = \alpha dt$. Notice that the previous relations are strict *equalities* where no average is taken.

To show that this expression is indeed equivalent to (99), it suffices to change the convention for the noise increments in (124). From (122), we have that

$$\left[a, c^\dagger\right] dW_t = \left[a, c^\dagger\right] \circ dW_t - \frac{\gamma}{2}\alpha\left[\left[a, c^\dagger\right], c\right] dt, \tag{125}$$

$$dW^\dagger_t \left[a, c\right] = dW^\dagger_t \circ \left[a, c\right] + \frac{\gamma}{2}\alpha\left[\left[a, c\right], c^\dagger\right] dt. \tag{126}$$

Plugging these expressions into (124) leads to

$$da(t) = -i[a, H_S]dt + \frac{\gamma^2}{2}\left(c^\dagger[a,c] - [a,c^\dagger]c\right)dt,$$
$$+\gamma\left(dW_t^\dagger \circ [a,c] - [a,c^\dagger]\circ dW_t\right), \tag{127}$$

which is the Stratonovich version of the SDE.

3.1.3 Some examples

WIP

3.2 Indirect, repeated measurements

We introduce here the concept of indirect measurements. In this setting, one does not perform the measurement on the system directly but instead on an ancilla A that is first entangled with the system S and then measured. Depending on the degree of intrication between the two systems, the projective measurement of A gives total or partial information about the state of S. in the later case, we talk about *weak* measurement. One paradigmatic example of

Let us put this idea into equations. Let $|\varphi\rangle \in \mathcal{H}_A$ be the state of the ancilla before interaction with the system and ρ_S the state of the system. After the interaction, the state is updated as follows.

$$\rho_S \otimes |\varphi\rangle\langle\varphi| \to U\left(\rho \otimes |\varphi\rangle\langle\varphi|\right)U^\dagger \tag{128}$$

where U is an unitary acting on $\mathcal{H}_S \otimes \mathcal{H}_A$. Let O_A be an observable living on $\mathcal{H}_A$ and s index the corresponding eigenvalues. A measurement of O_A updates the state of the system according to

$$\rho_S \to \frac{\langle a| U\left(\rho_S \otimes |\varphi\rangle\langle\varphi|\right)U^\dagger |a\rangle}{\pi(a)}, \tag{129}$$

$$\pi(a) := \mathrm{tr}\left(\langle a| U\left(\rho \otimes |\varphi\rangle\langle\varphi|\right)U^\dagger |a\rangle\right). \tag{130}$$

Eq.(129) has the form of a POVM with $F_a := \langle a| U |\varphi\rangle$ an operator acting on $\mathcal{H}_S$ alone. We can check that indeed $\sum_a F_a^\dagger F_a = \sum_a \langle\varphi| U^\dagger |a\rangle\langle a| U |\varphi\rangle = \mathbb{I}$.

In the repeated measurements formalism, one prepares a series of ancilla indexed by n, A_n that are all independent of one another. We first entangle the system with probe 1 and do the measurement of A_1, then the same for A_2, etc. A collection measurement results obtained this way is called a *quantum trajectory.* We will see below of to construct one explicitly. The treatment of the general case can be found in D.Bernard's lecture notes.

Non-Demolition condition In order for this procedure to reproduce a projective measurement, we need that a specific basis $\{|k\rangle\}$ of the system, that we call the *pointer basis* is preserved by the interaction with the ancilla, i.e if the system is in the pure state $\rho_S = |k\rangle\langle k|$, it is left invariant by the weak measurement procedure with probability one.

