Multivariate phantom distributions

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- We are going to present an original view into this apparently classic theory.
- This presentation is purely theoretical, but promising simulation studies and analysis of real data are coming.

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- This parallels the theory for sums, leads to the notion of max-stable distributions, domains of attraction etc.
- We claim that the asymptotics of $1 - F(a_n x_0 + b_n)$ for a single x_0 such that $0 < K(x_0) < 1$ determines everything.

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- $$\sup_{x \in \mathbb{R}^1} \big|G^n(x) - H^n(x)\big| \to 0, \text{ as } n \to \infty.$$
- H is regular and strictly tail-equivalent to G:

$$G_* = H_* \quad \text{and} \quad \frac{1 - H(x)}{1 - G(x)} \to 1, \text{ as } x \to G_*-.$$

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- The phantom distribution function is of essentially wider applicability.

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- If r_n is such that $r_n \ln n \to \rho > 0$, then there is no phantom distribution function.
- There are stationary sequences for which the extremal index is uninformative while phantom distribution functions do exist.

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- Following Leadbetter (1983) we say that $\{X_j\}$ has the extremal index $\theta = 0$ if $\mathbb{P}(M_n \leqslant u_n(\tau)) \to 1$ whenever $\{u_n(\tau)\}$ is such that $n(1 - F(u_n(\tau)) \to \tau \in (0, +\infty)$.

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- Asmussen (1998) The Lindley process

$$X_{j+1} = (X_j + Z_j)^+, \quad j = 1, 2, \dots,$$

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- In both cases a continuous phantom distribution function exists.

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- Clearly, the extremal index $\theta \in [0, 1]$ exists iff $\theta^+ = \theta^-(= \theta)$.

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- Let $v_n \nearrow F_*$ and define $m_k = [\log_4 k] \mod 2$, $k \geqslant 1$.
- $P(X_j \leqslant x \mid \xi = k) = F_k(x)$ is given by

$$F_k(x) = \begin{cases} 0, & x \leqslant v_1 , \\ 1 - 1/\sqrt{n}, & x \in (v_n, v_{n+1}] \text{, if } n < k^2 \text{ and } m_k = 0 , \\ 1 - 1/(2\sqrt{n}), & x \in (v_n, v_{n+1}] \text{, if } n < k^2 \text{ and } m_k = 1 , \\ 1 - 1/n, & x \in (v_n, v_{n+1}] \text{, } n \geqslant k^2 . \end{cases}$$

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- On the other hand, if we choose n such that $\sqrt{n}$ is an integer, then

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\begin{aligned}
n\,\mathbb{P}(X_1 > v_n) &= \sum_{1\leqslant k<\sqrt{n}} p_k + \sqrt{n} \sum_{\sqrt{n}\leqslant k} p_k(m_k+1)^{-1} \\
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while for $n = 4^{4\ell+2}$ we obtain

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- This can be clearly seen in the general criterion for existence of phantom distribution functions.

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- There exist: a non-decreasing sequence $\{v_n\}$ and a number $\gamma \in (0, 1)$ such that on some dense subset $\mathbb{Q} \subset \mathbb{R}^+$
$$\mathbb{P}(M_{\lfloor nt \rfloor} \leqslant v_n) \to \gamma^t, \ t \in \mathbb{Q}.$$

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with partial maxima

$$\mathbf{M}_n = (M_n^{(1)}, M_n^{(2)}) = \big(\max_{1 \leqslant j \leqslant n} X_j^{(1)}, \max_{1 \leqslant j \leqslant n} X_j^{(2)} \big),$$

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$$\sup_{\mathbf{u}=(u_1,u_2)\in\mathbb{R}^2} \left| \mathbb{P}(\mathbf{M}_n \leqslant \mathbf{u}) - G^n(\mathbf{u}) \right| \to 0, \text{ as } n \to \infty.$$

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$$\sup_{\mathbf{u}=(u_1,u_2)\in\mathbb{R}^2} \Big| \mathbb{P}(\mathbf{M}_n \leqslant \mathbf{u}) - G^n(\mathbf{u}) \Big| \to 0, \text{ as } n \to \infty.$$

- In fact, it is more convenient to take $\sup$ over $\overline{\mathbb{R}}^2$!

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- Consider

$$\mathcal{L} = \{\mathbf{s} \in [1, +\infty)^2 \,;\, s_1 \wedge s_2 = 1\}.$$

- Assume that for some $\rho : \mathcal{L} \to (0, 1)$

$$\mathbb{P}(\mathbf{M}_n \leqslant v_n(\mathbf{s})) \to \rho(\mathbf{s}), \quad \mathbf{s} \in \mathcal{L}.$$

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- Assume that $B_\infty(v_n(\mathbf{s}))$ holds for every $\mathbf{s} \in \mathcal{L}$, i.e. for all sequences p_n and q_n and as $n \to \infty$

$$\mathbb{P}(\mathbf{M}_{p_n+q_n} \leqslant v_n(\mathbf{s})) - \mathbb{P}(\mathbf{M}_{p_n} \leqslant v_n(\mathbf{s}))\mathbb{P}(\mathbf{M}_{\mathbf{q_n}} \leqslant v_n(\mathbf{s})) \to 0.$$

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Theorem

- Condition $B_\infty(v_n(\mathbf{s}))$ holds for every $\mathbf{s} \in [0, +\infty]^2$.
- There exists $H : [0, +\infty]^2 \to [0, 1]$ such that

$$\mathbb{P}(\mathbf{M}_n \leqslant v_n(\mathbf{s})) \to H(\mathbf{s}), \quad \mathbf{s} \in [0, +\infty]^2.$$

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$H(\mathbf{s})$ defined on $[0, +\infty)^2$ is the cumulative distribution function of a two-dimensional extreme value distribution.

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Moreover, if $H^{(1)}$ and $H^{(2)}$ are the marginal cumulative distribution functions, then

$$H^{(i)}((-\log \rho_i)s) = G_{2,1}(s), i = 1, 2,$$

where $G_{2,1}(s)$ is the CDF of the standard Fréchet extreme value distribution.

Phantom distribution function for random vectors

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Theorem

$$G(\mathbf{x}) = H(\mathbf{n}(\mathbf{x})),$$

where

$$n_i(\mathbf{x}) = \sup\{n \in \mathbb{N}\,;\; v_n^{(i)} \leqslant x_i\}, \quad i = 1, 2,$$

is a phantom distribution function for $\mathbf{X}_1, \mathbf{X}_2, \ldots$.

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- The task is then to find an efficient description of $\theta(x, y)$ by comparison of tails of G and F.