One way to enforce this is to impose that the unitary coupling the system and the ancilla can be written under the form

$$U = \sum_k |k\rangle\langle k| \otimes u_k \tag{131}$$

where u_k is a unitary acting on the ancilla space. This is called the *non-demolition* condition. Let's verify that this unitary indeed preserves the k basis. We start from a pure state $|\psi_S\rangle = |k_0\rangle$ before the interaction.

$$U|\psi_S\rangle \otimes |\varphi\rangle = |k_0\rangle \otimes u_{k_0}|\varphi\rangle. \tag{132}$$

After the measurement with output a, the system state is updated to

$$|\psi\rangle = \frac{|k_0\rangle\langle a| u_{k_0}|\varphi\rangle}{\sqrt{\pi(a)}} \tag{133}$$

where the probability $\pi(a)$ writes $\pi(a) = |\langle a| u_{k_0}|\varphi\rangle|^2$. Since the state is defined up to a phase

$$|\psi\rangle = |k_0\rangle \quad \forall a. \tag{134}$$

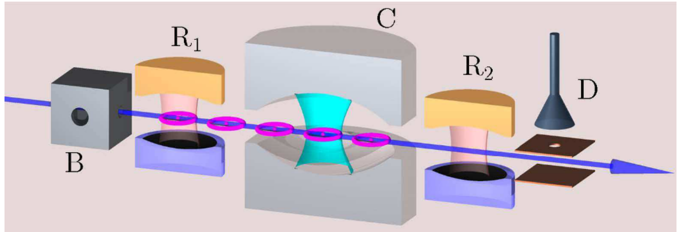

Figure 1: Scheme of the LKB experiment where photons are observed via Rydberg atoms that serve as ancilla. The photons in blue are trapped between the two mirrors of the cavity C. They are probed by two-level atoms (the small pink torus) flying out the preparation box B, passing through the cavity C and measured in D. Each atom is manipulated before and after C in Ramsey cavities R_1 and R_2, respectively. It is finally detected in D either in ground state g or in excited state e.

Repeated measurements and collapse We now want to repeat the previous procedure many times. We imagine that we prepare an ensemble of ancillas, all in the same initial state and index them by n. The ancilla are sent one by one to interact with the system and we measure everytime the state of the ancilla.

Let's assume that the non-demolition condition is satisfied. Let $\langle k| \rho_S^{(n)} |k'\rangle$ be the elements of the density matrix of the system in the preserved pointer basis. We give a special name to the diagonal elements $q_n(k) := \langle k| \rho_S^{(n)} |k\rangle$.

After each step, these elements are updated according to

$$q_n(k) \to q_{n+1}(k) = \frac{\langle k, a| U \left(\sum_{k',k''} \langle k'| \rho_S^{(n)} |k''\rangle |k'\rangle \langle k''| \otimes |\varphi\rangle \langle \varphi| \right) U^\dagger |k, a\rangle}{\pi_n(a)}, \tag{135}$$

$$= \frac{|\langle a| u_k |\varphi\rangle|^2 q_n(k)}{\pi_n(a)}. \tag{136}$$

Remark that $|\langle a| u_k |\varphi\rangle|^2$ can be thought as a conditioned probability, i.e the probability of getting the output a knowing that the system is in the state k, $p(a|k) := |\langle a| u_k |\varphi\rangle|^2$ and $\pi_n(a) = \sum_k p(a|k) q_n(k)$.

Theorem. *(Bauer-Bernard) Progressive collapse:*

Assume that the conditioned probabilities $p(a|k)$ are all disjoints, that is, there does not exist a disjoint pair of pointer states k and k' such that $p(a|k) = p(a|k')$ $\forall a$, then:

- *The sequences $n \to q_n(k)$ converge almost surely and in $\mathbb{L}^1$ for any k.*
- *The limit distribution is peaked: $q_\infty(k) = \delta_{k,k_\infty}$ for some random target pointer k_∞.*
- *The random target k_∞ is distributed according to the initial distribution $\mathbb{P}[k_\infty = k] = p_0(k)$.*
- *The convergence to the target is exponentially fast with $p_n(k)/p_\infty(k) \propto e^{-nS(k_\infty|k)}$, with $S(k_\infty|k)$ the relative entropy $S(k_\infty|k) = -\sum_s p(s|k_\infty) \log \left[\frac{p(s|k)}{p(s|k_\infty)} \right]$.*

Proof: Cf. lecture notes from D. Bernard. The key is to notice that $q_n(k)$ is a Martingale, i.e

$$\mathbb{E}\left[q_{n+1}(k)|\mathcal{F}_n\right] = q_n(k), \quad \forall k. \tag{137}$$

where $\mathcal{F}_n$ is the filtration up to the n-th step. One then uses Doob's martingale convergence theorem stipulating that a bounded Martingal necessarily converges almost surely and in $\mathbb{L}^1$. One can show that only the peaked distribution satisfies the "stationary" condition $q_\infty(k) = \frac{p(a|k) q_\infty(k)}{\pi_n(a)}$. The probability p_{k_∞} to get the pointer distribution $q_\infty(k) = \delta_{k,k_\infty}$ is then determined by the martingale property

$$p_{k_\infty} = \mathbb{E}[q_\infty(k)] = q_0(k). \tag{138}$$

The estimate on the rate of convergence can be obtained by applying successively the recursion relation.

Example Haroche's, cavity QED experiment Guerlin et al. [5] We now give an example of the previous QND collapse with Haroch's celebrated cavity QED experiment. The experimental setup is shown on Fig. 1.

Rydberg atoms are used as probes and are described as two-level atoms that can be either in the ground or excited state. They are generated in the excited state and then rotated by R_1 using a $\pi/2$ pulse onto the x axis. The interaction of the Rydberg atom with the photons in the cavity is such that the photon number rotates the Rydberg atom by a phase proportional to the number of photons in the cavity n.

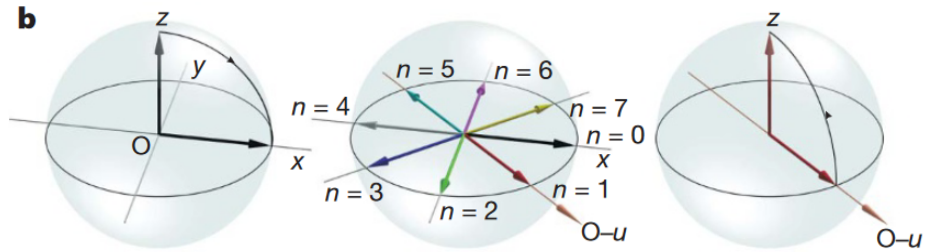

Figure 2: "Evolution of the atomic spin on the Bloch sphere in a real experiment: an initial pulse R_1 rotates the spin from O_z to O_x (left). Light shift produces a $\pi/4$ phase shift per photon of the spin's precession in the equatorial plane. Directions associated with $n = 0$ to 7 end up regularly distributed over 360 (centre). Pulse R_2 maps the direction O_u onto O_z, before the atomic state is read out (right)." From Guerlin et al. [5].

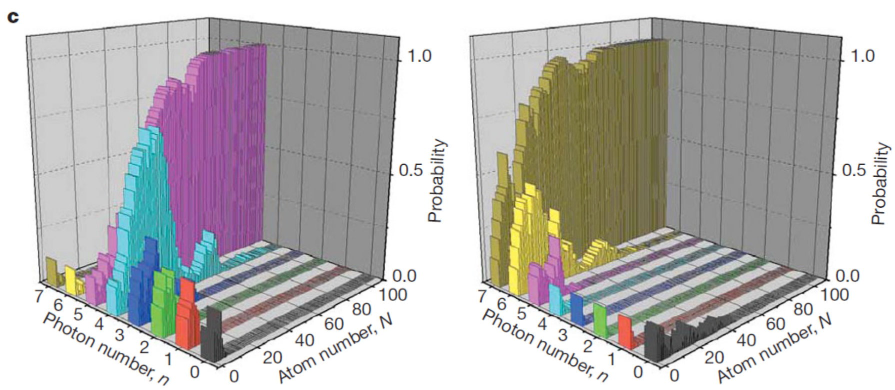

Figure 3: "Photon number probabilities plotted versus photon and atom numbers n and N. The histograms evolve, as N increases from 0 to 110, from a flat distribution into $n = 5$ and $n = 7$ peaks." From Guerlin et al. [5].

The resulting conditioned probability distribution is given by

$$p(a|\phi, n) = \frac{1}{2}\left(1 + \cos\left(\frac{\pi n}{q} - \phi + a\pi\right)\right) \tag{139}$$

with $a \in \{0, 1\}$, $\frac{\pi}{q}$ is the shift set by a photon set by the interaction time (with q integer) and ϕ is the angle of the axis O_u (see Fig. 2). Two realizations of the experiment are shown on Fig. 3.

3.2.1 Special case: spin 1/2

The system is a spin 1/2 and so are all the ancillas. All the spins are initially independent. All the ancillas are initially prepared in the state $|x\rangle^{(n)} = \frac{1}{\sqrt{2}}\left(|+\rangle^{(n)} + |-\rangle^{(n)}\right)$ while the state of the system is generically fixed to $\rho_{S,0}$.

The interaction Hamiltonian between the two spins is chosen to be

$$H = -\left|\uparrow\right\rangle\left\langle\uparrow\right| \otimes \sigma_y^{(n)} + \left|\downarrow\right\rangle\left\langle\downarrow\right| \otimes \sigma_y^{(n)}. \tag{140}$$

H is chosen such that it rotates the ancilla clockwise around the y axis of the Bloch sphere if the state of the system is $\uparrow$ and counterclockwise it is in $\downarrow$. For a short interaction time, the $+$ state of the ancilla is thus positively correlated with the $\uparrow$ state of the system and $-$ with the $\downarrow$ state. However, the correlation is not perfect, and a measurement of the ancilla state only gives "weak" information about the state of the system.

Let us see what happens when we start from an initial state ρ_n for the system and go through one step of the interaction and measurement procedure. To lighten notation, we drop the (n) superscript indexing the ancilla for the computations and will restore it when needed. Let ε be the interaction time. We suppose that we measure the state of the ancilla in the $\{|+\rangle, |-\rangle\}$ immediately after the interaction took place.

From the definition above

$$F_\pm = \langle \pm | U | x \rangle , \tag{141}$$

$$\pi_\pm = \mathrm{tr}\left(F_\pm \rho_n F_\pm^\dagger \right). \tag{142}$$

Let us perform the small ε expansion of $F_\pm$:

$$F_\pm \approx \langle \pm | \left(\mathbb{I} - iH\varepsilon - \frac{1}{2} H^2 \varepsilon^2 \right) | x \rangle , \tag{143}$$

$$= \frac{1}{\sqrt{2}} \mathbb{I} - i\varepsilon \left(- |\uparrow\rangle \langle\uparrow| + |\downarrow\rangle \langle\downarrow| \right) \otimes \langle \pm | \sigma_y^{(n)} | x \rangle + - \frac{1}{2\sqrt{2}} \varepsilon^2 \mathbb{I}, \tag{144}$$

$$= \frac{1}{\sqrt{2}} \left(\mathbb{I} \pm \varepsilon \sigma_z - \frac{\varepsilon^2}{2} \mathbb{I} \right), \tag{145}$$

where we used that $H^2 = \mathbb{I}$.

$$F_\pm \rho_n F_\pm^\dagger \approx \frac{1}{2} \left(\mathbb{I} \pm \varepsilon \sigma_z - \frac{\varepsilon^2}{2} \mathbb{I} \right) \rho_n \left(\mathbb{I} \pm \varepsilon \sigma_z - \frac{\varepsilon^2}{2} \mathbb{I} \right) \tag{146}$$

$$\approx \frac{1}{2} \left(\rho_n \pm \varepsilon \left\{ \sigma_z, \rho_n \right\} + \varepsilon^2 \left(\sigma_z \rho_n \sigma_z - \frac{1}{2} \left\{ \sigma_z^2, \rho_n \right\} \right) \right), \tag{147}$$

$$\pi_\pm \approx \frac{1}{2} \left(1 \pm 2\varepsilon \langle \sigma_z \rangle_n \right) \tag{148}$$

with $\langle \sigma_z \rangle_n := \mathrm{tr}(\rho_n \sigma_z)$. Finally,

$$\rho_{n+1} = \frac{F_\pm \rho_n F_\pm^\dagger}{\pi_\pm}, \tag{149}$$

$$\approx \rho_n + \Delta S^{(n)} \left\{ \left(\sigma_z - \langle \sigma_z \rangle_n \right), \rho_n \right\} + \varepsilon^2 \left(\sigma_z \rho_n \sigma_z - \frac{1}{2} \left\{ \sigma_z^2, \rho_n \right\} \right) - 2\varepsilon^2 \left\{ \left(\sigma_z - \langle \sigma_z \rangle_n \right), \rho_n \right\} \langle \sigma_z \rangle_n, \tag{150}$$

where we called $\Delta S^{(n)}$ a random variable taking value $\pm\varepsilon$ with probability $\pi_\pm^{(n)}$. Now notice that

$$\mathbb{E}[\Delta S^{(n)}] = \varepsilon \left(\pi_+^{(n)} - \pi_-^{(n)} \right) = 2\varepsilon^2 \langle \sigma_z \rangle_n, \tag{151}$$

$$\mathbb{E}[\left(\Delta S^{(n)} \right)^2] = \varepsilon^2 \left(\pi_+^{(n)} + \pi_-^{(n)} \right) = \varepsilon^2. \tag{152}$$

Now we define the *signal*

$$S_N = \sum_{n=0}^{N} \Delta S^{(n)}, \tag{153}$$

In the limit $N \to \infty$, $\varepsilon \to 0$ and keeping $t := N\varepsilon^2$ finite, the signal converges to the stochastic process S_t which follows the stochastic differential equation (SDE):

$$dS_t = 2\langle \sigma_z \rangle_t dt + dB_t, \tag{154}$$

where B_t is a Brownian process, i.e. dB_t is a random variable with mean 0 and variance dt. For $t \neq t'$, dB_t and $dB_{t'}$ are independent. The first term in (154) is called *the drift* and the second term is a white noise. We see that on average the signal evolves as the average value of the σ_z component of the system. Reinserting this expression and taking the continuous limit in (150), we obtain the SDE for the density matrix of the system:

$$d\rho_t = \left\{ \left(\sigma_z - \langle \sigma_z \rangle_t \right), \rho_t \right\} dB_t + \left(\sigma_z \rho_t \sigma_z - \frac{1}{2} \left\{ \sigma_z^2, \rho_t \right\} \right) dt. \tag{155}$$

The second part of the equation is of the GKLS form (**??**) that we are already familiar with. The first part is proportional to dB_t and comes from the randomness of the outcomes of the measurements on the ancilla. Importantly, remark it is *non-linear* in the density matrix.

In terms of matrix component, this gives

$$d\langle \sigma_z \rangle_t = 2 \left(1 - \langle \sigma_z \rangle_t^2 \right) dB_t, \tag{156}$$

$$d\langle \sigma_x \rangle_t = -2\langle \sigma_x \rangle_t \left(dt + \langle \sigma_z \rangle_t dB_t \right), \tag{157}$$

$$d\langle \sigma_y \rangle_t = -2\langle \sigma_y \rangle_t \left(dt + \langle \sigma_z \rangle_t dB_t \right). \tag{158}$$

Any process that can be written directly proportional to the noise is a Martingale, indeed $\mathbb{E}[d\langle\sigma_z\rangle_t|\mathcal{F}_t] = 0$ so that $\mathbb{E}[\langle\sigma_z\rangle_{t+dt}|\mathcal{F}_t] = \langle\sigma_z\rangle_t$. First equation is a bounded Martingale so it converges almost surely to a given random variable in the infinite time limit. In our case the only possible target random variables are the ones with peaked probability distributions $\delta\left(\langle\sigma_x\rangle \pm 1\right)$. To get the probability of attaining either of these distributions, we use the Martingale property to have $\mathbb{E}[\langle\sigma_z\rangle_\infty|\mathcal{F}_0] = \langle\sigma_z\rangle_{t=0}$. The latter is just Born rule!

For an experimental realization with superconducting qubit coupled to a radiating light field, see Campagne-Ibarcq et al. [6].

3.3 Quantum trajectories or how to win a factor of N in numerical simulation of open quantum systems

WIP...

A A bad introduction to stochastic differential equation (SDE)

In this appendix, we provide a bad first introduction to SDE. For a better introduction, you can read for instance Oksendal [7].

Consider the stochastic differential equation

$$\frac{dx}{dt} = f(x,t) + \text{"}\sigma(x,t)\xi_t\text{"} \tag{159}$$

where x is a random, continuous process in time, $f(x,t)$ and $\sigma(x,t)$ differentiable functions from $\mathbb{C}^2 \to \mathbb{C}$ and ξ_t a white noise, i.e $\mathbb{E}[\xi_t] = 0$ and $\mathbb{E}[\xi_t\xi_{t'}] = \delta(t-t')$. We put quotation marks because, as we will see, this equation taken as such is ill-defined. We want to make sense of the integral of such an equation, i.e:

$$x(t) = \int dt' f(x,t') + \text{"} \int dt' \sigma(x,t')\xi_{t'}^{\prime\prime}. \tag{160}$$

And to start, with we should try to make sense of

$$\text{"} \int dt' \xi_{t'} \text{"}. \tag{161}$$

Qualitatively, this represents a sum of random increments, that are all independent. The increments should average to 0 and their variance should be proportional to dt. However, we immediately run into a problem with this notation as the variance of the increment for small time step dt should be something like

$$\propto dt^2 \delta\left(0\right). \tag{162}$$

Something tempting would be to cancel the $\delta(0)$ contribution with one of the dt but it is not clear what it means from the mathematical point of view. A better starting point is to *first define the increment.* We call it dB_t and require that

$$\mathbb{E}\left[dB_t\right] = 0, \tag{163}$$

$$\mathbb{E}\left[dB_t^2\right] = dt, \tag{164}$$

$$\mathbb{E}\left[dB_t dB_{t'}\right] = 0 \text{ for } t \neq t'. \tag{165}$$

And our SDE would be rather written as

$$dx = f dt + \sigma dB_t. \tag{166}$$

Now the crucial problem is that if we try to construct integrals of dB_t using discrete Riemann summation, there is an a priori ill-definiteness that we expose now.

For the usual integration, if f is a $\mathcal{C}^1$ function, we have

$$\int_0^t f(t')dt' = \lim_{n\to\infty} \sum_i f(\tau_i)\left(t_{i+1} - t_i\right) \tag{167}$$

with $t = n\Delta t$, $t_i = i\Delta t$ and $\tau_i \in [t_i, t_{i+1}]$. The important point is that we are free to chose the point where we evaluate τ_i, it doesn't matter in the continuous limit as long as it belongs to the interval $[t_i, t_{i+1}]$. All possible prescriptions converge to the same quantity.

This is however *not true* for stochastic integration. Indeed, let's look at

$$\int_0^t B_{t'} dB_{t'} = \lim_{n\to\infty} \sum_{i=0}^{n-1} B_{\tau_i} \left(B_{t_{i+1}} - B_{t_i}\right). \tag{168}$$

We have that

$$\mathbb{E}\left[\int_0^t B_{t'} dB_{t'}\right] = \sum_{i=0}^{n-1} (\tau_i - t_i). \tag{169}$$

One sees that the result strongly depends on the choice of the prescription! For instance for $\tau_i = t_i$ we have

$$\mathbb{E}\left[\int_0^t B_{t'} dB_{t'}\right] = 0 \tag{170}$$

while for $\tau_i = \frac{t_i + t_{i+1}}{2}$, we have

$$\mathbb{E}\left[\int_0^t B_{t'} dB_{t'}\right] = t/2. \tag{171}$$

The most used conventions are precisely these two. The choice $\tau_i = t_i$ is called the *Itō* prescription while the choice $\tau_i = \frac{t_i + t_{i+1}}{2}$ is the *Stratonovich* prescription. Itō integrals will be noted $\int f dB_t$ while Stratonovich will be noted $\int f \circ dB_t$. They both have pros and cons that we list in the following table:

	Itō	Stratonovich
Pros	Easy to take averages	No obvious averaging
Cons	No chain rule, need additional Itō rules to differentiate	Chain rule works

Itō prescription In the Itō prescription taking average of the SDE is easy. Indeed:

$$dx = f(x,t)dt + \sigma(x,t)dB_t. \tag{172}$$

Since $\sigma(x,t)$ is independent of dB_t we have $\mathbb{E}[\sigma dB_t] = 0$ and

$$\mathbb{E}[dx] = [f]\, dt \tag{173}$$

We admit that the differentiation of a function $y(x)$ works as follows

$$dy = \partial_x y dx + \frac{1}{2}\partial_x^2 y \,(dx)^2, \tag{174}$$

$$= \partial_x y \,(f dt + \sigma dB_t) + \frac{1}{2}\partial_x^2 y \,(dB_t)^2, \tag{175}$$

$$= \left(\partial_x y f + \frac{1}{2}\partial_x^2 y \sigma^2\right) dt + \partial_x y \sigma dB_t. \tag{176}$$

To pass from the the second to the third line, we introduced an algebraic identity called the *Itō rule* $dB_t^2 = dt$. Importantly there is *no average* involved in the previous relation. What the Itō rule stipulates is that the stochastic process where all the dB_t^2 have been replaced by dt converge to the same process where they have been not when taking the continuous limit. For the proof of this statement, we refer to Oksendal [7].

A qualitative explanation for why we need to keep track of the second order expansion is that we have term proportional to dB_t in the SDE for x which scale like $\sqrt{dt}$ since B_t represents a Brownian process. Thus terms like dB_t^2 will give a contribution of order dt.

For the same reason, the Leibniz rule for the differentiation of a product has to be modified:

$$d(xy) = xdy + dxy + dxdy. \tag{177}$$

Conversion from Itō to Stratonovich. Let $\tilde{t}_i := \frac{t_i+t_{i+1}}{2}$, we can convert between the Itō and Stratonovich integral through the following steps:

$$\int_0^t x(B_{t'},t') \circ dB_{t'} = \lim_{n\to\infty} \sum_i x\left(B_{\tilde{t}_i},\tilde{t}_i\right)\left(B_{t_{i+1}}-B_{t_i}\right), \tag{178}$$

$$= \lim_{n\to\infty} \left[\sum_i x\left(B_{\tilde{t}_i},\tilde{t}_i\right)\left(B_{t_{i+1}}-B_{\tilde{t}_i}\right) + x\left(B_{\tilde{t}_i},\tilde{t}_i\right)\left(B_{\tilde{t}_i}-B_{t_i}\right)\right], \tag{179}$$

$$= \lim_{n\to\infty} \left[\sum_i x\left(B_{t_i},\tilde{t}_i\right)\left(B_{t_{i+1}}-B_{\tilde{t}_i}\right) + x\left(B_{t_i},t_i\right)\left(B_{\tilde{t}_i}-B_{t_i}\right) + \sigma\left(B_{t_i},t_i\right)\left(B_{\tilde{t}_i}-B_{t_i}\right)\left(B_{\tilde{t}_i}-B_{t_i}\right)\right], \tag{180}$$

$$= \int_0^t x\left(B_{t'},t'\right) dB_{t'} + \frac{1}{2}\sigma\left(B_{t'},t'\right) dt' \tag{181}$$

where in the last line we used the Itō rule $dB_t^2 = dt$.

In integrand form this relation is

$$x \circ dB_t = x dB_t + \frac{1}{2}\sigma dt \tag{182}$$

Hence, the equation in the Itō form

$$dx = f dt + \sigma dB, \tag{183}$$

corresponds to the Stratonovich equation

$$dx = f dt - \frac{1}{2}\partial_x\sigma\sigma dt + \sigma \circ dB_t, \tag{184}$$

where we used the fact that the SDE followed by σ is given by

$$d\sigma(x,t) = \left(\partial_x\sigma f + \frac{1}{2}\partial_x^2\sigma\sigma^2\right) dt + \partial_x\sigma\sigma dB_t. \tag{185}$$

We can now easily check that the chain rule is verified for Stratonovich. Recall the differential of $y(x)$ for Itō:

$$dy = \left(\partial_x y f + \frac{1}{2}\partial_x^2 y\sigma^2\right) dt + \partial_x y\sigma dB_t. \tag{186}$$

The SDE for $\partial_x y\sigma$ is given by

$$d\left(\partial_x y\sigma\right) = \left(\partial_x^2 y\sigma^2 + \partial_x y\partial_x\sigma\sigma\right) dB_t + O(dt), \tag{187}$$

so that

$$\left(\partial_x y\sigma\right) dB_t = \left(\partial_x y\sigma\right) \circ dB_t - \frac{1}{2}\left(\partial_x^2 y\sigma^2 + \partial_x y\partial_x\sigma\sigma\right) dt \tag{188}$$

Replacing this into the expression (186), we end up with

$$dy = \left(\partial_x y f\right) dt - \frac{1}{2}\left(\partial_x y\partial_x\sigma\sigma\right) dt + \left(\partial_x y\sigma\right) \circ dB_t, \tag{189}$$

and we see that indeed

$$dy = \partial_x y dx \tag{190}$$

for Stratonovich.

Finally, let us check that Leibniz rule is fulfilled for Stratonovich. Let x and y be two stochastic processes following the SDEs

$$dx = f_x dt + \sigma_x dB_t, \tag{191}$$

$$dy = f_y dt + \sigma_y dB_t, \tag{192}$$

$$dx = f_x dt - \frac{1}{2}\partial_x\sigma_x\sigma_x + \sigma_x \circ dB_t, \tag{193}$$

$$dy = f_y dt - \frac{1}{2}\partial_y\sigma_y\sigma_y + \sigma_y \circ dB_t, \tag{194}$$

$$d(xy) = xdy + dxy + dxdy, \tag{195}$$
$$= (x\sigma_y + y\sigma_x)\, dB_t + + (xf_y + yf_x)\, dt + \sigma_x\sigma_y dt, \tag{196}$$

and

$$d(x\sigma_y + y\sigma_x) = (\sigma_x\sigma_y + x\partial_y\sigma_y\sigma_y + \sigma_y\sigma_x + y\partial_x\sigma_x\sigma_x)\, dB_t + O(dt) \tag{197}$$

and we indeed have that

$$d(xy) = (x\sigma_y + y\sigma_x) \circ dB_t - \frac{1}{2}(\sigma_x\sigma_y + x\partial_y\sigma_y\sigma_y + \sigma_y\sigma_x + y\partial_x\sigma_x\sigma_x)\, dt + (xf_y + yf_x)\, dt + \sigma_x\sigma_y dt, \tag{198}$$
$$= x\left(\sigma_y \circ dB_t - \frac{1}{2}\partial_y\sigma_y\sigma_y dt + f_y dt\right) + y\left(\sigma_x \circ dB_t - \frac{1}{2}\partial_x\sigma_x\sigma_x dt + f_x dt\right), \tag{199}$$
$$= x \circ dy + y \circ dx. \tag{200}$$

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